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L^2-INDEX THEOREMS ON CERTAIN COMPLETE MANIFOLDS

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1. Introduction

Consider a Riemannian manifold M, Hermitian vector bundles E and F over M, and a first order elliptic differential operator $D: C^\infty(E) \to C^\infty(F)$. Such operators arise naturally from the Riemannian structure like the Gauss-Bonnet and the signature operator; more generally, one can consider the Dirac operators in the sense of [10]. Being a differential operator, D has closed extensions $\overline{D}$ mapping the Hilbert space $\mathscr{D}(\overline{D})$ (with the graph norm) to $L^2(F)$. In particular, there is the closure $D_{\min}$ and the maximal extension $D_{\max} = (D'_{\min})^*$, where $D': C^\infty(F) \to C^\infty(E)$ is the formal adjoint. If M is complete, then $D_{\max} = D_{\min}$ for all Dirac operators. Moreover, if M is compact, then $D_{\max}$ is a Fredholm operator, and its index is given by the celebrated Atiyah-Singer index formula. In general, D may or may not have a Fredholm extension. In this work we deal with a class of operators which need not be Fredholm but have a finite L^2-index in the sense that $\ker D \cap L^2(E)$ and $\ker D' \cap L^2(F)$ both have finite dimension; then we define

$$(1.1) \qquad L^2\text{-}\operatorname{ind} D := \dim\ker D \cap L^2(E) - \dim\ker D' \cap L(F).$$

We will also assume that M is complete and $D_{\max} = D_{\min}$. Then if $D_{\min}$ is Fredholm, we have $\operatorname{ind} D_{\min} = L^2\text{-}\operatorname{ind} D$, but our assumptions will not imply the Fredholm property. Note that if D has a finite L^2-index, then a closed extension $\overline{D}$ is Fredholm if and only if the essential spectrum $\sigma_e(\overline{D}^*\overline{D})$ of the self-adjoint operator $\overline{D}^*\overline{D}$ has a positive lower bound. Still, the situation which we treat should be regarded as a type I case in the sense of [13].

Our model case is a complete manifold with finitely many ends which are all warped products. It follows from simple examples that the L^2-cohomology for such manifolds can be infinite, so we need a condition on the warping function f (formula (2.14) below) which allows at most

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linear growth. Then we observe that a geometric operator D on a warped product has a particularly simple normal form, as an operator valued ordinary differential equation (cf. (2.3) and (2.4)). This allows us to construct a weight function g with the property that gDg is a Fredholm operator, even if D is not. Moreover, gDg is unitarily equivalent to a regular singular operator in the sense of [5], which enables us to compute the index of all closed extensions. This means that we produce a normal form, for the weighted operators, which does not involve the warping function any more. On the other hand, we introduce boundary conditions for gDg (in most cases), but it turns out that we always have a very natural choice. Moreover, the transformation avoids the analysis of boundary integrals. To obtain an index formula we have to relate the L^2-index of D to the index of a suitable closed extension of gDg. Whereas it is easy to see that under our assumptions the L^2-index is always finite we do not succeed in computing it in all cases. It seems that the difficulty arises whenever D is itself not Fredholm and the operator S_0 occuring in its normal form has small eigenvalues. The structure of the index formula is as follows. It contains interior terms, involving the geometry of the whole manifold, the spectral invariants of the cross-section such as the η-invariant, and the global contributions which can be expressed in terms of the solutions of an ordinary differential equation $\mathbf{R}_+$ (cf. Theorem 4.3).

We derive an abstract version of the described geometric situation, allowing for perturbations. Then the most definite result is Theorem 4.3. We apply this to various geometric situations and obtain a unified and sometimes more general treatment of the known results in these cases. It may be of interest to note that we also obtain nonlocal contributions for the L^2-index of the Gauss-Bonnet operator similar to the conic case treated by Cheeger and, more completely, in [5] (cf. Corollary 5.4).

The plan of the paper is as follows. In §2 we introduce the class of operators to be considered. Then we reduce the problem to an index calculation for a regular singular operator in the sense of [5] by introducing a suitable weight function. The necessary analysis is carried out in §3 and we prove the index theorem in §4. §5 contains the applications to manifolds which are asymptotically warped products.

List of notations.

$M_1 \subset M$ is a compact manifold with boundary (cf. (2.1));
H is a Hilbert space, and $H_1 \subset H$ is a dense subspace;
E and F are Hermitian vector bundles over M;
$D\colon C_0^\infty(E) \to C_0^\infty(F)$ is an elliptic first order differential operator;

$D'\colon C_0^\infty(F) \to C_0^\infty(E)$ is the formal adjoint;
$D_g = gDg$ is a weighted operator obtained from D;
$D_g' = gD'g$;
$\mathscr{H} = L^2(E|M_1) \oplus L^2(\mathbf{R}_+, H)$, $\mathscr{H}' = L^2(F|M_1) \oplus L^2(\mathbf{R}_+, H)$;
$\widetilde{\mathscr{H}} = L^2(E|M_1) \oplus L^2([0, 1], H)$, $\widetilde{\mathscr{H}}' = L^2(F|M_1) \oplus L^2([0, 1], H)$;
X is the bounded operator on $L^2([0, 1], H)$ defined by $Xf(x) = xf(x)$.

2. The class of operators

The class of operators which we consider is suggested by the example of warped products. Therefore, we describe first the model situation in some detail. Thus assume that M is complete and that there is an open subset $U \subset M$ such that

(2.1) $\quad M_1 := M \setminus U$ is a compact manifold with boundary,

(2.2) $\quad$ U is isometric to $(0, \infty) \times N$ with metric $g = dy^2 + f(y)^2 g_N$, where $N = \partial M_1$ is (compact) Riemannian with metric g_N.

Then it is a matter of calculation to obtain unitary representations of the geometric operators on U as simple ordinary differential operators with operator coefficients; this is, of course, done by separating the canonical variable y on $(0, \infty)$. We will present a general scheme for this in a future publication. For the time being we simply mention two important examples which will suffice for our applications.

Example 1. Assume $m = \dim M \equiv 0 \bmod 2$. The Gauss-Bonnet operator

$$D_{GB} := d + \delta\colon \Omega^{\mathrm{ev}}(U) \to \Omega^{\mathrm{odd}}(U)$$

is unitarily equivalent to

$$\partial_y + \frac{1}{f(y)}\left(S_0 + S_1(y)\right) : C_0^\infty\left((0, \infty), H_1\right) \to C_0^\infty\left((0, \infty), H\right), \tag{2.3}$$

where

$$\begin{gathered} H := L^2\left(\Lambda^* N\right), \qquad H_1 := H^1\left(\Lambda^* N\right), \\ S_0 := d_N + \delta_N\colon H_1 \to H, \\ S_1(y) := f'(y)\operatorname{diag}\left((-1)^j\left(j - \frac{n}{2}\right)\right)_{0 \le j \le n}, \end{gathered} \tag{2.4}$$

and $n := \dim N$ (cf. [5] for more details).

Example 2. Assume $m \equiv 0 \bmod 4$ and denote by $\Omega^{\pm}(U)$ the ± 1 eigenspace of the involution τ on $\Omega(U)$ given by multiplication with $\sqrt{-1}^{m/2+j(j-1)} *$ on $\Omega^j(U)$. Then the signature operator

$$D_S = d + \delta : \Omega^+(U) \to \Omega^-(U)$$

is unitarily equivalent to an operator of the form (2.3), where H and H_1 are as in Example 1, but

$$\begin{aligned} S_0\omega &:= (-1)^{(m/4)+1+[(j+1)/2]}\left((-1)^j *_N d_N - d_{N^*N}\right)\omega \\ &\qquad\qquad \text{for } \omega \in \Omega^j(N), \\ S_1(y) &:= f'(y)\,\mathrm{diag}(n/2 - j)_{0\le j\le n}. \end{aligned} \tag{2.5}$$

Again, more details can be found in [5].

In the spirit of [4], [5] we introduce an abstract version of these examples. We assume again (2.1) and consider a first order elliptic differential operator $D\colon C^\infty(E) \to C^\infty(F)$ on M. We replace (2.2) by the following assumption.

(2.6) There is a Hilbert space H with isometries $\Phi_E\colon L^2(E|U) \to L^2((0,\infty), H)$, $\Phi_F\colon L^2(F|U) \to L^2((0,\infty), H)$ such that Φ_E, Φ_F induce isomorphisms $H_0^1(E|\overline{U}) \simeq H_0^1([0,\infty), H) \cap L^2((0,\infty), H_1) \simeq H_0^1(F|\overline{U})$. Moreover, there is a self-adjoint operator S_0 in H with domain $\mathscr{D}(S_0) := H_1$, a smooth function $(0,\infty) \ni y \mapsto S_1(y) \in \mathscr{L}(H_1, H)$, a positive function $f \in C^\infty(\mathbf{R}_+)$, and smooth functions
$$(0,\infty) \ni y \mapsto A_j(y) \in \mathscr{L}(H) \cap \mathscr{L}(H_1), \qquad j = 1, 2,$$
such that for $u \in C_0^\infty((0,\infty), H_1)$ and $y \in (0,\infty)$, we have
$$\Phi_F D \Phi_E^{-1} u(y) = A_1(y)\,\partial_y A_2(y)\,u(y) + \tfrac{1}{f(y)}\left(S_0 + S_1(y)\right)u(y).$$

A Dirac operator on a complete manifold has a unique closed extension [10, Theorem 5.7], so it is reasonable to assume

$$D_{\max} = D_{\min}. \tag{2.7}$$

This implies that the L^2-kernel and L^2-cokernel of D are respectively the kernel and cokernel of the unique closed extension. In what follows, the unique closed extension will also be denoted by D. From (2.6) we derive unitary isomorphisms

$$\begin{aligned} \Phi &\colon L^2(E) \to L^2(E|M_1) \oplus L^2((0,\infty), H) =: \mathscr{H}, \\ \Phi' &\colon L^2(F) \to L^2(F|M_1) \oplus L^2((0,\infty), H) =: \mathscr{H}', \end{aligned} \tag{2.8}$$

and the domain $\mathscr{D}(D)$ of D can be identified with a subspace of $\mathscr{H}$. In order to localize the analysis on U, we want to multiply by C^∞ functions. As in [4] we put $C^\sim(\mathbf{R}_+) := \{\varphi \in C^\infty(\mathbf{R}^+) | \varphi$ is constant near 0 and near $\infty\}$. For $\varphi \in C^\sim(\mathbf{R}^+)$ and $u = (u_i, u_b) \in \mathscr{H}^{(\prime)}$ we define

$$\varphi u := \left(\varphi(0)\, u_i, \varphi_{u_b}\right) \in \mathscr{H}^{(\prime)},$$

and require that

$$\Phi^{(\prime)-1} \varphi u = \overline{\varphi} \Phi^{(\prime)-1} u \tag{2.9}$$

for some $\overline{\varphi} \in C^\infty(M)$, with $\overline{\varphi} \in C_0^\infty(M)$ if $\varphi \in C_0^\infty[0, \infty)$. Clearly, elements $u = (u_i, u_b)$ of $\mathscr{D}(D)$ will have to satisfy a "transmission condition" at $N = \partial M_1$. To formulate it we observe that for $u \in H_0^1(E)$ with $\Phi u = (u_i, u_b)$, and for $v \in H_0^1(F)$ with $\Phi' v = (v_1, v_b)$ we have

$$u_i \Big| N = \Phi_E^{-1} u_b \Big| N, \qquad v_i \Big| N = \Phi_F^{-1} v_b \Big| N. \tag{2.10}$$

Now we define the "boundary space"

(2.11)
$$\mathscr{D}_b := \Big\{ u \in L^2_{\text{loc}}([0, \infty), H_1) \cap H^1_{\text{loc}}([0, \infty), H) \cap L^2((0, \infty), H) \\ | u' + \tfrac{1}{j}(S_0 + S_1) u \in L^2((0, \infty), H) \Big\},$$

and we obtain that

$$\begin{aligned} \mathscr{D}(D) &= H^1(E|M_1) \oplus_t \mathscr{D}_b \\ &:= \left\{ (u_i, u_b) \in H^1(E|M_1) \oplus \mathscr{D}_b | u_i | N = \Phi_E^{-1} u_b | N \right\}. \end{aligned} \tag{2.12}$$

It is easily checked that in the above examples the assumptions (2.6), (2.7), and (2.9) are satisfied. In addition, there is a Hermitian vector bundle G over N such that

(2.13)
$H = L^2(G), \qquad H_1 = H^1(G)$,
S_0 is a symmetric first order elliptic differential operator on $C^\infty(G)$, and
$S_1(y)$ is a smooth family of first order differential operators on $C^\infty(G)$.

(2.13) will not be necessary for most of our arguments.

In the setting just described we now proceed to derive an L^2-index theorem. This will not hold without further restrictions on our data as can be seen from the example of the Gauss-Bonnet operator for rotationally invariant metrics in $\mathbf{R}^n$ (cf. [6]). It will be necessary to have

$$\int_0^\infty \frac{dy}{f(y)} = \infty,$$

which will follow if we assume

$$f'(y) = a + o(1) \quad \text{as } y \to \infty, \text{ for some } a \geq 0. \tag{2.14}$$

Moreover, S_1 is thought of as a small perturbation of S_0, which is expressed by

$$\left\| S_1(y)\left(|S_0| + 1\right)^{-1} \right\|_H + \left\| \left(|S_0| + 1\right)^{-1} S_1(y) \right\|_H = o(1) \quad \text{as } y \to \infty. \tag{2.15}$$

Finally, A_1 and A_2 have to be close to the identity in the following sense:

$$\begin{aligned} &\left\| \left(f(y)\partial_y\right)^i \left(A_j(y) - I\right) \right\|_{H_1} + \left\| \left(f(y)\partial_y\right)^i \left(A_j(y) - I\right) \right\|_H \\ &\qquad = o(1) \quad \text{as } y \to \infty \text{ for } i = 0, 1, \ j = 1, 2. \end{aligned} \tag{2.16}$$

Remarks. (1) From (2.14) it follows that

$$f(y) = ay + o(y) \quad \text{as } y \to \infty. \tag{2.17}$$

(2) Being elliptic on a compact manifold, S_0 has a discrete spectrum. For $\alpha > 0$ we may replace f by αf and S_0, S_1 by αS_0, αS_1 without changing the assumptions (2.14), (2.15), and (2.16). Thus we may assume that

$$0 \leq a < 1 \quad \text{and} \pm \tfrac{1}{2} \notin \operatorname{spec} S_0 \tag{2.18}$$

which will make it possible to apply the analysis of the next section.

(3) All our conditions are translation invariant, i.e., invariant under the change of variable $y \mapsto y + R$, $R > 0$.

Under these assumptions we are going to show that the L^2-index of D is finite, and we will obtain an inequality for it which, in some interesting cases, is an equality. Set

$$F(y) := \int_0^y \frac{du}{f(u)} \tag{2.19}$$

and let $\overline{g} \in C^\infty(M)$ be a positive function such that

$$g^2(y) = f(y)\, e^{F(y)} \quad \text{for } y \text{ sufficiently large}, \tag{2.20}$$

using (2.9). A convenient way to construct g^2 is as follows. Select $y_0 > 0$ and choose $\psi = \psi_{y_0} \in C^\infty(\mathbf{R})$ such that

$$0 \le \psi \le 1, \qquad \psi(y) = 0 \quad \text{if } y \le y_0, \qquad \psi(y) = 1 \quad \text{if } y \ge 2y_0.$$

Then put

$$g^2(y) := ((1-\psi)(y) f(0) + \psi f(y))\, e^{\psi F(y)} \tag{2.20$'$}$$

and $\overline{g}|M_1 := f(0)^{1/2}$.

