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AN EXOTIC 4-MANIFOLD

SELMAN AKBULUT

In [1] we have constructed a fake smooth structure on a contractible 4-manifold W^4 relative to boundary. This is a smooth manifold V with $\partial V = \partial W$ such that the identity map $\partial V \to \partial W$ extends to a homeomorphism but not to a diffeomorphism $V \to W$. This is a relative result in the sense that V itself is diffeomorphic to W, even though no such diffeomorphism can extend the identity map on the boundary. Here we strengthen this result by dropping the boundary hypothesis at the expense of slightly enlarging W: We construct two compact smooth 4-manifolds Q_1, Q_2 which are homeomorphic but not diffeomorphic to each other. In particular *no* diffeomorphism $\partial Q_1 \to \partial Q_2$ can extend to a diffeomorphism $Q_1 \to Q_2$.

Let Q_i^4, $i = 1, 2$, be the 4-manifolds obtained by attaching 2-handles to B^4 along knots K_i, $i = 1, 2$, with $+1$-framing (see Figures 1 and 2). Clearly Q_1 and Q_2 are homotopy equivalent to $\mathbf{C}P_0^2 = \mathbf{C}P^2 - \text{int}(B^4)$, and it will be shown that $\partial Q_1 = \partial Q_2$.

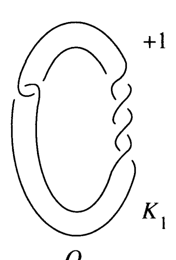

FIGURE 1

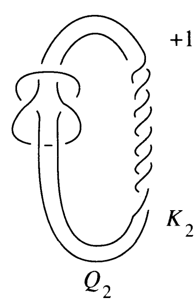

FIGURE 2

Theorem 1. *Q_1 and Q_2 are homeomorphic but not diffeomorphic to each other. In fact, even their interiors are not diffeomorphic to each other.*

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The fact that they are homeomorphic to each other follows from [2]. The proof of Theorem 1 uses the method of [1]; as a by-product we will get the following result: Let Q_3 be the 4-manifold obtained by attaching a 2-handle to B^4 along the knot K_3 with $+1$-framing (Figure 3).

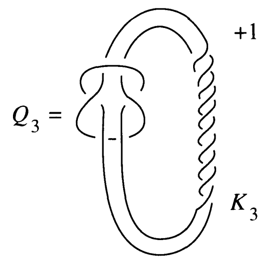


FIGURE 3

Theorem 2. *There is a diffeomorphism $f\colon \partial Q_3 \to \partial Q_3$ which extends to a self-homeomorphism of Q_3, but f cannot extend to a self-diffeomorphism of Q_3.*

It is an easy exercise to show that f extends to a homotopy equivalence, hence by [2] it extends to a homeomorphism. The existence of Q_2 is already contained in [1] as $W_1 \# \mathbf{C}P^2$, but not as B^4 with a single 2-handle. We use the usual conventions: $\simeq$ for homotopy equivalence and $\approx$ for diffeomorphism. For every oriented manifold M, we denote the oppositely oriented manifold by $-M$. Also we denote $-\mathbf{C}P^2$ by $\overline{\mathbf{C}P}^2$.

In [1] we constructed a 1-connected compact smooth 4-manifold M_1 with $\partial M_1 = \partial Q_1$. M_1 has the properties: M_1 is even with signature 16 and has second betti number $b_2(M) = 22$. If V is any smooth contractible manifold with $\partial M_1 = \partial V$, then if we call $\widetilde{M} = M_1 \cup_\partial (-Q_1)$ and $M' = M_1 \cup_\partial (-V)$ we have:

$$(1) \qquad \widetilde{M} \approx (3\mathbf{C}P^2) \# (20\overline{\mathbf{C}P}^2),$$

$$(2) \qquad (M' \# \overline{\mathbf{C}P}^2) \# (k\overline{\mathbf{C}P}^2) \not\approx \widetilde{M} \# (k\overline{\mathbf{C}P}^2), \qquad k = 0, 1, 2, \ldots.$$

Furthermore $\widetilde{M}$ is obtained from $M' \# \overline{\mathbf{C}P}^2$ (for some choice of V) by removing a contractible manifold W and regluing with a diffeomorphism $f\colon \partial W \to \partial W$ as described in [1]. That is, for some smooth N with $\partial N = \partial W$ we have:

(i) $M' \# \overline{\mathbf{C}P}^2 = N \cup_\partial (-W)$,

(ii) $\widetilde{M} = N \cup_f (-W)$.

Let W_k be the contractible manifold of Figure 4. By [1] $\partial M_1 = \partial W_1$. We claim that $W_k \# \mathbf{C}P^2 \approx N_k$, where N_k is the manifold of Figure 8.

This can be seen as follows: Figure 5 is $W_k \# \mathbf{C}P^2$, by a handle slide and an isotopy we obtain Figures 6 and 7. After cancelling the 1 and 2 handle pair in Figure 7 we obtain Figure 8 ($-k+5$ in the figure indicates that many full twists across the two strands). Since $N_1 = Q_2$, $N_0 = Q_3$, and $W_0 = W$ of [1] we have

(a) $W_1 \# \mathbf{C}P^2 = Q_2$,
(b) $W \# \mathbf{C}P^2 = Q_3$.