Lemma 2.1. $\lim_{y\to\infty} f(y) = \infty$, *the function*

$$s(y) := \int_y^\infty \frac{du}{g(u)^2} \tag{2.21}$$

is a diffeomorphism $(0, \infty) \to (0, s(0))$, *and for* y *sufficiently large we have*

$$s(y) = e^{-F(y)}. \tag{2.22}$$

Moreover,

$$\Psi\colon C_0^\infty((0, s(0)), H) \ni u \mapsto \tfrac{1}{g} u \circ s \in C_0^\infty((0, \infty), H) \tag{2.23}$$

is unitary with respect to the obvious L^2*-structures.*

Proof. For y large by (2.20) we have

$$s(y) = \int_y^\infty f(u)^{-1} e^{-F(u)}\, du = e^{-F(y)};$$

thus

$$s'(y) = -\frac{1}{g^2(y)} = -\frac{1}{f(y)} s(y) < 0. \tag{2.24}$$

From (2.17) and (2.18) we conclude that for y sufficiently large

$$F(y) \ge \log y^{\tilde{a}} + C \tag{2.25}$$

for some $C \in \mathbf{R}$ and some $\tilde{a}$ with $\tilde{a} > 1$, proving that $s\colon (0, \infty) \to (0, s(0))$ is a diffeomorphism. That Φ is unitary is obvious from the definition. q.e.d.

Next we define a first order elliptic differential operator $C_0^\infty(E) \to C_0^\infty(F)$ by

$$D_{\overline{g}} := \overline{g} D \overline{g}, \tag{2.26}$$

and study its transformation under Ψ. If $u \in C_0^\infty((0, s(0)), H_1)$, then $\Psi u \in C_0^\infty((0, \infty), H_1)$ and

$$\begin{aligned} D_{\overline{g}}\Psi u(y) &= g(y)\left[A_1(y)\partial_y A_2(y) + f(y)^{-1}(S_0 + S_1(y))\right] u \circ s(y) \\ &= g(y)^{-1}\left[g^2(y)s'(y)A_1A_2(y)u' \circ s(y)\right. \\ &\qquad \left. + \left(g^2/f\right)(y)\left(S_0 + S_1(y) + A_1 f A_2'(y)\right) u \circ s(y)\right] \\ &=: \Psi T_g u(y). \end{aligned}$$

Using (2.24) we obtain

$$\begin{aligned} T_g u(x) &= \Big[-A_1A_2 \circ s^{-1}(x)\partial_x + a(x)\Big(S_0 + S_1 \circ s^{-1}(x) \\ &\qquad\qquad + A_1 f A_2' \circ s^{-1}(x)\Big)\Big] u(x) \qquad (2.27) \\ &=: -\left[B_1(x)\partial_x B_2(x) + a(x)\left(\widetilde{S}_0 + \widetilde{S}_1(x)\right)\right] u(x), \end{aligned}$$

where we have written

$$B_j(x) = A_j \circ s^{-1}(x), \qquad j = 1, 2, \tag{2.28a}$$

$$\widetilde{S}_0 = -S_0, \tag{2.28b}$$

$$\widetilde{S}_1(x) = -S_1 \circ s^{-1}(x), \tag{2.28c}$$

$$a(x) = \left(g^2/f\right) \circ s^{-1}(x). \tag{2.28d}$$

By (2.24), $f/g^2(y) = s(y) = x$ for y large; so $\widetilde{S}_1(x) = -S_1(s^{-1}(x)) = -S_1(y)$ for x small. It is apparent from (2.27) that $D_{\overline{g}}$ satisfies similar assumptions as D but now the boundary operator T_g is the model operator for conic singularities as treated in [5]. Somewhat surprisingly, we will be able to reduce the index calculation for D to this case. The relevant analysis of such operators will be carried out in §3. Assuming for the moment that all closed extensions of $D_{\overline{g}}$ are Fredholm (this will be proved in Theorem 3.3 below) our aim is to compare $L^2\text{-}\operatorname{ind} D$ with the index of a suitable closed extension of $D_{\overline{g}}$. It follows from (2.20) that

$$g(y)^{-1} = Ce^{-1/2\int_0^y ((1+f'(u))/f(u))\,du} \quad \text{for large } y,$$

so that by (2.14)

$$\overline{g}^{-1} \in L^\infty(M). \tag{2.29}$$

Thus the map

$$\beta : L^2\text{-}\ker D \ni f \mapsto \overline{g}^{-1} f \in \ker D_{\overline{g}, \max} \tag{2.30a}$$

is well defined, and obviously linear and injective. Similarly, we have a linear injective map

(2.30b) $$\beta' : L^2\text{-}\ker D' \to \ker D'_{\overline{g},\max}.$$

Thus we obtain

Theorem 2.1. *Under conditions* (2.14) *and* (2.15) D *has a finite* L^2-*index.*

We want to obtain a formula for $L^2\text{-}\operatorname{ind} D$. The first task is to construct a suitable closed extension of $D_{\overline{g}}$. We define the space

(2.31) $$W := \left[\mathscr{D}\left(D_{\overline{g},\max}\right) \cap g^{-1}\mathscr{H}\right] / \mathscr{D}\left(D_{\overline{g},\min}\right),$$

and the operator

(2.32) $$D_{\overline{g},W} := D_{\overline{g},\max}\Big|\mathscr{D}\left(D_{\overline{g},\max}\right) \cap g^{-1}\mathscr{H}.$$

To see that this makes sense we need

Lemma 2.2. $\mathscr{D}(D_{\overline{g},\min}) \subset \mathscr{D}(D_{\overline{g},\max}) \cap g^{-1}\mathscr{H}$.

Proof. We only have to show that $gu \in \mathscr{H}$ if $u \in \mathscr{D}(D_{\overline{g},\min})$ or $gu_b \in L^2(\mathbf{R}_+, H)$. Now $u_b = \Psi w_b$ for some $w_b \in L^2((0,1), H)$, and, by Lemma 3.2 and (3.11) below, $u \in \mathscr{D}(D_{\overline{g},\min})$ implies

$$\|w_b(x)\|_H = o\left((x\,|\log x|)^{1/2}\right), \qquad x \to 0.$$

But $gu_b = \Psi(g \circ s^{-1} w_b)$, so by (2.24)

$$\left(g \circ s^{-1}\right)^2(x)\, \|w_b(x)\|_H^2 = O\left(f \circ s^{-1}(x)\,|\log x|\right).$$

Now for large R it follows from (2.22) that

$$\int_0^{s(R)} f \circ s^{-1}(x)\,|\log x|\, dx = \int_R^\infty e^{-F(y)} F(y)\, dy,$$

and (2.25) implies the convergence of the integral. So $gu_b = \Psi(g \circ s^{-1} w_b) \in L^2(\mathbf{R}_+, H)$. q.e.d.

Thus $\mathscr{D}(D_{\overline{g},W})$ is all u in $\mathscr{D}(D_{\overline{g},\max})$ such that $gu \in \mathscr{H}$. We will show below (Theorem 3.4) that $D_{\overline{g},W}$ is a closed Fredholm extension of $D_{\overline{g}}$ with index

(2.33) $$\operatorname{ind} D_{\overline{g},W} = \operatorname{ind} D_{\overline{g},\min} + \dim W.$$

On the other hand we have

Lemma 2.3. *With* β, β' *defined in* (2.30a,b) *we have*

$$\beta\left(L^2\text{-}\ker D\right) = \ker D_{\overline{g},W}, \qquad \beta'\left(L^2\text{-}\ker D'\right) \subset \ker D^*_{\overline{g},W}.$$

Proof. If $v \in L^2\text{-}\ker D$, then $\beta(v) = g^{-1}v \in \mathscr{D}(D_{\overline{g},\max}) \cap g^{-1}\mathscr{H}$, hence $\beta(v) \in \ker D_{\overline{g},W}$. Conversely, if $u \in \ker D_{\overline{g},W}$, then $v := gu \in \ker D \cap \mathscr{H}$, so β is bijective.

Consider next $v \in L^2\text{-}\ker D'$; by (2.30b) we know that $\beta'(v) \in \ker D'_{\overline{g},\max}$ and to obtain $\beta'(v) \in \mathscr{D}(D^*_{\overline{g},W})$ it suffices to show that for all $u \in \mathscr{D}(D_{\overline{g},W})$

$$\left(D_{\overline{g}}u,\, \beta'(v)\right) = \left(u,\, D'_{\overline{g}}\beta'(v)\right) = 0. \tag{2.34}$$

Now

$$\left(D_{\overline{g}}u,\, \beta'(v)\right) = (D\overline{g}u,\, v)$$

and $gu \in \mathscr{H}$ by construction. By interior regularity we may assume in (2.34) that $u = (0, u_b)$. Choose $\psi \in C_0^\infty(\mathbf{R})$ such that

$$\psi(y) = \begin{cases} 1, & |y| \le 1, \\ 0, & |y| \ge 2, \end{cases}$$

and put $\psi_n(y) := \psi(y/n)$. Then we find

$$\begin{aligned}
(Dgu, v) &= \int_0^\infty \langle Dgu(y),\, v(y)\rangle_H\, dy \\
&= \lim_{n\to\infty} \int_0^\infty \langle Dgu(y),\, \psi_n(y)v(y)\rangle_H\, dy \\
&= \lim_{n\to\infty} \int_0^\infty \Big[\langle gu(y),\, \psi_n(y)D'v(y)\rangle_H \\
&\qquad + \tfrac{1}{n}\langle gu(y),\, \psi'(y/n)v(y)\rangle_H\Big]\, dy \\
&= 0.
\end{aligned}$$

Hence the proof is complete. q.e.d.

We can treat D' in the same way; we introduce

$$W' := \left[\mathscr{D}\left(D'_{\overline{g},\max}\right) \cap g^{-1}\mathscr{H}'\right] / \mathscr{D}\left(D'_{\overline{g},\min}\right)$$

and

$$D'_{\overline{g},W'} := D'_{\overline{g},\max}|\mathscr{D}\left(D'_{\overline{g},\max}\right) \cap g^{-1}\mathscr{H}',$$

which makes sense by Lemma 2.2 applied to D'. Applying Lemma 2.3 to D' we obtain

$$\operatorname{ind} D_{\overline{g},W} \le L^2\text{-}\operatorname{ind} D \le -\operatorname{ind} D'_{\overline{g},W'}. \tag{2.35}$$

In general, it seems quite difficult to compute $L^2\text{-}\operatorname{ind} D$ in terms of D, f, S_0, and S_1. Ind $D_{\overline{g},W}$ can be computed by the methods in [5], and this will be carried out in §4. Thus the problem lies with the difference

$$\begin{aligned} h_1 &:= L^2\text{-}\operatorname{ind} D - \operatorname{ind} D_{\overline{g},W} \\ &= \dim\ker D^*_{\overline{g},W} - \dim L^2\text{-}\ker D' \\ &= \dim\ker D^*_{\overline{g},W} - \dim\ker D'_{\overline{g},W'}. \end{aligned} \tag{2.36}$$

Writing

$$h_0 := \dim W \tag{2.37}$$

we have by (2.33) and (2.36) the following index theorem:

Theorem 2.2. *Assume conditions* (2.1), (2.6), (2.7), (2.9), (2.14), (2.15), *and* (2.16). *Then D has a finite L^2-index given by*

$$L^2\text{-}\operatorname{ind} D = \operatorname{ind} D_{\overline{g},\min} + h_0 + h_1. \tag{2.38}$$

A priori, h_i depends on the choice of g. We have, however,

Lemma 2.4. *For $i = 0, 1$, h_i is the same for all positive $\overline{g} \in C^\infty(M)$ satisfying* (2.20) *for sufficiently large y.*

Proof. By (2.31), W, and thus h_0, is independent of $\overline{g}$; for g is fixed for large y, and a change of $\overline{g}$ in a compact set does not affect $\mathscr{D}(D_{\overline{g},\max})$, $\mathscr{D}(D_{\overline{g},\min})$, $g^{-1}\mathscr{H}$ or $\operatorname{ind} D_{\overline{g},W}$, so h_1 is also independent. q.e.d.

To obtain a more explicit formula we have to compute the various terms in (2.38). This will be done below for $\operatorname{ind} D_{\overline{g},\min}$ and h_0 whereas we have only an inequality for h_1. We will show, however, that $h_1 = 0$ in many interesting cases; thus we arrive at a satisfying L^2-index theorem.

3. Regular singular operators

The operators $D_{\overline{g}}$ introduced in the previous section belong to the class arising from the study of conic singularities (cf. [5]). Though their properties are quite analogous to our assumptions on D above, we write them out explicitly for convenience. Thus let M be a Riemannian manifold, not necessarily complete, let E and F be Hermitian vector bundles over M, and let $\widetilde{D}\colon C^\infty(E) \to C^\infty(F)$ be a first order elliptic differential operator.

We assume again (2.1) but we replace (2.6) by the following assumption:

(3.1) there is a Hilbert space H with a dense subspace H_1 and isometries $\tilde{\Phi}_E: L^2(E|U) \to L^2((0,1), H)$, $\tilde{\Phi}_F: L^2(F|U) \to L^2((0,1), H)$ such that $\tilde{\Phi}_E$, $\tilde{\Phi}_F$ induce isomorphisms $H_0^1(E|\overline{U}) \simeq H_0^1((0,1), H) \cap L^2((0,1), H_1) \simeq H_0^1(F|\overline{U})$. Moreover, there is a self-adjoint operator $\tilde{S}_0$ in H with domain $\mathscr{D}(\tilde{S}_0) := H_1$, a smooth function $(0,1) \ni x \mapsto \tilde{S}_1(x) \in \mathscr{L}(H_1, H)$, and smooth functions $(0,1) \ni x \mapsto B_j(x) \in \mathscr{L}(H) \cap \mathscr{L}(H_1)$, $\quad j = 1, 2$, such that for $u \in C_0^\infty((0,1), H_1)$ and $x \in (0,1)$,
$$\tilde{\Phi}_F \tilde{D} \tilde{\Phi}_E^{-1} u(x) = B_1(x)\, \partial_x B_2(x)\, u(x) + x^{-1}\left(\tilde{S}_0 + \tilde{S}_1(x)\right) u(x).$$

As before, we think of $\tilde{S}_1$ as a perturbation. Thus we require an estimate similar to (2.15):

$$\left\| \tilde{S}_1(x)\left(\left|\tilde{S}_0\right| + 1\right)^{-1} \right\| + \left\| \left(\left|\tilde{S}_0\right| + 1\right)^{-1} \tilde{S}_1(x) \right\| = o(1) \quad \text{as } x \to 0. \tag{3.2}$$

We also need the analogue of (2.16):

$$\left\| (x\partial_x)^i \left(B_j(x) - I\right) \right\|_{H_1} + \left\| (x\partial_x)^i \left(B_j(x) - I\right) \right\|_H = o(1) \quad \text{as } x \to 0 \text{ for } i = 0, 1 \text{ and } j = 1, 2. \tag{3.3}$$

As in (2.8) we derive unitary isomorphisms

$$\tilde{\Phi}: L^2(E) \to L^2(E|M_1) \oplus L^2((0,1), H) := \tilde{\mathscr{H}},$$
$$\tilde{\Phi}': L^2(F) \to L^2(F|M_1) \oplus L^2((0,1), H) := \tilde{\mathscr{H}}'.$$

We need, furthermore, the analogue of (2.9): we put $C^\sim[0,1] := \{\varphi \in C^\infty[0,1] | \varphi$ is constant near 0 and 1$\}$ and define for $u = (u_i, u_b) \in \tilde{\mathscr{H}}^{(\prime)}$

$$\varphi u = \left(\varphi(1)\, u_i, \varphi u_b\right) \in \tilde{\mathscr{H}}^{(\prime)}.$$

Then we require that

$$\tilde{\Phi}^{(\prime)-1} \varphi u = \overline{\varphi} \tilde{\Phi}^{(\prime)-1} u \tag{3.4}$$

for some $\overline{\varphi} \in C^\infty(M)$, with $\overline{\varphi} \in C_0^\infty(M)$ if $\varphi \in C_0^\infty(0,1]$. Finally, we have again for $u \in H_0^1(E)$ with $\tilde{\Phi} u = (u_i, u_b)$, and $v \in H_0^1(F)$ with $\tilde{\Phi}' v = (v_i, v_b)$

$$u_i \big| N = \tilde{\Phi}_E^{-1} u_b \big| N, \qquad v_i \big| N = \tilde{\Phi}_F^{-1} v_b \big| N. \tag{3.5}$$

Operators $\widetilde{D}$ satisfying the above assumptions will be called *regular singular*. Thus the geometric operators on manifolds with conic singularities are regular singular (cf. [5]) but also the weighted operator $D_{\bar{g}}$ introduced in §2. The analysis of [5] has to be extended since we allow much weaker perturbations. This has to be paid for by assuming that either

(3.6a) $$\pm\tfrac{1}{2} \notin \operatorname{spec} \widetilde{S}_0$$

or

(3.6b) $$\left\|\widetilde{S}_1(x)\left(\left|\widetilde{S}_0\right|+1\right)^{-1}\right\| + \left\|\left(\left|\widetilde{S}_0\right|+1\right)^{-1}\widetilde{S}_1(x)\right\| = O\left(x^{\delta}\right)$$

and

(3.6c) $$\left\|x\partial_x\left(B_j(x)-I\right)\right\|_{H_1} + \left\|x\partial_x\left(B_j(x)-I\right)\right\|_{H} = O\left(x^{\delta}\right),$$
$$\text{as } x \to 0,\ j = 1, 2,$$

for some $\delta > 0$. This is not a restriction in dealing with $D_{\bar{g}}$, in view of (2.18).

We start our investigation of the closed extensions of $\widetilde{D}$ with the observation that

(3.7a) $$\begin{aligned}\mathscr{D}\left(\widetilde{D}_{\max}\right) &\simeq H^1(E|M_1) \oplus_t \widetilde{\mathscr{D}}_b \\ &= \left\{(u_i, u_b) \in H^1(E|M_1) \oplus \widetilde{\mathscr{D}}_b | u_i|N = \widetilde{\Phi}_E^{-1} u_b|N\right\},\end{aligned}$$

where

(3.7b) $$\begin{aligned}\widetilde{\mathscr{D}}_b = \Big\{u \in L^2_{\mathrm{loc}}((0,1], H_1) \cap H^1_{\mathrm{loc}}((0,1], H) \cap L^2((0,1), H) \mid \\ x \mapsto B_1\partial_x B_2 u(x) + x^{-1}\left(\widetilde{S}_0 + \widetilde{S}_1(x)\right)u(x) \in L^2((0,1), H)\Big\}.\end{aligned}$$

In what follows we identify $\widetilde{D}$ with $\widetilde{\Phi}'\widetilde{D}\widetilde{\Phi}^{-1}$.

To construct a boundary parametrix we introduce

(3.8a) $$P_{0,s}f(x) := \int_0^x (y/x)^s f(y)\,dy, \qquad s > -\tfrac{1}{2},$$

(3.8b) $$P_{1,s}f(x) := \int_1^x (y/x)^s f(y)\,dy, \qquad s < \tfrac{1}{2},$$

and note that
$$\left(\partial_x + X^{-1}s\right)P_{0,s} = \left(\partial_x + X^{-1}s\right)P_{1,s} = I.$$

From Lemma 2.1 in [5] we have the following estimates for $x \in (0, 1)$ and $f \in L^2(0, 1)$:

(3.9a)
$$\left|P_{0,s}f(x)\right| + \left|P_{1,-s-1}f(x)\right| \le 2x^{1/2}\,|2s+1|^{-1/2}\,\|f\|_{L^2} \quad \text{if } s > -\tfrac{1}{2},$$

$$\left|P_{1,-1/2}f(x)\right| \le \varepsilon\,(x\,|\log x|)^{1/2} \quad \text{if } x \le x(\varepsilon)\,, \ \text{ for all } \varepsilon > 0. \tag{3.9b}$$

Lemma 2.2 in [5] has to be extended to include the case $\beta = -1$, at the expense of loosing the ε-decay.

Lemma 3.1. *In* $L^2(0, 1)$ *we have the norm estimates*

$$\left\|X^{-1}P_{0,s}\right\| + \left\|P_{1,-s}X^{-1}\right\| \le \left|s+\tfrac{1}{2}\right|^{-1}, \qquad s > -\tfrac{1}{2}, \tag{3.10a}$$

$$\left\|X^{-1}P_{1,s}\right\| + \left\|P_{0,-s}X^{-1}\right\| \le \left|s+\tfrac{1}{2}\right|^{-1}, \qquad s < -\tfrac{1}{2}. \tag{3.10b}$$

In $L^2(0, \varepsilon)$, $0 < \varepsilon \le 1$, *we have for* $\delta > 0$

$$\left\|X^{\delta-1}P_{1,s}\right\| + \left\|P_{0,-s}X^{\delta-1}\right\| \le \varepsilon^{\delta}\left|s+\tfrac{1}{2}-\delta\right|^{-1}, \qquad s < \delta - \tfrac{1}{2}. \tag{3.10c}$$

Moreover, $P_{0,s}$ *and* $P_{1,-s}$ *are compact for* $s > -\frac{1}{2}$.

Proof. The proof follows as before from Schur's test, with $p(x) = q(x) = x^{-1/2}$ (cf. [8]). By [5, (2.8)], $P^*_{1,-s} = -P_{0,s}$, and $P_{0,s}$ is Hilbert-Schmidt for $s > -\frac{1}{2}$, thus $P_{0,s}$ and $P_{1,-s}$ are compact for $s > -\frac{1}{2}$.

In what follows it is convenient to rewrite the operator in (3.1):

$$\begin{aligned}
B_1(x)\,\partial_x B_2(x) + x^{-1}\left(\tilde{S}_0 + \tilde{S}_1(x)\right) &= B_1B_2(x)\,\partial_x + x^{-1}\left(\tilde{S}_0 + \tilde{S}_1(x) + xB_1B_2'(x)\right)\\
&= B_1B_2(x)\left[\partial_x + x^{-1}\left(\tilde{S}_0 + \left(B_2^{-1}B_1^{-1}(x) - I\right)\tilde{S}_0\right.\right.\\
&\qquad\left.\left. + B_2^{-1}B_1^{-1}\tilde{S}_1(x) + xB_2^{-1}B_1^{-1}B_1B_2'(x)\right)\right]\\
&=: B(x)\left[\partial_x + x^{-1}\left(\tilde{S}_0 + \tilde{\tilde{S}}_1(x)\right)\right].
\end{aligned} \tag{3.11}$$

It follows from (3.2) and (3.6) that $\tilde{\tilde{S}}_1$ satisfies either (3.2) or (3.6b) whereas B satisfies (3.3b).

Now we define an extension $\tilde{D}_0$ of $\tilde{D}$ as a restriction of $\tilde{D}_{\max}$ to the domain

$$\left\{u \in \mathscr{D}\left(\tilde{D}_{\max}\right) \,\middle|\, X^{-1}\tilde{\tilde{S}}_1 u_b \in L^2((0, 1), H),\ \|u_b(x)\|_H = o\left((x\,|\log x|)^{1/2}\right) \text{ as } x \to 0\right\}. \tag{3.12}$$

We will show below that $\tilde{D}_0 = \tilde{D}_{\min}$, which in general differs from $\tilde{D}_\delta$ introduced in [5]. The corresponding boundary parametrix is now

$$P_0 := \bigoplus_{\substack{s\in\operatorname{spec}\tilde{S}_0\\ s>-1/2}} P_{0,s} \oplus \bigoplus_{\substack{s\in\operatorname{spec}\tilde{S}_0\\ s\le-1/2}} P_{1,s}. \tag{3.13}$$

Repeating the proof of Lemma 2.3 in [5] and using (3.9) and (3.10) we obtain

Lemma 3.2. *If* $\psi \in C_0^\infty(-1, 1)$, *then* ψP_0 *maps* $L^2((0, 1), H)$ *into* $\mathscr{D}(\widetilde{D}_0)$.

Lemma 3.3. *If* $u = (u_i, u_b) \in \mathscr{D}(\widetilde{D}_0)$ *with* $u_i = 0$ *and* $u_b(1) = 0$, *then*

$$P_0 B^{-1}\widetilde{D}u = u + P_0 X^{-1}\widetilde{\widetilde{S}}_1 u.$$

Proof. Denote by $(e_s)_{s\in\operatorname{spec}\widetilde{S}_0}$ an orthonormal basis of H satisfying $S_0 e_s = s e_s$ (with a slight abuse of notation in the case of multiple eigenvalues) and put

$$h(x) := \left(\partial_x + x^{-1}\widetilde{S}_0\right) u(x) = B^{-1}\widetilde{D}u(x) - x^{-1}\widetilde{\widetilde{S}}_1(x)\, u(x)\,, \tag{3.14}$$

hence $h \in L^2((0, 1), H)$ by definition of $\widetilde{D}_0$. Proceeding as in Lemma 2.4 of [5] we obtain

$$u_s(x) := \langle u(x)\,, e_s\rangle = P_{1,s} h_s(x) \quad \text{for all } s.$$

Now if $s > -\frac{1}{2}$ we find

$$\begin{aligned} u_s(x) = P_{1,s} h_s(x) &= -x^{-s}\int_0^1 y^s h_s(y)\, dy + P_{0,s} h_s(x) \\ &=: c_s x^{-s} + P_{0,s} h_s(x)\,, \end{aligned}$$

using $h_s \in L^2$. Since $\|u(x)\|_H = o((x|\log x|)^{1/2})$, $x \to 0$, it follows from (3.8) that $c_s = 0$, so $u = P_0 h = P_0\widetilde{D}u - P_0\widetilde{\widetilde{S}}_1 u$ and the lemma is proved.

Lemma 3.4. *There is* $0 < \varepsilon \le 1$ *such that for* φ, $\psi \in C_0^\infty(-\varepsilon, \varepsilon)$ *with* $\psi\varphi = \varphi$ *and* $u \in \mathscr{D}(\widetilde{D}_0)$

$$\varphi u = \psi P_0 V \widetilde{D}_0 \varphi u \tag{3.15}$$

for some bounded operator V *in* $L^2((0, 1), H)$. *As a consequence,*

$$\left\|\varphi X^{-1}\left(\left|\widetilde{S}_0\right| + 1\right) u\right\| \le C\left\|\widetilde{D}_0 \varphi u\right\|. \tag{3.16}$$

Proof. Note first that $\varphi u \in \mathscr{D}(\widetilde{D}_{\max})$ by (3.7a,b), hence $\varphi u \in \mathscr{D}(\widetilde{D}_0)$ by construction. Then the proof differs from that of Lemma 2.5 in [5] only in so far that $P_0 X^{-1}\widetilde{\widetilde{S}}_1$ is not necessarily bounded in L^2. But for $u \in \mathscr{D}(\widetilde{D}_0)$ we have

$$\left(\psi P_0 X^{-1}\widetilde{\widetilde{S}}_1 \chi\right)^{n+1} \varphi u = \psi P_0 \left(\psi X^{-1}\widetilde{\widetilde{S}}_1 P_0 \chi\right)^n X^{-1}\widetilde{\widetilde{S}}_1 \varphi u,$$

where $\chi \in C_0^\infty(-\varepsilon, \varepsilon)$ with $\chi\psi = \psi$, and hence by iteration

$$\varphi u = \psi P_0 \chi \sum_{j=0}^{n} \left(-\psi X^{-1}\tilde{\tilde{S}}_1 P_0 \chi\right)^j B^{-1} f$$
$$+ (-1)^{n+1} \psi P_0 \left(\psi X^{-1}\tilde{\tilde{S}}_1 P_0 \chi\right)^n X^{-1}\tilde{\tilde{S}}_1 \varphi u.$$

By Lemma 3.1 and assumption (3.2) or (3.6b), $\|\psi X^{-1}\tilde{\tilde{S}}_1 P_0 \chi\| < 1$ if ε is sufficiently small, and $X^{-1}\tilde{\tilde{S}}_1 \varphi u \in L^2$ by definition of $\tilde{D}_0$. So we reach the same conclusion. q.e.d.

Recalling that $\tilde{D}\varphi u = \varphi \tilde{D} u + C_\varphi u$ for some bounded operator C_φ in $\tilde{\mathscr{H}}$, we obtain exactly as in [5] from Lemmas 3.4 and 3.1, and from (3.8)

Lemma 3.5. *$\tilde{D}_0$ is a closed operator.*

Theorem 3.1. $\tilde{D}_0 = \tilde{D}_{\min}$.

Proof. Since $\tilde{D}_0$ is a closed extension we have $\tilde{D}_0 \supset \tilde{D}_{\min}$. It remains to show the reverse inclusion. If $u \in \mathscr{D}(\tilde{D}_{\max})$ and $\psi \in C^\infty[0, 1]$ with $\psi(0) = 0$, then $\psi u \in \mathscr{D}(\tilde{D}_{\min})$ by interior regularity. Therefore, it is enough to prove the following: if $u \in \mathscr{D}(\tilde{D}_0)$ there is a sequence $(u_n)_{n\in\mathbf{N}} \subset \mathscr{D}(\tilde{D}_{\min})$ such that

(3.17a) $\quad u_n \to \varphi u$ in $\tilde{\mathscr{H}}$ for some $\varphi \in C_0^\infty(-1, 1)$ with $\varphi = 1$ near 0,

(3.17b) $\quad \left(\tilde{D} u_n\right)_{n\in\mathbf{N}}$ is a Cauchy sequence in $\tilde{\mathscr{H}}'$.