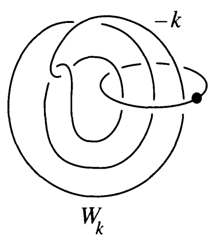


FIGURE 4

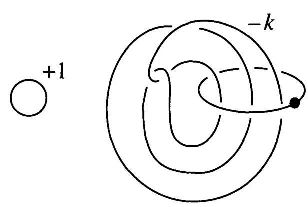


FIGURE 5

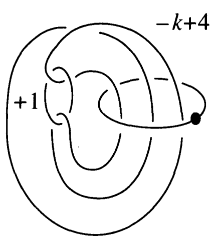


FIGURE 6

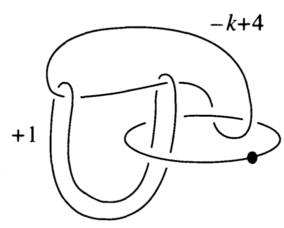


FIGURE 7

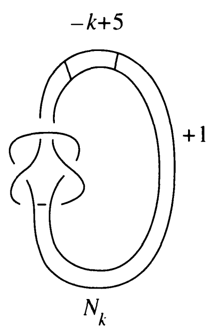


FIGURE 8

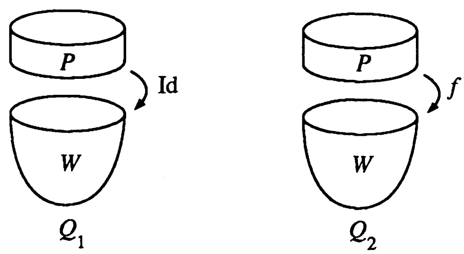
PROOF OF THEOREM 1. If $\operatorname{Int}(Q_1) \approx \operatorname{Int}(Q_2)$, then by (a) $\operatorname{Int}(Q_1) \approx \operatorname{Int}(W_1 \# \mathbf{C}P^2)$. Then Q_1 would have a smoothly imbedded $S^2 \hookrightarrow Q_1$ with self-intersection $+1$. This would imply $Q_1 \approx W' \# \mathbf{C}P^2$ for some contractible W' with $\partial W' = \partial Q_1$. Hence

$$\widetilde{M} \approx M_1 \cup_\partial (-W' \# \overline{\mathbf{C}P}^2) \approx M' \# \overline{\mathbf{C}P}^2,$$

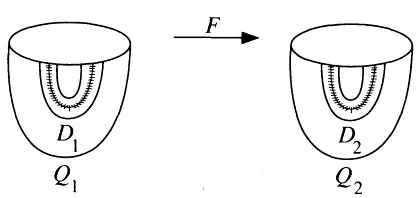
where $M' = M_1 \cup_\partial (-W')$. This contradicts (2). q.e.d.

PROOF OF THEOREM 2. By (b) $W \# \mathbf{C}P^2 = Q_3$, hence by (i) and (ii) $M' \# \overline{\mathbf{C}P}^2 \# \overline{\mathbf{C}P}^2 = N \cup_\partial (-W \# \overline{\mathbf{C}P}^2) \approx N \cup_\partial (-Q_3)$, and $\widetilde{M} \# \overline{\mathbf{C}P}^2 = N \cup_f (-W \# \overline{\mathbf{C}P}^2) = N \cup_f (-Q_3)$. So if $f\colon \partial(-Q_3) \to \partial(-Q_3)$ extended to a diffeomorphism of $-Q_3$, $M' \# (2\overline{\mathbf{C}P}^2)$ would be diffeomorphic to $\widetilde{M} \# \overline{\mathbf{C}P}^2$ contradicting (2). q.e.d.

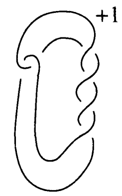
Remark. Q_1 is obtained from Q_2 by removing the contractible manifold W from the interior and glueing it back by a diffeomorphism (i.e., Gluck contraction to W). That is, we can write $Q_1 = P \cup_\partial W$ and $Q_2 = P \cup_f W$ for some smooth P with $\partial P = \partial Q_1 \coprod \partial W$.



This can be seen as follows. Q_1 (Figure 9) is diffeomorphic to Figure 10, and Q_2 (Figure 12) is diffeomorphic to Figure 11. There is a diffeomorphism F between the boundaries of Figures 10 and 11, induced by the obvious involution f (of [1]). F carries the loop γ of Figure 10 to the loop $F(\gamma)$ of Figure 11. γ and $F(\gamma)$ bound obvious discs D_1, D_2 in Q_1 (Figure 10) and Q_2 (Figure 11), respectively. We can extend F across these discs:



Let $(D_i \times B^2, \partial D_i \times B^2) \hookrightarrow (Q_i, \partial Q_i)$, $i = 1, 2$, be the tubular neighborhoods of these discs. Then obviously $Q_i - D_i \times B^2 \approx W$ for $i = 1, 2$, and F induces $f\colon \partial W \to \partial W$. q.e.d.



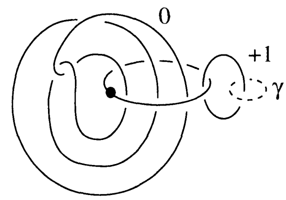
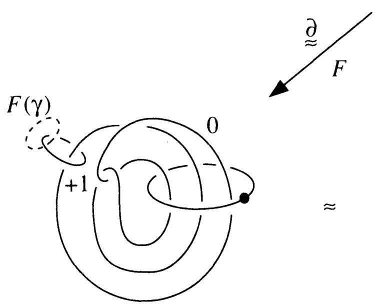


FIGURE 9 FIGURE 10



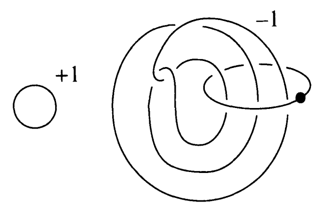


FIGURE 11 FIGURE 12

References

[1] S. Akbulut, *A fake compact contractible 4-manifold*, J. Differential Geometry **33** (1991).
[2] M. Freedman, *The topology of 4-dimensional manifolds*, J. Differential Geometry **17** (1982) 357–453.

MICHIGAN STATE UNIVERSITY