Now we proceed as in [4, Theorem 6.1]: choose $\varphi \in C_0^\infty(-1, 1)$ with $0 \le \varphi \le 1$ and $\varphi(x) = 1$ if $|x| \le 1/2$, put

$$\alpha_n := (\log n)^{-1/2}, \qquad n \ge 2,$$

and let

$$\psi_n(x) := x^{\alpha_n}(1 - \varphi(nx))\varphi(x), \qquad \psi_{nm}(x) := \psi_n(x) - \psi_m(x).$$

Then we put $u_n := \psi_n u$ and $u_{nm} := \psi_{nm} u$, such that $u_n \in \mathscr{D}(\tilde{D}_{\min})$ and satisfies (3.17a), $n \ge 2$. It remains to show that $\|\tilde{D} u_{nm}\|_{\tilde{\mathscr{H}}'}$ tends to zero as $m \ge n \to \infty$. Now

$$\left\|\tilde{D} u_{nm}\right\|^2_{\tilde{\mathscr{H}}'} = \left\|\psi'_{nm} B u + \psi_{nm} \tilde{D} u\right\|^2_{\tilde{\mathscr{H}}'} \le C \left\|\psi'_{nm} u\right\|^2_{\tilde{\mathscr{H}}'} + o(1).$$

For $0 < \delta \leq 1$ we have

$$\alpha_n^2 \int_0^\delta x^{2\alpha_n - 1} \log x \, dx = \frac{\alpha_n}{2} \delta^{2\alpha_n} \log \delta - \frac{1}{4} \delta^{2\alpha_n} = \left(\frac{\log \delta}{2 (\log n)^{1/2}} - \frac{1}{4} \right) e^{2 \log \delta / (\log n)^{1/2}},$$

so this term is uniformly bounded in $0 < \delta \leq 1$ and $n \geq 2$. Moreover,

$$\begin{aligned} n^2 \int_0^{1/n} x^{2\alpha_n + 1} \log x \, dx &= -\frac{n^2}{2\alpha_n + 2} n^{-2\alpha_n - 2} \log n - \frac{n^2}{(2\alpha_n + 2)^2} n^{-2\alpha_n - 2} \\ &= -\frac{e^{-2(\log n)^{1/2}}}{(2\alpha_n + 2)^2} ((2\alpha_n + 2) \log n + 1) \\ &= o(1) \quad \text{as } n \to \infty. \end{aligned}$$

Combining these estimates with (3.11) and the obvious fact that

$$\lim_{n, n \to \infty} \int_\delta^1 \psi'_{n,m}(x)^2 \|u(x)\|_H^2 \, dx = 0$$

for all $\delta \in (0, 1]$, we arrive at

$$\lim_{n, m \to \infty} \left\| \psi'_{n,m} u \right\|_{\widetilde{\mathscr{H}}'}^2 = 0$$

as desired. q.e.d.

Since we have only used that $\|u(x)\|_H = o(x^{1/2} |\log x|^{1/2})$ if $u \in \mathscr{D}(\widetilde{D}_0)$ we thus obtain

Corollary 3.2. $\mathscr{D}(\widetilde{D}_{\min}) = \{u \in \mathscr{D}(\widetilde{D}_{\max}) | \|u(x)\|_H = o(x^{1/2} |\log x|^{1/2})\}$. If (3.6a) holds, then we can replace "$o(x^{1/2} |\log x|^{1/2})$" by "$O(x^{1/2})$".

Now we are ready to deal with the Fredholm properties of $\widetilde{D}_0$.

Theorem 3.3. *$\widetilde{D}_{\min} : \mathscr{D}(\widetilde{D}_{\min}) \to \widetilde{\mathscr{H}}'$ is a Fredholm operator.*

Proof. To show that $\widetilde{D}_{\min}$ is Fredholm it is enough to construct a right parametrix for $\widetilde{D}_{\min}$ and $\widetilde{D}^*_{\min}$; since $\widetilde{D}$ and $\widetilde{D}'$ have the same structure it is enough to deal with $\widetilde{D}$. So we have to construct an operator $P \in \mathscr{L}(\widetilde{\mathscr{H}}', \widetilde{\mathscr{H}})$ with $P(\widetilde{\mathscr{H}}') \subset \mathscr{D}(\widetilde{D}_{\min})$ and

$$\widetilde{D} P = I + K, \qquad K \text{ compact in } \widetilde{\mathscr{H}}'.$$

Since $\widetilde{D}$ is elliptic, we have interior parametrices, i.e., given ψ, $\tilde{\psi} \in C_0^\infty(M)$ with $\tilde{\psi} = 1$ in a neighborhood of $\operatorname{supp} \psi$ we can find compact operators $P_\psi \in \mathscr{L}(L^2(F), L^2(E))$ and $K_\psi \in \mathscr{L}(L^2(F))$ such that $P_\psi(L^2(F)) \subset H_0^1(E)$ and

$$\widetilde{D} P_\psi = \psi + K_\psi. \tag{3.18}$$

Now let ε be sufficiently small and choose $\varphi \in C_0^\infty(-\varepsilon, \varepsilon)$ with $\varphi = 1$ in a neighborhood of 0. Put $\psi = 1 - \varphi$ and pick $\tilde\varphi \in C_0^\infty(-\varepsilon, \varepsilon)$ such that $\tilde\varphi = 1$ in a neighborhood of $\operatorname{supp}\varphi$. Define

$$\widetilde{P} := \widetilde{\Phi} P_\psi \widetilde{\Phi}'^{-1} + \tilde\varphi P_0 B^{-1}\varphi.$$

By (3.11) and Lemma 3.2, $P \in \mathscr{L}(\widetilde{\mathscr{H}}', \widetilde{\mathscr{H}})$ with $P(\widetilde{\mathscr{H}}') \subset \mathscr{D}(\widetilde{D}_{\min})$. Moreover,

$$\begin{aligned}\widetilde{D}\widetilde{P} &= \psi + \widetilde{\Phi}' K_\psi \widetilde{\Phi}'^{-1} + \varphi + \tilde\varphi' B P_0 B^{-1}\varphi + \tilde\varphi X^{-1} B \tilde{\widetilde{S}}_1 P_0 B^{-1}\varphi \\ &=: I + \widetilde{K} + \tilde\varphi X^{-1} B \tilde{\widetilde{S}}_1 P_0 B^{-1}\varphi =: I + \widetilde{K} + R.\end{aligned}$$

By (3.18) and Lemma 3.1, $\widetilde{K}$ is compact in $\widetilde{\mathscr{H}}'$. If ε is sufficiently small we conclude from (3.2) or (3.6b) and Lemma 3.1 that $\|R\| < 1$. Putting $P := \widetilde{P}(I + R)^{-1}$ and $K := \widetilde{K}(I + R)^{-1}$ we obtain $\widetilde{D}P = I + K$ which completes the proof. q.e.d.

Next we study the closed extensions of $\widetilde{D}$ besides $\widetilde{D}_{\min}$.

Theorem 3.4. *The closed extensions of $\widetilde{D}$ are all Fredholm operators, which correspond bijectively to the subspaces of the finite-dimensional space*

$$\mathscr{D}\left(\widetilde{D}_{\max}\right)/\mathscr{D}\left(\widetilde{D}_{\min}\right) := W_0.$$

Moreover, denoting by $\widetilde{D}_W$ the closed extension corresponding to the space $W \subset W_0$ we have

$$\operatorname{ind}\widetilde{D}_W = \operatorname{ind}\widetilde{D}_{\min} + \dim W.$$

Proof. The proof of Theorem 3.3 actually works for every closed extension of $\widetilde{D}$. Hence all closed extensions of $\widetilde{D}$ are Fredholm operators. Thus it follows that $W_0 = \mathscr{D}(\widetilde{D}_{\max})/\mathscr{D}(\widetilde{D}_{\min})$ is finite-dimensional. If $W \subset W_0$ is an arbitrary subspace, we obtain a closed Fredholm extension $\widetilde{D}_W$ by restricting $\widetilde{D}_{\max}$ to the inverse image $\mathscr{D}(\widetilde{D}_W)$ of W under the projection $\mathscr{D}(\widetilde{D}_{\max}) \to W_0$. The inclusion map $i_W : \mathscr{D}(\widetilde{D}_{\min}) \hookrightarrow \mathscr{D}(\widetilde{D}_W)$ is then Fredholm with $\operatorname{ind} i_W = -\dim W$, and from $\widetilde{D}_{\min} = \widetilde{D}_W \circ i_W$ we find

$$\operatorname{ind}\widetilde{D}_W = \operatorname{ind}\widetilde{D}_{\min} + \dim W. \quad \text{q.e.d.}$$

In §4 we will use the following facts.

Lemma 3.6. *Put*

$$P_{\max} := \bigoplus_{s \geq 1/2} P_{0,s} \oplus \bigoplus_{s < 1/2} P_{1,s}. \tag{3.19}$$

If $u = u_b \in \widetilde{\mathscr{D}}_b$ and $\varphi \in C_0^\infty(-1, 1)$, then we have

$$P_{\max} B^{-1}\widetilde{D}\varphi u = \varphi u + P_{\max} X^{-1}\tilde{\widetilde{S}}_1 \varphi u. \tag{3.20}$$

Proof. We use the notation introduced in the proof of Lemma 3.3. It follows from Lemma 3.1 that $P_{\max} X^{-1}\tilde{\tilde{S}}_1$ is bounded in L^2. Setting

$$h(x) := B^{-1}\tilde{D}\varphi u(x) - X^{-1}\tilde{\tilde{S}}_1 \varphi u(x)$$

we have

$$(\varphi u_s)'(x) + x^{-1}s\varphi u_s(x) = h_s(x),$$

where $h_s \in L^2_{\mathrm{loc}}(0, 1]$, hence from $\varphi u_s(1) = 0$

$$\varphi u_s(x) = P_{1,s}h_s(x), \qquad x \in (0, 1].$$

Now if $s \geq \frac{1}{2}$, then we have $P_{0,s}h_s \in L^2$ by (3.2) or (3.6b) and Lemma 3.1. Thus

$$x^{-s}\int_0^1 y^s h_s(y)\, dy = P_{0,s}h_s(x) - P_{1,s}h_s(x) = 0$$

since the right-hand side is in L^2, but the left-hand side is in L^2 only if the integral is 0.

Lemma 3.7. *Let Q be the orthogonal projection in H onto*

$$\bigoplus_{|s|<1/2} \ker(S_0 - s)$$

and assume that

$$\left\| x^{-1}\left(Q\tilde{\tilde{S}}_1(x) - \tilde{\tilde{S}}_1(x)Q\right)\right\| = O(1) \quad \text{as } x \to 0. \tag{3.21}$$

Then for $\varphi \in C_0^\infty(-1, 1)$ we have

$$\varphi(I - Q)\widetilde{\mathscr{D}}_b \subset \mathscr{D}\left(\tilde{D}_{\min}\right).$$

In particular, $\tilde{D}$ has a unique closed extension if $Q = 0$.

Proof. For $u = u_b \in \widetilde{\mathscr{D}}_b$ we have

$$B^{-1}\tilde{D}\varphi(I - Q)u = (I - Q)B^{-1}\tilde{D}\varphi u + X^{-1}\left(Q\tilde{\tilde{S}}_1 - \tilde{\tilde{S}}_1 Q\right)\varphi u,$$

so $\varphi(I - Q)u \in \widetilde{\mathscr{D}}_b$ by (3.6b) and (3.21). Observe that in view of (3.13) and (3.19), $(I - Q)P_{\max} = P_0(I - Q)$. Multiplying (3.20) by $I - Q$ from the left and letting $f := B^{-1}\tilde{D}\varphi u \in L^2$ we find

$$P_0(I - Q)f = \varphi(I - Q)u + P_0(I - Q)X^{-1}B\tilde{\tilde{S}}_1\varphi u.$$

By Lemma 3.1 we have $D(x) := P_0(I - Q)x^{-1}\tilde{\tilde{S}}_1(x)$ bounded in H, and from (3.2) it follows that

$$\|D(x)\| = o(1) \quad \text{as } x \to 0.$$

Hence if φ has sufficiently small support, we obtain as in the proof of Lemma 3.4 that

$$\varphi (I - Q) u = P_0 W f$$

for some bounded operator W in $L^2((0, 1), H)$. Thus the assertion follows from Lemma 3.2. q.e.d.

We remark that (3.21) is always satisfied if $\tilde{\tilde{S}}_1(x)$ commutes with Q for x near 0; otherwise it is a decay condition on the perturbation $\tilde{\tilde{S}}_1$.

4. The index formula

Theorem 2.2 will be made more explicit in this section by computing $\operatorname{ind} D_{\overline{g}, \min}$. This will be done by a Fredholm deformation to an operator with computable index, using essentially the methods of [5]. To do so it is convenient to introduce the following assumption:

(4.1) in (2.6) we have $S_1(y) = 0$ and $A_1(y) = A_2(y) = I$ for y near 0.

Note that this assumption has also been used in [1] and is satisfied if M has the product metric near ∂M_1. In concrete situations, however, it is easy to remove (cf. §5).

Now construct g^2 by (2.20′) and s by (2.21). Using (2.23) and (2.27) we find that $D_{\overline{g}, \min}$ transforms unitarily to an operator T in $\mathscr{H}$ with boundary part

$$\begin{aligned} & - B_1 \partial_x B_2 + a \left(S_0 + \tilde{S}_1\right) \\ & = B_1 B_2 \left[-\partial_x + a \left(S_0 + \left((B_1 B_2)^{-1} - I\right) S_0 \right.\right. \\ & \qquad\qquad \left.\left. + (B_1 B_2)^{-1} \tilde{S}_1 - a^{-1} B_2^{-1} B_2'\right)\right] \\ & =: -B \left[\partial_x + a \left(\tilde{S}_0 + \tilde{\tilde{S}}_1\right)\right], \qquad \tilde{S}_0 = -S_0. \end{aligned} \tag{4.2}$$

Here B_1, B_2, $\tilde{S}_1$, and a are given by (2.28) which implies (3.2), (3.3), and (3.6a). Note that

$$B_1(x) = B_2(x) = I, \quad \tilde{S}_1(x) = \tilde{\tilde{S}}_1(x) = 0 \quad \text{near } s(0),$$

and that

$$a(x) = \begin{cases} 1 & x \text{ near } s(0), \\ 1/x, & x \text{ near } 0. \end{cases}$$

We can now apply the results of §3 to T, with obvious modifications due to the fact that $xa(x) \not\equiv 1$ in $(0, s(0)]$. Thus with $\mathscr{D}_0$ as in (3.12) we see that $D_{\overline{g},\min}$ is unitarily equivalent to T with domain

$$\begin{aligned} &\mathscr{D}(T) = H^1(E|M_1) \oplus_t \widetilde{\mathscr{D}}_0, \\ (4.3) \qquad &(u_i, u_b) \mapsto \left(f(0)Du_i, -B\left[\partial_x + a\left(\widetilde{S}_0 + \widetilde{\widetilde{S}}_1\right)\right]u_b\right) \in \widetilde{\mathscr{H}}'. \end{aligned}$$

We now introduce

(4.4)

$$A := \sup_{x\in(0,s(0)]} \sum_{\substack{1\le i\le 2\\ 0\le j\le 1}} \left[\left\|(x\partial_x)^j(B_i(x)-I)\right\|_H + \left\|(x\partial_x)^j(B_i(x)-I)\right\|_{H_1} + \left\|(|S_0|+1)^{-1}\widetilde{S}_1(x)\right\|_H + \left\|\widetilde{S}_1(x)(|S_0|+1)^{-1}\right\|_H\right].$$

Theorem 4.1. *Assume* (2.1), (2.6), (2.7), *and* (4.1). *If* $A < A(S_0)$, *then each of the operators*

$$\begin{aligned} &T_\alpha : \mathscr{D}(T) \mapsto \widetilde{\mathscr{H}}', \\ &(u_i, u_b) \mapsto \left(f(0)Du_i, -B_\alpha\left[\partial_x + a\left(\widetilde{S}_0 + \alpha\widetilde{\widetilde{S}}_1\right)\right]u_b\right) \end{aligned}$$

is a Fredholm operator, where $\alpha \in [0, 1]$ *and* $B_\alpha(x) := \alpha B(x) + (1-\alpha)I$.

Moreover, the function $[0, 1] \ni \alpha \mapsto T_\alpha \in \mathscr{L}(\mathscr{D}(T), \widetilde{\mathscr{H}}')$ *is continuous. In particular,*

$$\operatorname{ind} T = \operatorname{ind} T_0.$$

Proof. For sufficiently small A, $B_\alpha(x)$ is a continuous family of invertible operators on $[0, 1] \times [0, s(0)]$. To prove that T_α is well defined and continuous in α it is thus enough to prove that $a\widetilde{\widetilde{S}}_1$ is continuous on $\mathscr{D}(T)$. Now from (the proof of) Lemma 3.4 we obtain (3.16) for all $\varphi \in C_0^\infty(-s(0), s(0))$ if $A(s_0)$ is small enough. Since $\widetilde{\widetilde{S}}_1(x) = 0$ near $s(0)$, the desired continuity follows from (3.2) and (3.16).

It remains to show that each T_α is a Fredholm operator. As remarked above, (4.1) implies that for some $\delta \in (0, s(0))$

$$(4.5) \qquad T_\alpha u = Tu \quad \text{for all } u \in \mathscr{D}(T) \text{ with } \operatorname{supp} u_b \cap (0, \delta] = \emptyset.$$

Choose $\varphi \in C_0^\infty(-s(0), s(0))$ with $\varphi = 1$ in a neighborhood of $[-\delta, \delta]$ and $\psi \in C_0^\infty(-s(0), s(0))$ with $\psi\varphi = \varphi$. Since $D_{\overline{g}}$ is elliptic, we can find an operator $P_\varphi \in \mathscr{L}(\widetilde{\mathscr{H}}', \mathscr{D}(T))$ and a compact operator $K_\varphi \in \mathscr{L}(\widetilde{\mathscr{H}}')$ such that

$$(4.6) \qquad T_\alpha P_\varphi = TP_\varphi = (1-\varphi) + K_\varphi.$$

Denote next by P_b the boundary parametrix for $\partial_x + a\widetilde{S}_0$, as constructed in (3.13). Then we find

$$T_\alpha\left(-\varphi P_b \psi B_\alpha^{-1}\right) = \varphi - B_\alpha \varphi' P_b \psi B_\alpha^{-1} - \alpha a B_\alpha \widetilde{\widetilde{S}}_1 \varphi P_b B_\alpha^{-1} =: \varphi + K_\varphi^\alpha + \alpha R_\varphi^\alpha. \tag{4.7}$$

By Lemma 3.1, K_φ^α is compact in $\mathscr{L}(\widetilde{\mathscr{H}}')$ and $\|R_\varphi^\alpha\| < 1$ if $A(S_0)$ is sufficiently small. Hence

$$\begin{aligned} T_\alpha\left(P_\varphi - \varphi P_b \psi B_\alpha^{-1}\right) &= I + K_\varphi + K_\varphi^\alpha + \alpha R_\varphi^\alpha =: I + \widetilde{K}^\alpha + R^\alpha, \\ T_\alpha\left(P_\varphi - \varphi P_b \psi B_\alpha^{-1}\right)\left(I + R^\alpha\right)^{-1} &= I + \widetilde{K}^\alpha\left(I + R^\alpha\right)^{-1} =: I + K^\alpha. \end{aligned} \tag{4.8}$$

Thus we have constructed a right parametrix for T_α. To complete the proof we have to construct a right parametrix for T_α^*, too. But our assumptions imply that T_α^* is also regular singular, and the parametrix just constructed maps into the minimal domain. q.e.d.

We proceed to compute the index of T_0 using the heat kernel method as in [5]. The only modifications in the argument arise from the fact that $xa(x) \not\equiv 1$. With the notation in [5, Theorem 4.1] we have

Theorem 4.2. *Assume* (2.1), (2.6), (2.7), (2.9), *and* (4.1), *and let* A *in* (4.4) *be sufficiently small. Then* $D_{\overline{g}\min}$ *is Fredholm with*

$$\begin{aligned} \operatorname{ind} D_{\overline{g}\min} = \int_{M_1} \omega_D &+ \tfrac{1}{2}\left(\eta\left(S_0\right) - \dim\ker S_0\right) \\ &- \sum_{-1/2<s<0} \dim\ker\left(S_0 - s\right) - \sum_{k\geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k). \end{aligned}$$

Proof. The closed extensions of $T_0 = T_{0,\min}$ are classified by the subspaces of the space

$$W_0 = \bigoplus_{|s|<1/2} \ker\left(S_0 - s\right),$$

as in [5, Lemma 3.2]. The "Dirichlet extension" $T_{0,\delta}$ constructed there corresponds to

$$W_< := \bigoplus_{-1/2<s<0} \ker\left(S_0 - s\right),$$

so

$$\operatorname{ind} T_0 = \operatorname{ind} T_{0,\delta} - \sum_{-1/2<s<0} \dim\ker\left(S_0 - s\right). \tag{4.9}$$

It remains to determine the constant term, $\beta(\chi)$, in the asymptotic expansion as $t \to 0$ of

$$\operatorname{tr} \chi\left(e^{-tT_{0,\delta}^* T_{0,\delta}} - e^{-tT_{0,\delta} T_{0,\delta}^*}\right) := F_\chi(t), \tag{4.10}$$

for various choices of χ. Choose $0 < s_1 < s_2 < s(0)$ such that $g^2 \circ s^{-1}(x) = f(0)$ for $x \in [s_2, s(0)]$ and $a(x) = x^{-1}$ for $x \in (0, s_1]$. Then choose $\chi_1 \in C_0^\infty(-s_1, s_1)$, $\chi_2 \in C^\infty(0, s(0))$ such that $\chi_1 = 1$ in a neighborhood of 0, and $\chi_1 + \chi_2 = 1$ in a neighborhood of $[0, s_2]$, and put $\chi_3 := 1 - \chi_1 - \chi_2$.

If we let $\chi = \chi_3$ in (4.10), then we obtain, by unitary equivalence with $D_{\overline{g}}$, that the constant term equals

$$\beta(\chi_3) = \int_M \overline{\chi}_3 \omega_D,$$

where ω_D is the usual index form. The proof of Lemma 4.4 in [5] shows next that

$$\beta(\chi_1) = \tfrac{1}{2}(\eta(S_0) - \dim\ker S_0) - \sum_{k\geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k). \tag{4.11}$$

For the remaining coefficient, $\beta(\chi_2)$, we use [3, Theorem 4.1] to find that

$$\beta(\chi_2) = C(\chi_2) \operatorname{Res} \eta_{S_0}(0).$$

It is readily seen from the explicit formula for $C(\chi_2)$ that we can increase $C(\chi_2)$ without affecting $\beta(\chi_3)$ or $\beta(\chi_1)$. Thus we conclude

$$\beta(\chi_2) = 0; \qquad \beta(\chi_3) = \int_{M_1} \omega_D. \tag{4.12}$$

Combining (4.9), (4.11), and (4.12) yields the theorem.

Corollary 4.3. *We have*

$$\operatorname{ind} D_{\overline{g},\max} = \operatorname{ind} D_{\overline{g},\min} + \sum_{|s|<1/2} \dim\ker(S_0 - s). \tag{4.13}$$

Proof. Note that $(D_{\overline{g}})' = (D')_{\overline{g}}$ and that Theorem 4.2 applies to $-D'_{\overline{g}}$ as well, with S_0 replaced by $-S_0$. Then we compute, recalling $\omega_{D'} = -\omega_D$,

$$\begin{aligned}
\operatorname{ind} D_{\overline{g},\max} &= \operatorname{ind}\left(-D_{\overline{g},\max}\right) = \operatorname{ind}\left(-D'^{*}_{\overline{g},\min}\right) \\
&= -\operatorname{ind}\left(-D'_{\overline{g},\min}\right) \\
&= \int_{M_1} \omega_D + \tfrac{1}{2}(\eta(S_0) + \dim\ker S_0) \\
&\quad + \sum_{0<s<1/2} \dim\ker(S_0 - s) - \sum_{k\geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k) \\
&= \operatorname{ind} D_{\overline{g},\min} + \sum_{|s|<1/2} \dim\ker(S_0 - s). \quad \text{q.e.d.}
\end{aligned}$$

Now we turn to the computation of h_0 and h_1. Their analysis depends on the small eigenvalues of S_0 and S_1. We introduce the orthogonal projection Q onto the eigenspaces of S_0 with eigenvalues $\bigoplus_{|s|<1/2}\ker(S_0-s)$ between $-\frac{1}{2}$ and $\frac{1}{2}$. By construction, there is s_1 with $0 \le s_1 < \frac{1}{2}$ such that

$$\operatorname{spec} S_0 \cap \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \subset [-s_1, s_1]. \tag{4.14}$$

We will use Lemma 3.7 so we want the decay condition (3.21) for the transformed operator obtained from $\overline{g}D\overline{g}$. In view of (2.23) it takes the form

$$f^{-1}(y)\, g^2(y) \left\| Q\overline{S}_1(y) - \overline{S}_1(y)\, Q \right\| = O(1) \quad \text{as } y \to \infty, \tag{4.15}$$

and this will be assumed in what follows. Here we have written (cf. (3.13))

$$D \simeq A(y)\left[\partial_y + \frac{1}{f(y)}\left(S_0 + \overline{S}_1(y)\right)\right]$$

with

$$\begin{aligned} A(y) &:= A_1 A_2(y), \\ \overline{S}_1(y) &:= \left(A^{-1}(y) - I\right) S_0 + A^{-1} S_1(y) + f A_2^{-1} A_2'(y). \end{aligned}$$

We will have to study the reduced matrix operators

$$D_Q := \partial_y + f(y)^{-1}\left(QS_0Q + Q\overline{S}_1(y)\, Q\right), \tag{4.16a}$$

$$D_Q' := -\partial_y + f(y)^{-1}\left(QS_0Q + Q\overline{S}_1(y)^*\, Q\right). \tag{4.16b}$$

For y, $y_1 \ge 0$ we denote by $W(y, y_1)$ and $W'(y, y_1)$ the respective solution operators, i.e., the matrix functions with

$$\begin{aligned} D_Q^{(\prime)} W^{(\prime)}(y, y_1) &= 0, \qquad y \ge 0, \\ W^{(\prime)}(y_1, y_1) &= I. \end{aligned} \tag{4.17}$$

Lemma 4.1. (a) *For y, $y_1 \ge 0$ and any $s_2 \in (s_1, \frac{1}{2})$ we have*

$$\left\| W^{(\prime)}(y, y_1) \right\| \le C e^{s_2 |F(y) - F(y_1)|},$$

where s_1 is defined in (4.14), *and F in* (2.19).
(b) *For y, $y_1 \ge 0$ we have*

$$W(y, y_1)^*\, W'(y, y_1) = I.$$

(c) *Let* $v \in C^1(\mathbf{R}_+, QH)$ *be a solution of the equation*

$$D_Q v = g^{-1} w, \qquad w \in L^2(\mathbf{R}_+, H). \tag{4.18a}$$

Then for $y_1 \geq 0$ *there is* $\gamma_{y_1} v \in QH$ *and* $v_{y_1} \in C^1(\mathbf{R}_+, QH)$ *such that*

$$v(y) = W_Q(y, y_1)\, \gamma_{y_1} v + v_{y_1}(y), \tag{4.18b}$$

$$\left\| v_{y_1}(y) \right\|_H^2 = O\left(e^{-F(y)}\right). \tag{4.18c}$$

Moreover, the map $v \mapsto \gamma_{y_1} v$ *is linear, and* $\gamma_{y_1} v$ *and* v_{y_1} *are uniquely determined by the properties* (4.18b) *and* (4.18c).

Proof. (a) Since $W_Q(y, y_1)$ solves the equation $D_Q W_Q(\cdot, y_1) = 0$, for $e \in QH$ from (4.16a), (2.15), and (4.14) we have
(4.19)

$$\begin{aligned}\left\| W_Q(y, y_1)\, e \right\|^2 &= \|e\|^2 + 2 \int_{y_1}^{y} \operatorname{Re}\left\langle \partial_{y'} W_Q\left(y', y_1\right) e,\, W_Q\left(y', y_1\right) e \right\rangle dy' \\ &\leq \|e\|^2 + 2 s_2 \int_{y_1}^{y_2} f\left(y'\right)^{-1} \left\| W_Q\left(y', y_1\right) e \right\|^2 dy',\end{aligned}$$

if $s_2 \in (s_1, \frac{1}{2})$, and y, y_1 are sufficiently large. Hence the assertion follows from Gronwall's Lemma (cf. [9, p. 24]). The proof for W_Q' is analogous.

(b) It follows from a straightforward computation that

$$\frac{\partial}{\partial_y}\left(W(y, y_1)^* \, W'(y, y_1)\right) = 0. \tag{4.20}$$

(c) We choose $y_1 = 0$, for simplicity of notation. Then

$$v(y) = W_Q(y, 0)\, v(0) + \int_0^y W_Q\left(y, y'\right) g^{-1} W\left(y'\right) dy'. \tag{4.21}$$

Since $g^{-1}(y') = f(y')^{-1/2} e^{-F(y')/2}$ for y' large, we deduce from part (a) and $w \in L^2(\mathbf{R}_+, QH)$ that the integrand in (4.20) is in $L^1(\mathbf{R}_+, QH)$. Since $W_Q(y, y') = W_Q(y, 0) W_Q(0, y')$, we may write

$$\begin{aligned} v(y) &=: W_Q(y, 0)\, \gamma_0 v - \int_y^\infty W_Q\left(y, y'\right) g^{-1} w\left(y'\right) dy' \\ &=: W_Q(y, 0)\, \gamma_0 v + v_0(y). \end{aligned}$$

We estimate, again with part (a),

$$\|v_0(y)\|_H^2 \le \|w\|^2_{L^2(\mathbf{R}_+, H)} \int_y^\infty f\left(y'\right)^{-1} e^{-F(y')} \left\|W_Q\left(y, y'\right)\right\|^2 dy'$$
$$\le \|w\|^2_{L^2(\mathbf{R}_+, H)} e^{-F(y)} \int_y^\infty e^{(2s_1-1)(F(y')-F(y))} \frac{dy'}{f(y)'} = O\left(e^{-F(y)}\right),$$

which proves (4.18c).

Now assume a second representation

$$v(y) = W_Q(y, 0)\overline{\gamma_0 v} + \overline{v}_0(y).$$

It follows from part (b) that

$$\overline{\gamma_0 v} - \gamma_0 v = W_Q'(y, 0)^* \left(v_0(y) - \overline{v}_0(y)\right),$$

hence from part (a) and (4.18c) that

$$\left\|\overline{\gamma_0 v} - \gamma_0 v\right\| \le Ce^{(s_1 - 1/2)F(y)} \to 0, \qquad y \to \infty.$$

Thus $\overline{\gamma_0 v} = \gamma_0 v$ and $v_0 = \overline{v}_0$. q.e.d.

We can now calculate h_0. For some $y_1 \ge 0$ we introduce

$$K_{y_1}^{(l)} := \left\{e \in QH | W^{(l)}(\cdot, y_1) e \in L^2\right\} \subset QH.$$

Since $W^{(l)}(y, y_1) = W^{(l)}(y, y_2)W^{(l)}(y_2, y_1)$, it is easily seen that the dimensions of these spaces are independent of y_1.

Lemma 4.2. *For* $y_1 \ge 0$ *we have*

$$h_0 = \dim \mathscr{D}\left(D_{\overline{g}, \max}\right) \cap g^{-1}\mathscr{H}/\mathscr{D}\left(D_{\overline{g}, \min}\right) = \dim K_{y_1}. \tag{4.22}$$

Proof. We want to construct a linear map $\gamma: \mathscr{D}(D_{\overline{g}, \max}) \to QH$ such that

$$\ker \gamma | \mathscr{D}\left(D_{\overline{g}, \max}\right) \cap g^{-1}\mathscr{H} = \mathscr{D}\left(D_{\overline{g}, \min}\right).$$

To achieve this, observe first that $u \in \mathscr{D}(D_{\overline{g}, \min})$ if and only if $u \in \mathscr{D}(D_{\overline{g}, \max})$ and

$$\|gu(y)\|_H = O\left(e^{-F(y)/2}\right), \qquad y \to \infty. \tag{4.23}$$

In fact, this follows from the remark after Corollary 3.2 and (2.23). Next we note that for $\overline{\varphi} \in C_0^\infty(M)$ with $\overline{\varphi} = 1$ in a neighborhood of M_1 we have $\overline{\varphi}u \in \mathscr{D}(D_{\overline{g}, \min})$ for $u \in \mathscr{D}(D_{\overline{g}, \max})$, by interior regularity. Also, $(I - Q)(1 - \varphi)u \in \mathscr{D}(D_{\overline{g}; \min})$, by Lemma 3.7. So it remains to study $Q(1 - \varphi)u := \tilde{u}$. We compute

$$D_Q g\tilde{u}(y) = g^{-1}A^{-1}QD_g(1 - \varphi)u(y) - f^{-1}Q\left(Q\overline{S}_1 - \overline{S}_1 Q\right) gu(y)$$
$$:= w(y),$$

with $w \in L^2(\mathbf{R}_+, H)$. Hence from Lemma 4.1 we obtain the decomposition

$$g\tilde{u}(y) = W_Q(y, y_1)\gamma_{y_1} g\tilde{u} + \tilde{v}(y). \tag{4.24}$$

Now choose $y_1 > 0$ such that $\varphi(y) = 0$ if $y \geq y_1$ and define

$$\gamma(u) := \gamma_{y_1} g\tilde{u} = \gamma_{y_1}(g(1-\varphi)Qu) \in QH.$$

Next we want to show that

$$\ker \gamma = \mathscr{D}\left(D_{\overline{g},\min}\right), \qquad \operatorname{im}\gamma = K_{y_1}.$$

Let $u \in \ker\gamma$. Then we obtain from (4.24) and (4.18c)

$$\|g\tilde{u}(y)\| = O\left(e^{-F(y)/2}\right),$$

hence $\tilde{u} \in \mathscr{D}(D_{\overline{g},\min})$ by (4.23) and $u \in \mathscr{D}(D_{\overline{g},\min})$. Consequently, if $u \in \mathscr{D}(D_{\overline{g},\min})$, from (4.24) and Lemma 4.1(c) we obtain

$$\gamma(u) = W_Q'(y, y_1)^*(g\tilde{u}(y) - \tilde{v}(y)).$$

If we substitute the above equation in the estimates (4.23) and (4.18c), and use Lemma 4.1(a) we obtain for $y \geq y_1$

$$\|\gamma(u)\| \leq C_{y_1} e^{(s_2 - 1/2)F(y)} \to 0, \qquad y \to \infty,$$

hence $\gamma(u) = 0$ as claimed.

It remains to show that $\operatorname{im}\gamma = K_{y_1}$. Pick $e \in K_{y_1}$ and $\psi \in C^\infty(\mathbf{R})$ such that $\psi(y) = 1$ for $y \geq y_1/2$ and $\psi(y) = 0$ for $y \leq y_1/3$. Define

$$u(y) := \psi g^{-1}(y) W_Q(y, y_1)e.$$

Since ψ' has compact support and $e \in K_{y_1}$, we have $u \in \mathscr{D}(D_{\overline{g},\max})$ and $gu \in \mathscr{H}$. It follows that

$$g\tilde{u}(y) = W_Q(y, y_1)e + ((1-\varphi)\psi(y) - 1)W_Q(y, y_1)e,$$

so the uniqueness of the decomposition (4.24) implies

$$e = \gamma(u). \quad \text{q.e.d.}$$

We can make (4.22) even more precise in the following special case which covers the examples given in §1. Thus we now assume the following:

$$\operatorname{spec} S_0 \cap \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] = \{0\}, \tag{4.25}$$

(4.26) in (2.6) we have $A_1 = A_2 = I$ and $S_1(y) = f'(y)S_1$ for some self-adjoint operator in H with domain H_1.

$$(I - Q)S_1 Q = 0. \tag{4.27}$$

In view of Remark (2) after (2.16), only (4.27) requires verification in the warped product case. But this is straightforward from the explicit formulas (2.3) and (2.4). Note that (4.26) implies $a = 0$ in (2.14).

We write the spectral decomposition of $\overline{S}_1 := QS_1Q$ as

$$\overline{S}_1 := \bigoplus_t tQ_t. \tag{4.28}$$

Then we obtain

Lemma 4.3. *Under the assumptions* (4.25), (4.26), *and* (4.27) *we have*

$$h_0 = \sum_{f^{-t} \in L_2} \dim Q_t.$$

Proof. It is readily seen that in this case,

$$D_Q = \bigoplus_t \left[\partial_y + \frac{f'}{f}(y)\, t \right] Q_t$$

and

$$W(y, 0) = \bigoplus_t (f(y)/f(0))^{-t} Q_t.$$

Therefore

$$K_0 = \bigoplus_{f^{-t} \in L^2} Q_t. \quad \text{q.e.d.}$$

As mentioned before, h_1 is more difficult to deal with. So we only obtain an inequality for this quantity, which implies, however, $h_1 = 0$ in many interesting cases. To formulate it we observe first that K_{y_1} and K'_{y_1} are orthogonal: let $e^{(\prime)} \in K^{(\prime)}_{y_1}$ and put $u^{(\prime)}(y) := W^{(\prime)}(y, y_1)e^{(\prime)}$. Then

$$\begin{aligned} \left\langle u(y), u'(y) \right\rangle &= \left\langle W(y, y_1)e, W'(y, y_1)e' \right\rangle \\ &= \left\langle e, W(y, y_1)^* W'(y, y_1)e' \right\rangle = \left\langle e, e' \right\rangle, \end{aligned}$$

by Lemma 4.1(b). Since $u, u' \in L^2$ we obtain

$$\left\langle e, e' \right\rangle = \lim_{T \to \infty} \int_T^{T+1} \left\langle u(y), u'(y) \right\rangle dy = 0.$$

Denote by L_{y_1} the orthogonal complement of $K_{y_1} \oplus K'_{y_1}$ in QH, and by Q_1 the orthogonal projection onto L_{y_1}. It is easy to see (cf. the proof of

Lemma 4.4 below) that with

$$\mathscr{H}_W^* := \left\{ v \in C^\infty(F) \,|\, g^{-1}v \in \mathscr{H}',\, D'v = 0,\, (Du, v) = 0 \text{ for all } u \in \mathscr{H} \text{ with } gDu \in \mathscr{H}' \right\}, \tag{4.29}$$

we have

$$\ker D^*_{\bar{g}, W} = \beta'\left(\mathscr{H}_W^*\right).$$

With $y_1 > 0$ arbitrary but fixed we introduce the map

$$\tau_{y_1} : \mathscr{H}_W^* \ni v \mapsto Q_1 \gamma' g^{-1} v \in L_{y_1},$$

where γ' is the map analogous to γ for D'_Q. Note that this is well defined by $g^{-1}v \in \mathscr{D}(D'_{\bar{g}, \max})$ and Lemma 4.1(c).

Lemma 4.4. L^2-$\ker D' = \ker \tau_{y_1}$. *Consequently,*

$$\begin{aligned} h_1 &= \dim \ker D^*_{\bar{g}, W} - \dim L^2\text{-}\ker D' \\ &= \dim \operatorname{im} \tau_{y_1} \le \dim L_{y_1}. \end{aligned}$$

Proof. Let $v \in L^2$-$\ker D'$. Then clearly $v \in C^\infty(F)$, $g^{-1}v \in \mathscr{H}'$, and $D'v = 0$. Moreover, it follows as in the proof of Lemma 2.3 that $(Du, v) = 0$ for all $u \in \mathscr{H}$ with $gDu \in \mathscr{H}'$ so $v \in \mathscr{H}_W^*$. Next we obtain $D'_Q(1 - \varphi)v = g^{-1}w$ with $w \in L^2$, hence from (the analogue of) Lemma 4.1(c)

$$Q(1 - \varphi)v(y) = W'(y, y_1)\gamma'_{y_1} g^{-1} v + \tilde{v}(y, y_1), \tag{4.30}$$

where v, $\tilde{v} \in L^2$. Thus $\gamma'_{y_1} g^{-1} v \in K'_{y_1}$ and $v \in \ker \tau_{y_1}$.

Conversely, let $v \in \ker \tau_{y_1}$; we have to show that $v \in \mathscr{H}'$. We have $\tilde{v} := g^{-1}v \in \mathscr{D}(D'_{\bar{g}, \max})$ and consequently (with φ as in the proof of Lemma 4.2) by Lemma 3.7 $(I - Q)(1 - \varphi)\tilde{v} \in \mathscr{D}(D'_{\bar{g}, \min})$ which is, by (4.23), equivalent to

$$\|(I - Q)(1 - \varphi)v(y)\|_H = O\left(e^{-F(y)/2}\right),$$

implying $(I - Q)(1 - \varphi)v \in \mathscr{H}'$, by (2.25). From Lemma 4.1 we obtain the decomposition (4.30) so it remains to show that $\gamma'_{y_1}\tilde{v} \in K'_{y_1}$. By assumption, $e' := \gamma'_{y_1}\tilde{v} \in K_{y_1} \oplus K'_{y_1}$. Choose $e \in K_{y_1}$; then we can find $u \in \mathscr{D}(D_{\bar{g}, \max}) \cap g^{-1}\mathscr{H}$ with $\gamma_{y_1} u = e$. Write $\tilde{u} := gu$ such that $\tilde{u} \in \mathscr{H}$

and $gD\tilde{u} = D_g u \in \mathscr{H}'$, hence $0 = (D\tilde{u}, v)$. From Lemma 4.1 again we have

$$Q(1-\varphi)\tilde{u}(y) =: \psi(y) W(y, y_1) e + W(y),$$

so we obtain, in consequence of (4.30), Lemma 4.1(a), and Lemma 3.7,

$$\begin{aligned} 0 = (D\tilde{u}, v) &= \lim_{T\to\infty} \langle \tilde{u}(T), v(T)\rangle \\ &= \lim_{T\to\infty} \left\langle W(T, y_1) e + W(T), W'(T, y_1) e' + \tilde{v}(T, y_1)\right\rangle + o(1) \\ &= \left\langle e, e'\right\rangle + o(1). \end{aligned}$$

Thus $e' \perp K_{y_1}$ which completes the proof. q.e.d.

Let us again assume (4.25), (4.26), and (4.27). The proof of Lemma 4.3 shows that

$$K_0 = \bigoplus_{f^{-t}\in L^2} Q_t, \qquad K_0' = \bigoplus_{f^t\in L^2} Q_t,$$

hence in this case

$$L_0 = \bigoplus_{f^t, f^{-t}\notin L^2} Q_t, \tag{4.31}$$

which gives as a useful special case

Lemma 4.5. *Suppose that* $f^t \in L^2$ *or* $f^{-t} \in L^2$ *for all* $t \neq 0$ *and that* $Q_0 = \{0\}$ *in* (4.28). *Then* $h_1 = 0$.

Note that in the examples of §2, $Q_0 = \{0\}$ is always satisfied so that Lemma 4.5 applies.

We combine the results of this section with Theorem 2.2 to formulate our main result.

Theorem 4.3. *Let* M *be a complete Riemannian manifold, let* E *and* F *be Hermitian vector bundles over* M, *and let* $D\colon C^\infty(E) \to C^\infty(F)$ *be a first order elliptic differential operator. Assume conditions* (2.1), (2.6), (2.7), (2.9), (2.14), (4.1), (4.15), *and that the constant* A *in* (4.4) *is sufficiently small. Then* D *has a finite* L^2*-index given by*

$$\begin{aligned} L^2\text{-}\operatorname{ind} D = \int_{M_1} \omega_D + \tfrac{1}{2}\left(\eta(s_0) - \dim\ker S_0\right) - \sum_{-1/2<s<0} \dim\ker(S_0 - s) \\ - \sum_{k\geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k) + \dim K_0 + \dim \operatorname{im} \tau_0. \end{aligned}$$

Here ω_D *is the usual index form,* η_{S_0} *is the* η*-function associated with the operator* S_0 *in* (2.6), *and* K_0 *and the map* τ_0 *are defined in Lemmas* 4.2 *and* 4.4 *respectively.*

5. Asymptotically warped products

Asymptotically warped products will be studied in this section. By this we mean a complete orientable Riemannian manifold M with (2.1) and

$$(5.1)\qquad \begin{gathered} U \text{ is isometric to } (0, \infty) \times N \text{ with metric} \\ g = dy^2 + f(y)^2 g_N(y). \end{gathered}$$

Here f is a smooth positive function satisfying (2.14), and $g_N(y)$ is a smooth family of metrics on $N = \partial M_1$ converging to a limiting metric $g_N = g_N(\infty)$ as $y \to \infty$. This defines a warped metric $g^0 := dy^2 + f(y)^2 g_N$ on U. We denote by ∇ and ∇^0 the Levi-Civita connections for the metrics g and g^0, and by ω, ω^0 and Ω, Ω^0 the respective connection and curvature forms. Then we want that with $\theta := \omega - \omega^0$

$$(5.2)\qquad \sup_{p\in N}\left(\left|g - g^0\right|^0_{(y,p)} + f(y)\,|\theta|^0_{(y,p)}\right) = o(1) \quad \text{as } y \to \infty.$$

Here $|\cdot|^0$ denotes the norm defined by g^0. These conditions are enough to ensure that D_{GB} and D_S have a finite L^2-index. To obtain a more convenient formula for $\operatorname{ind} D_{\overline{g},\min}$ in some cases we will need in addition that

$$(5.3)\qquad \sup_{p\in N} f(y)^2\,|\Omega|^0_{(y,p)} = O(1) \quad \text{as } y \to \infty.$$

To handle h_0 and h_1 we will also impose the decay condition (4.15). The calculations for D_S and D_{GB} are almost identical, so we will give proofs only for the latter operator. We begin by establishing (2.6). We recall first the method used in [5, §5] for the warped product case (actually only for $f(y) = y$, but the generalization is obvious). With $c_j := j - (m-1)/2$, $m = \dim M$, we introduce the maps $\Phi_{\mathrm{ev}/\mathrm{odd}}\colon C_0^\infty((0,\infty), \Omega(N)) \to \Omega_{\mathrm{ev}/\mathrm{odd}}(U)$ by

(5.4a)
$$\Phi_{\mathrm{ev}}\left(\sum_{j\geq 0}\omega_j(y)\right) = \sum_{j\geq 0}\left(f(y)^{c_{2j}}\,\omega_{2j}(y) + f(y)^{c_{2j+1}}\,\omega_{2j+1}(y)\wedge dy\right),$$

(5.4b)
$$\Phi_{\mathrm{odd}}\left(\sum_{j\geq 0}\omega_j(y)\right) = \sum_{j\geq 0}\left(f(y)^{c_{2j}}\,\omega_{2j}(y)\wedge dy + f(y)^{c_{2j+1}}\,\omega_{2j+1}(y)\right),$$

which define unitary maps between $L^2((0,\infty))$, $L^2(\Lambda^* N))$ and $L_0^2(\Lambda^*_{\mathrm{ev}/\mathrm{odd}} U)$, where N has the metric g_N, and the subscript 0 refers to the metric g^0. Thus we obtain as in Example 1 of §2

$$
\begin{aligned}
\Phi_{\mathrm{odd}}^{-1} D_{GB}^0 \Phi_{\mathrm{ev}} &= \Phi_{\mathrm{odd}}^{-1}\left(d+\delta^0\right)\Phi_{\mathrm{ev}} \\
&= \partial_y + \frac{1}{f(y)}\left(S_0 + f'(y)\, S_1\right) \\
&= \partial_y + \frac{1}{f(y)}\left(S_0 + aS_1 + \left(f'(y) - a\right) S_1\right) \\
&=: \partial_y + \frac{1}{f(y)}\left(\widetilde{S}_0 + \widetilde{S}_1(y)\right),
\end{aligned}
\tag{5.5}
$$

where S_0 and S_1 are given by (2.3), $a \geq 0$ is defined by (2.14), and we may assume $a < 1$ and $\pm\frac{1}{2} \notin \operatorname{spec} \widetilde{S}_0$. Now we introduce an endomorphism $B \in C^\infty$ (End $\Lambda^* U$) with the property that

$$\langle \omega_1, \omega_2\rangle (p) = \langle \omega_1, B\omega_2\rangle_0 (p), \qquad \omega_1, \omega_2 \in \Omega(U), \ p \in U.$$

Here $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_0$ denotes the pointwise scalar product with respect to g and g_0, respectively.

Then we find smooth functions $B_{\mathrm{ev}/\mathrm{odd}} \in C^\infty((0,\infty), \mathscr{L}(H) \cap \mathscr{L}(H_1))$, where $H = L^2(\Lambda^* N)$, $H_1 = H^1(\Lambda^* N)$, such that

$$B\Phi_{\mathrm{ev}/\mathrm{odd}} = \Phi_{\mathrm{ev}/\mathrm{odd}} B_{\mathrm{ev}/\mathrm{odd}}.$$

Moreover, $B_{\mathrm{ev}/\mathrm{odd}}$ satisfies (2.16a), i.e.,

$$\left(f(y)\,\partial_y\right)^j \left(B_{\mathrm{ev}/\mathrm{odd}} - I\right)(y) = o, \quad \text{as } y \to \infty, \ j \leq 1. \tag{5.6}$$

To see this we choose a local orthonormal frame $(e_i)_{1\leq i\leq m-1}$ for (N, g_N) such that $f_0 := \partial/\partial y$, $f_i := f(y)^{-1} e_i$, $i \geq 1$, is a local orthonormal frame for (U, g^0), parallel with respect to ∇^0 along the geodesics normal to N. From this frame we construct a local orthonormal frame $(\tilde{f}_i)_{0\leq i\leq m-1}$ for (U, g) by the Gram-Schmidt procedure. Then it is readily seen that the coefficients of $B_{\mathrm{ev}/\mathrm{odd}}$ with respect to the frame (f_i) are smooth functions in the variables $g_{ij} := \langle f_i, f_j\rangle$. But

$$
\begin{aligned}
\frac{\partial}{\partial y}\left\langle f_i, f_j\right\rangle &= \left\langle \left(\nabla_{f_0} - \nabla^0_{f_0}\right) f_i, f_j\right\rangle + \left\langle f_i, \left(\nabla_{f_0} - \nabla^0_{f_0}\right) f_j\right\rangle \\
&= \sum_k \left(\theta_{ik}(f_0)\, g_{kj} + \theta_{jk}(f_0)\, g_{ki}\right).
\end{aligned}
$$

So (5.6) follows from (5.2). Moreover, $B_{\mathrm{ev}/\mathrm{odd}}$ satisfies (2.16b) which follows from (5.2) and the explicit formulas for the Levi-Civita connection ∇^0 (cf. [11, p. 206]). Thus we obtain a unitary equivalence

$$D_{GB} = d + \delta \simeq \Phi_{\mathrm{odd}}^{-1} B^{1/2} D_{GB} B^{-1/2} \Phi_{\mathrm{ev}} \\ = B_{\mathrm{odd}}^{1/2} \Phi_{\mathrm{odd}}^{-1} (d + \delta) \Phi_{\mathrm{ev}} B_{\mathrm{ev}}^{-1/2}. \tag{5.7}$$

Recall that [10, Lemma 5.13]

$$\delta = -\sum_k \tilde{f}_k \llcorner \nabla_{\tilde{f}_k},$$

where $\llcorner$ denotes interior multiplication, and $(\tilde{f}_i)$ is the local orthonormal frame for (U, g) constructed above. We write

$$\tilde{f}_k = \sum_l a_{kl} f_l,$$

and observe that the a_{kl} are smooth functions in the variables g_{ij}. Then it follows as above that

$$\sup_{p \in \mathbf{N}} \left(f(y)\, \partial_y\right)^j (a_{kl} - \delta_{kl})(y, p) = o(1) \quad \text{as } y \to \infty, \ j \le 1. \tag{5.8}$$

Hence we have

$$\begin{aligned} \delta = & - \sum_{k,l,l'} a_{kl} a_{kl'} f_l \llcorner \left(\nabla_{f_{l'}} - \nabla^0_{f_{l'}}\right) \\ & - \sum_{k,l,l'} (a_{kl} a_{kl'} - \delta_{kl}\delta_{kl'}) f_l \llcorner \nabla^0_{f_{l'}} + \delta^0 \\ =: & \sum_{k,l,l'} a_{kl} a_{kl'} C_{ll'} + \sum_{l,l'} b_{l,l'} E_{l,l'} + \delta^0 \\ =: & \ C + E + \delta^0. \end{aligned}$$

So we must study the transformation of C and E under Φ. Note that C and E are independent of the choice of frame.

Lemma 5.1. *We have*

$$\tilde{C} := \Phi_{\mathrm{odd}}^{-1} C \Phi_{\mathrm{ev}} \in C^\infty\left((0, \infty), \mathscr{L}(H) \cap \mathscr{L}(H_1)\right)$$

and

$$f(y)\left(\left\|\tilde{C}(y)\left(|\tilde{S}_0| + 1\right)^{-1}\right\| + \left\|\left(|\tilde{S}_0| + 1\right)^{-1} \tilde{C}(y)\right\|\right) = o(1) \tag{5.10}$$

as $y \to \infty$.

Proof. It is enough to prove the assertion locally for each C_{lk}. Using the frame (f_i) constructed above we have

$$\left(\nabla_{f_k} - \nabla^0_{f_k}\right) f_i = \sum_j \theta_{ij}(f_k) f_j =: \sum_j \theta^k_{ij} f_j.$$

If $(e_i^*)_{1\le i\le m-1}$ and $(f_i^*)_{0\le i\le m-1}$ denote the respective dual frames, $f_0^* = dy$, we find

$$\left(\nabla_{f_k} - \nabla^0_{f_k}\right) e_i^* = f(y)^{-1}\,\theta^k_{0i}\,dy + \sum_{j\ge 1}\theta^k_{ji} e_j^*, \qquad i \ge 1,$$

$$\left(\nabla_{f_k} - \nabla^0_{f_k}\right) dy = \theta^k_{00}\,dy + f(y)\sum_{j\ge 1}\theta^k_{j0} e_j^*.$$

Thus for $p \in \mathbf{N}$ and $1 \le i_1 < \cdots < i_r \le m-1$

(5.11a)

$$\begin{aligned}
&\left(\nabla_{f_k} - \nabla^0_{f_k}\right) e^*_{i_1}\wedge\cdots\wedge e^*_{i_r}\\
&\quad = f(y)^{-1}\sum_l (-1)^{r-l}\,\theta^k_{0i_l} e^*_{i_1}\wedge\cdots\wedge\widehat{e^*_{i_l}}\wedge\cdots\wedge e^*_{i_r}\wedge dy\\
&\qquad + \sum_{l,j}(-1)^{r-l}\,\theta^k_{ji_l} e^*_{i_1}\wedge\cdots\wedge\widehat{e^*_{i_l}}\wedge\cdots\wedge e^*_{i_r}\wedge e^*_j,
\end{aligned}$$

(5.11b)

$$\begin{aligned}
&\left(\nabla_{f_k} - \nabla^0_{f_k}\right) e^*_{i_1}\wedge\cdots\wedge e^*_{i_r}\wedge dy\\
&\quad = \left(\left(\nabla_{f_k} - \nabla^0_{f_k}\right) e^*_{i_1}\wedge\cdots\wedge e^*_{i_r}\right)\wedge dy\\
&\qquad + \theta^k_{00} e^*_{i_1}\wedge\cdots\wedge e^*_{i_r}\wedge dy + f(y)\sum_{j\ge 1}\theta^k_{j0} e^*_{i_1}\wedge\cdots\wedge e^*_{i_r}\wedge e^*_j.
\end{aligned}$$

Combining (5.4a,b) with (5.11a,b) and (5.2) gives the lemma. q.e.d.

Lemma 5.2. *We have, with b_{ij} in (5.9),*

$$\begin{aligned}
\widetilde{E} &:= \Phi^{-1}_{\mathrm{odd}} E \Phi_{\mathrm{ev}}\\
&= b_{00}\,\mathrm{diag}\left((-1)^r\right)\partial_y + f(y)^{-1} b_{00}\,\mathrm{diag}(c_r)\\
&\quad + \sum_{l\ge 1}\left(b_{l0} e_l \llcorner \partial_y + f(y)^{-1} b_{l0} e_l \llcorner \mathrm{diag}(c_r)\right)\\
&\quad + f(y)^{-1}\sum_{l,l'\ge 1} b_{ll'} e_l \llcorner \nabla^0_{e_{l'}}\\
&=: \widetilde{E}_1(y)\,\partial_y + f(y)^{-1}\widetilde{E}_2(y).
\end{aligned}$$

Here $\widetilde{E}_1 \in C^\infty((0,\infty), \mathscr{L}(H) \cap \mathscr{L}(H_1))$ *with*

$$\left\|\widetilde{E}_1(y)\right\|_{H_1} = o(1) \quad \text{as } y \to \infty,$$
$$\left\|\left(f(y)\partial_y\right)^j \widetilde{E}_1(y)\right\|_H = o(1) \quad \text{as } y \to \infty,\ j \le 1,$$

and $\widetilde{E}_2 \in C^\infty((0,\infty), \mathscr{L}(H_1, H))$ *with*

$$\left\|\widetilde{E}_2(y)\left(\left|\widetilde{S}_0\right| + 1\right)^{-1}\right\|_H + \left\|\left(\left|\widetilde{S}_0\right| + 1\right)^{-1}\widetilde{E}_2(y)\right\|_H = o(1) \quad \text{as } y \to \infty.$$

Proof. The lemma follows by straightforward computations, together with (5.4a,b), [11, p. 206] and (5.8).

Now we combine (5.7), (5.5), and (5.9) with Lemmas 5.1 and 5.2 to derive the unitary representation

(5.12)

$$\begin{aligned}
D_{GB} &\simeq \Phi_{\text{odd}}^{-1} B^{1/2} D_{GB} B^{-1/2} \Phi_{\text{ev}} \\
&= B_{\text{odd}}^{1/2}(y)\Big[\left(I + \widetilde{E}_1(y)\right)\partial_y \\
&\qquad + f(y)^{-1}\left(\widetilde{S}_0 + \widetilde{S}_1(y) + f(y)\widetilde{C}(y) + \widetilde{E}_2(y)\right) B_{\text{ev}}^{-1/2}(y) \\
&\qquad + f(y)^{-1}(\widetilde{S}_0 + \widetilde{S}_1(y) + f(y)\widetilde{C}(y) + \widetilde{E}_2(y))\Big] B_{\text{ev}}^{-1/2}(y) \\
&= \left[B_{\text{odd}}^{1/2}(y)\left(I + \widetilde{E}_1(y)\right) B_{\text{ev}}^{-1/2}(y)\right]\partial_y \\
&\quad + \frac{1}{f(y)}\Big[\widetilde{S}_0 + B_{\text{odd}}^{1/2}(y)\left(\widetilde{S}_1(y) + f(y)\widetilde{C}(y) + \widetilde{E}_2(y)\right) B_{\text{ev}}^{-1/2}(y) \\
&\qquad + \left(B_{\text{odd}}^{1/2}(y) - I\right)\widetilde{S}_0 B_{\text{ev}}^{-1/2}(y) + \widetilde{S}_0\left(B_{\text{ev}}^{-1/2}(y) - I\right) \\
&\qquad + f(y) B_{\text{odd}}^{1/2}(y)\left(B_{\text{ev}}^{-1/2}(y)\right)'\Big] \\
&=: A(y)\partial_y + f(y)^{-1}\left(\widetilde{S}_0 + S_1(y)\right),
\end{aligned}$$

where $S_1(y)$ satisfies (2.15), and $A(y)$ satisfies (2.16). Now we put $E := \Lambda_{\text{ev}}^* M$, $F := \Lambda_{\text{odd}}^* M$, $D := D_{GB} = d + \delta$, and

$$\Phi_E := \left(\Phi_{\text{ev}} B_{\text{ev}}^{-1/2}\right)^{-1}, \qquad \Phi_F := B_{\text{odd}}^{1/2}\Phi_{\text{odd}}^{-1}.$$

Then it is easily checked that (2.6), (2.9), and (2.10) are satisfied. (2.7) holds since M is complete, and (2.1), (2.14) hold by assumption.

Theorem 5.1. *Let $D = D_{GB}$ or $D = D_S$, and assume $m = \dim M$ is even or divisible by four, respectively. Under the assumptions* (2.1), (5.1), (2.14), *and* (5.2), D *has a finite* L^2*-index, given by*

$$L^2\text{-}\operatorname{ind} D$$

$$(5.13)\qquad = \int_{M_1} \omega_D + \int_{\partial M_1} \alpha_D + \tfrac{1}{2}\left(\eta\left(\widetilde{S}_0\right) - \dim\ker \widetilde{S}_0\right) - \sum_{-1/2<s<0} \dim\ker\left(\widetilde{S}_0 - s\right) + \sum_{k\geq 1} \alpha_k \operatorname{Res} \eta_{\widetilde{S}_0}(2k) + h_0 + h_1,$$

where the various notation of (5.13) *is defined as follows.*

(a) ω_D *is the "index form" of* D *(defined in* [5] *after* (4.32)), *equal to the Chern-Gauss-Bonnet form for* D_{GB} *and to the Hirzebruch* $L_{m/4}$*-polynomial for* D_S. α_D *is the transgression of the characteristic form* ω_D *from* ∇ *to* ∇^a, *where* ∇^a *is the Levi-Civita connection for the metric* $g^a := dy^2 + f_a(y)^2 g_N(\infty)$ *with* $f_a(y) := ay$ *if* $a > 0$, *and* $f_a(y) := 1$ *if* $a = 0$.

(b) $\widetilde{S}_0$ *is given by* (5.5), *with* S_0, S_1 *in* (2.3) *for* D_{GB} *and* S_0', S_1' *in* (2.5) *for* D_S.

(c) h_0 *and* h_1 *are defined by* (2.37) *and* (2.36). *If the decay condition* (4.15) *holds, then they are given by Lemmas* 4.2 *and* 4.4, *respectively. If* $g = g^0$, *then Lemmas* 4.3 *and* 4.5 *apply.*

Proof. Again, we consider only the case $D = D_{GB}$. From the above considerations and Theorem 2.2 it is clear that D has a finite L^2-index given by (2.38). To derive (5.13) we deform the given metric g near ∂M_1 to the metric g^a. By the description of $D_{\overline{g},\min}$ resulting from (3.11), (3.7a), and Corollary 3.2 it is easily seen that this deformation does not change the index of $D_{\overline{g},\min}$. For the new metric we have (4.1) in view of (2.3) and (2.5), hence Theorem 4.2 applies. The proof of (5.13) is completed by recalling the definition of the transgression (cf. [7, Chapter 2.1]). q.e.d.

We can now derive extensions of known L^2-index theorems in some special situations.

Corollary 5.2 (*the asymptotically Euclidean case*). *Assume* M *is asymptotically warped with* $f(y) = y$, $N = S^{m-1}$, *and* $m = \dim M > 2$. *If* (5.2) *holds and*

$$(5.14)\qquad \sup_{p\in N} y^2\,|\Omega|^0_{(y,p)} = O(1) \quad \text{as } y \to \infty,$$

then

$$(5.15)\qquad L^2\text{-}\operatorname{ind} D_{GB} = \lim_{R\to\infty} \int_{y\leq R} \omega_{GB},$$

and

$$L^2\text{-}\operatorname{ind} D_S = \lim_{R\to\infty} \int_{y\le R} \omega_S. \tag{5.16}$$

Proof. We have $a = 1$ in (2.14), and, in consequence of [5, Lemma 5.1; 12],

$$\operatorname{spec} \widetilde{S}_0 \cap [-\tfrac{1}{2}, \tfrac{1}{2}] = \varnothing$$

if $m > 2$. In view of (2.18) we may thus assume that $0 \le a < 1$ in (2.14), and

$$\operatorname{spec} \widetilde{S}_0 \cap [-\tfrac{1}{2}, \tfrac{1}{2}] = \varnothing.$$

Hence $Q = 0$ in (4.15), and $h_0 = h_1 = 0$ by Lemmas 4.2 and 4.4.

Now let $R > 0$ and apply our construction and Theorem 5.1 to

$$M_1^R := M_1 \cup \{p \in U | y(p) \le R\}, \qquad U^R := \{p \in U | y(p) > R\}$$

(cf. Remark (3) after (2.16)). Then we obtain (5.13) with M_1 replaced by M_1^R and $h_0 = h_1 = 0$. We want to prove next that

$$\lim_{R\to\infty} \int_{\partial M_1^R} \alpha_D = 0. \tag{5.17}$$

Observe that in this case the metrics g^0 and g^a coincide on U. Thus, if Ω_s denotes the curvature two-form of the connection $s\nabla^0 + (1-s)\nabla$, then

$$\alpha_D = (m-1) \int_0^1 P(\theta, \Omega_s, \dots, \Omega_s)\, ds,$$

where P is the complete polarization of the invariant polynomial defining ω_D (cf. [7, Lemma 2.1.2]) and θ as in (5.2). Since $\Omega^0 = 0$, it follows from (5.2) and (5.3) that

$$\sup_{p\in S^{m-1}} R^{m-1} |\alpha_D|^0_{(R,p)} \to 0, \qquad R \to \infty,$$

which implies (5.17). The proof of (5.15) is completed if we compare the resulting index formula with the formula for $M = \mathbf{R}^m$.

In the case $D = D_S$, from [12], and [5, Lemma 5.3] with $\widetilde{S}_0 = S_0' + S_1'$ and S_i' as in (2.5), as before we conclude that $\operatorname{spec} \widetilde{S}_0 \cap [-\frac{1}{2}, \frac{1}{2}] = \varnothing$. Hence again $h_0 = h_1 = 0$ and (5.13) holds. Letting $R \to \infty$ as before and comparing again with $M = \mathbf{R}^m$ we obtain the assertion (5.16). q.e.d.

Corollary 5.2 extends Theorem 5.2 in [2], which in turn extends Theorem 1 in [14]. It should be noted, however, that in [2] the case $m = 2$ is also treated but not in [14]. By [5, Lemma 5.1] we have $\operatorname{spec} \widetilde{S}_0 \cap [-\frac{1}{2}, \frac{1}{2}] \ne \varnothing$ and consequently $Q \ne 0$, so we have to deal with h_1. Note that this is not a Fredholm problem since in the asymptotically Euclidean case D_{GB} is not Fredholm in any dimension. We hope to return to this question in a future publication.

Corollary 5.3 (*the cylindrical case*). *Assume* $f(y) \equiv 1$ *in* (5.1) *and* $g = g^0$ *on* U. *Then*

$$L^2\text{-}\operatorname{ind} D_S = \int_{M_1} \omega_s + \tfrac{1}{2}\left(\eta\left(S_0'\right) - \dim\ker S_0'\right) + h_1,$$

where h_1 *is the dimension of the space of limiting values of elements in the extended* L^2*-kernel of* D_S' *(as defined in* [1, p. 58]).

Proof. It is clear that (5.2) and (5.3) hold and that $\alpha_S = 0$. Thus Theorem 5.1 applies. By [1, Theorem (4.14)], $\eta_{S_0'}$ is regular in $\operatorname{Re} z > -\frac{1}{2}$, so all residues in (5.10) vanish. Moreover, since $a = 0$ in (2.14), we have $\widetilde{S}_0 = S_0'$ and may assume by the usual scaling that $Q = \ker S_0' = \bigoplus_{j\geq 0} H^j(N)$. Furthermore, it follows from (2.5) that (4.15) holds, and Lemma 4.2 gives $h_0 = 0$. To prove the assertion concerning h_1 recall the definition of the extended L^2-kernel of D':

$$\overline{L^2\ker D'} := \Big\{v \in C^\infty\left(\Omega^-(M)\right) | D'v = 0,\ \lim_{y\to\infty} v(y) := v_\infty$$
$$\text{exists in } H = L^2\left(\Lambda^* N\right) \text{ and } v - v_\infty \in L^2((0,\infty), H)\Big\},$$

where we identify v with its image under Φ_- in $L^2((0,\infty), H)$. We claim that

$$h_1 = \dim\left\{v_\infty | v \in \overline{L^2\ker D'}\right\}. \tag{5.18}$$

To see this we show that $\overline{L^2\ker D'} = \mathscr{H}_W^*$, where $\mathscr{H}_W^*$ is defined in (4.29), and that the map τ_y in Lemma 4.4 is given by $\overline{L^2\ker D'} \ni v \mapsto g^{-1}(y)v_\infty \in H$ for all y; then (5.18) follows from Lemma 4.4. Consider $v \in \overline{L^2\ker D'}$; we have to show that $g^{-1}v \in L^2((0,\infty), H)$ and $(D_S u, v) = 0$ for all $u \in L^2((0,\infty), H)$ with $gD_S u \in L^2((0,\infty), H)$. Since $D_S = \partial_y + S_0'$ we can decompose v in the eigenspaces of S_0',

$$v(y) = \sum_{\lambda\in\operatorname{spec} S_0'} v_\lambda e^{\lambda y}. \tag{5.19}$$

Since $v \in \overline{L^2\ker D'}$, we have $v_\lambda = 0$ for $\lambda > 0$ and $v_\infty = v_0 \in \ker S_0' = Q$. Moreover, $g(y) = e^{y/2}$ so $g^{-1}v \in L^2((0,\infty), H)$. Now consider u as above and assume without loss of generality that $u(y) = 0$

for $y \leq 1$. By assumption and (2.31), $g^{-1}u \in \mathscr{D}(D_{S,\bar{g},W})$, and by (2.37) and Lemma 4.2, $D_{S,\bar{g},W} = D_{S,\bar{g},\min}$. Thus Corollary 3.2 gives $\|u(y)\|_H = O(e^{-F(y)/2}) = O(e^{-y/2})$, and it follows that $(D_S u, v) = 0$. Hence $\overline{L^2 \ker D'} \subset \mathscr{H}_W^*$.

Conversely, if $v \in \mathscr{H}_W^*$, we have the decomposition (5.19) and only have to show that $v_\lambda = 0$ for $\lambda > 0$. Fix $\lambda > 0$ and put $u(y) := \varphi(y) v_\lambda e^{-\lambda y}$ with $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(y) = 0$ for $y \leq 1$ and $\varphi(y) = 1$ for $y \geq 2$. Clearly, $u \in L^2((0,\infty), H)$ and $gD_S u(y) = g\varphi'(y) v_\lambda e^{-\lambda y} \in L^2((0,\infty), H)$, hence

$$0 = (D_S u, v) = \int_0^\infty \varphi'(y) \|v_\lambda\|_H^2 \, dy = \|v_\lambda\|_H^2 .$$

Thus $\overline{L^2 \ker D'} = \mathscr{H}_W^*$. We now recall the definition of the map τ_y: introduce $\gamma_y : \mathscr{D}(D_{S,\bar{g},\max}) \to QH$ as in Lemma 4.1. Then $\tau_y(v) = \gamma_y(g^{-1}v)$ since $Q = Q_1$. Choose $y_1 > 0$ such that in (4.18a) $\varphi(y_1) = 1$ and write for $v \in \mathscr{H}_W^*$

$$Qv(y) = Qg \cdot g^{-1} v(y) = v_0 = W(y, y_1) v_0 .$$

Comparing this with (4.18a) we conclude from the uniqueness of γ_y that

$$\gamma_{y_1} v = g(y_1) \tau_{y_1}(v) = v_\infty$$

which completes the proof of (5.18). q.e.d.

Corollary 5.3 is Corollary (3.14) in [1]; note the difference in orientation which leads to $A = -S_0'$ in Theorem (3.10). Finally, we treat the cusp case.

Corollary 5.4 (*the cusp case*). *Assume* (2.1), (5.1), (5.2), *and in addition*

$$\operatorname{vol}^0 U < \infty \tag{5.20}$$

and

$$\operatorname{Ricc}^0 \leq 0 \quad \text{on } U, \tag{5.21}$$

where Ricc^0 *denotes the Ricci tensor of the metric* g^0. *Then* (2.14) *holds with* $a = 0$ *and Theorem* 5.1 *applies. If* $g = g^0$ *on* U, (5.3) *holds, and* (5.21) *is strengthened to*

$$\operatorname{Ricc}^0 \leq -\varepsilon^2 \quad \text{on } U, \tag{5.22}$$

then

$$L^2\text{-}\operatorname{ind} D_S = \lim_{R\to\infty} \int_{y \leq R} \omega_S + \tfrac{1}{2}\eta\left(S_0'\right), \tag{5.23}$$

and

$$L^2\text{-}\operatorname{ind} D_{GB} = \lim_{R\to\infty} \int_{y\le R} \omega_{GB} + \sum_{0\le j<n/2} (-1)^j \dim H^j(N). \tag{5.24}$$

Proof. From (5.21) together with [11, p. 211] we find

$$\operatorname{Ricc}^0\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = -n\frac{f''}{f}(y) \le 0, \qquad n = \dim N, \tag{5.25}$$

hence f' is increasing. Since, by (5.20),

$$\int_0^\infty f(y)^n\, dy < \infty,$$

we must have $f'(y) < 0$ for all y, and $a = \lim_{y\to\infty} f'(y) = 0$ since f is positive. Thus we may also assume $\operatorname{spec} S_0 \cap [-\frac{1}{2}, \frac{1}{2}] = \{0\}$.

Assume next $g = g^0$ on U and (5.22) instead of (5.21). Then (5.25) implies $nf''(y) \ge \varepsilon^2 f(y)$, hence

$$\int_0^\infty f(y)\, dy \le -\frac{n}{\varepsilon^2} f'(0).$$

Since $\lim_{y\to\infty} f(y) = 0$, we have $f^\alpha \in L^2(0, \infty)$ for all $\alpha \ge \frac{1}{2}$. As noted before, (4.15) also holds in this case if $D = D_S$ or $D = D_{GB}$, so we obtain (5.13). For $R > 0$ we can also derive (5.13) with M_1 replaced by $M_1^R = M_1 \cup \{p \in U | y(p) \le R\}$. To study the transgression α_D we denote by ω^0 and ω^a the connection one-forms for ∇^0 and ∇^a respectively, and we put $\tilde\theta := \omega^0 - \omega^a$. It follows from [11, p. 206] that

$$\sup_{p\in N} f(y) \left|\tilde\theta\right|^0_{(y,p)} = o(1) \quad \text{as } y \to \infty.$$

Moreover, since f is bounded, using (5.3) and [11, p. 211] we conclude that

$$\sup_{p\in N} f(y)^2 \left(\left|\Omega^0\right|^0_{(y,p)} + \left|\Omega^a\right|^0_{(y,p)}\right) = O(1) \quad \text{as } y \to \infty.$$

Hence as in the proof of Corollary 5.2 we have $\lim_{R\to\infty} \int_{\partial M_1^R} \alpha_D = 0$. Since $a = 0$, $\widetilde{S}_0 = S_0'$ if $D = D_S$, and we may assume that

$$Q = \ker S_0' = \bigoplus_{j\ge 0} H^j(N).$$

Then it is easily seen that (4.15) holds and h_0 and h_1 are given by Lemmas 4.3 and 4.4 respectively. Since $n = m - 1$ is odd and N is orientable,

$$\dim\ker S_0' = \sum_{j=0}^n \dim H^j(N) = 2 \sum_{j=m/2}^n \dim H^j(N). \tag{5.26}$$

From (2.5) we have the spectral resolution

$$QS_1'Q = \bigoplus_{j=0}^{n} \left(\frac{n}{2} - j\right) H^j(N).$$

Now $j - n/2 \geq \frac{1}{2}$ if and only if $j \geq m/2$, hence by Lemma 4.3 we conclude

$$h_0 = \sum_{j=m/2}^{n} \dim H^j(N). \tag{5.27}$$

Note that the contributions from h_0 and $\ker S_0'$ to the index cancel with each other. Moreover, $|n/2 - j| \geq \frac{1}{2}$ for all j, hence $f^{-|t|} \in L^2(0, \infty)$ for all $t \in \operatorname{spec} QS_1'Q$. Thus Lemma 4.5 implies

$$h_1 = 0. \tag{5.28}$$

Now (5.23) follows from (5.26) through (5.28) since $\eta_{S_0'}$ is regular in $\operatorname{Re} z > -\frac{1}{2}$ by Theorem (4.14) in [1].

If $D = D_{GB}$, from (2.4) as before we have that

$$\ker S_0 = \bigoplus_{j=0}^{n} H^j(N) \tag{5.29}$$

and

$$QS_1Q = \bigoplus_{j=0}^{n} (-1)^j \left(j - \frac{n}{2}\right) H^j(N).$$

Since $(-1)^{j+1}(j - n/2) \geq \frac{1}{2}$ if and only if $(-1)^{n-j+1}(n - j - n/2) \geq \frac{1}{2}$, using Lemma 4.3 we obtain

$$h_0 = 2 \sum_{0 \leq 2j < n/2} \dim H^{2j}(N). \tag{5.30}$$

As before we conclude $h_1 = 0$ and $\lim_{R \to \infty} \int_{\partial M_1^R} \alpha_D = 0$. Finally, from [5, Lemma 5.1] it follows that in this case

$$\eta_{S_0} \equiv 0. \tag{5.31}$$

The proof is completed by combining (5.30) and (5.31). q.e.d.

Corollary 5.4 substantially generalizes Theorem 2 in [14].

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