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UNIQUENESS OF MINIMAL SURFACES EMBEDDED IN $\mathbb{R}^3$, WITH TOTAL CURVATURE 12Π

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1. Introduction

A well-known general uniqueness question for minimal surfaces in a Euclidean 3-space $\mathbb{R}^3$ is the determination of all the embedded complete minimal surfaces with finite total curvature. Until recently the only known surfaces were the plane and the catenoid. In [1], the author has constructed an example of a complete minimal immersion of genus one and three embedded ends. Later in [3], D. Hoffman and W. Meeks have proved that this surface is embedded. Inspired by this surface they were able to construct, for each genus $\gamma > 1$, a complete minimal embedded surface with three ends.

The plane and the catenoid have total curvature zero and 4Π, respectively, and they are the only embedded complete minimal surfaces with such properties. In [10], R. Schoen shows that there does not exist a complete minimal surface embedded in $\mathbb{R}^3$ with total curvature 8Π.

In this work we make a contribution to the classification of minimal embedded surfaces in $\mathbb{R}^3$ with total curvature 12Π.

We say that two minimal surfaces M and M' in $\mathbb{R}^3$ are the same if there exists a rigid motion and a homothety in $\mathbb{R}^3$ that carries M onto M'.

The main result to be proved in this paper is the following theorem.

Theorem 1. *There exists a unique complete minimal immersion in $\mathbb{R}^3$ of genus one and three ends with finite total curvature such that:*

(a) *the ends are embedded and parallel,*

(b) *two ends are catenoid type and one end is flat.*

We prove, also, the following corollary.

Corollary 1. *Let M be a complete minimal surface embedded in $\mathbb{R}^3$ with total curvature 12Π. Then, we have two possibilities:*

(a) *M is the surface that appears in* [1] *and* [3].

(b) *M is of genus one with three ends of catenoid type.*

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Remark 1. We know that D. Hoffman and W. Meeks in [4] have constructed a 1-parameter family M_t, $t \in (1, \infty)$, of embedded minimal surfaces in $\mathbb{R}^3$ of genus one and three ends of catenoid type. M_t is conformally equivalent to C/L_t, where $L_t = \{m + nti \in \mathbf{C}; m, n \in Z\}$, punctured in the three half-lattice points. Recently, in [2], we have proved the uniqueness of this family; that is, for each $r \in (1, \infty)$, there exists, at most, one complete minimal surface M_t, embedded in $\mathbb{R}^3$ with three ends of catenoid type, where M_t is conformally equivalent to $\mathbf{C}/L_t$ punctured at three points.

To prove Theorem 1, we consider the compact Riemann surfaces of genus one, $\overline{M}_1$, with complex structures induced by lattices L of $\mathbf{C}$. Through $\pi\colon \mathbf{C} \to \mathbf{C}/L = \overline{M}_1$, we identify elliptic functions and elliptic differentials of L with meromorphic functions and meromorphic differentials of $\overline{M}_1$, respectively. Then, in this way, we consider the Weierstrass representation (g, w) of the immersions $x\colon M \to \mathbb{R}^3$ with the properties of Theorem 1, where $M \subset \overline{M}_1$ is not compact.

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2. Complete minimal immersions in $\mathbb{R}^3$

In [7] or [9], we have the following theorem of representation of complete minimal surfaces, called Weierstrass-Osserman's representation:

Theorem 2. *Let $x\colon M \to \mathbb{R}^3$ be a complete minimal immersion of finite total curvature. Then the following hold.*

(a) *M is conformally equivalent to a compact Riemann surface of genus γ, $\overline{M}_\gamma$, punctured at the points $q_1, \cdots, q_N$.*

(b) *There exist a meromorphic function g and a meromorphic differential w in $\overline{M}_\gamma$ such that w is holomorphic in M and $q \in M$ is a pole of order m of g if and only if q is a zero of order $2m$ of w. g is the Gauss normal of the immersion.*

(c) *If δ is a closed path in M, then*

$$\operatorname{Re} \int_\delta gw = 0 \quad \textit{and} \quad \overline{\int_\delta w} = \int_\delta g^2 w.$$

(d) *Every divergent path in M has infinite length.*

Conversely, let $\overline{M}\gamma$ be a compact Riemann surface, let $M = \overline{M}_\gamma - \{q_1, \cdots, q_N\}$, where $q_1, \cdots, q_N \in \overline{M}_\gamma$, and let g, w be a meromorphic function and a meromorphic differential in $\overline{M}_\gamma$, respectively. If (g, w) satisfies

(b) *and* (c) *then* $x\colon M \to \mathbb{R}^3$,

$$x(q) = \frac{1}{2}\operatorname{Re}\int_{q_0}^{q}((1-g^2)w, i(1+g^2)w, 2gw)$$

is a minimal immersion with finite total curvature. Furthermore if x satisfies (d), *x is complete.*

If $D_j \subset \overline{M}_\gamma$, $j = 1, \cdots, N$, is a small topological disk with $q_j \in D_j$, then $F_j = x(M \cap D_j)$ is an end of the immersion. Let $1 \le j \le N$ be fixed. Since g is the Gauss map of x, we can suppose, after a rotation of x in $\mathbb{R}^3$, that $g(q_j) = 0$. In this situation, [5], F_j is an embedded end if and only if q_j is a pole of order two of w. Then, around q_j, we have the local expressions

$$g(z) = a_n z^n + o(z)^{n+1}, \qquad a_n \neq 0, \quad n \ge 1,$$

$$w(z) = \frac{b}{z^2} + o(z)^{-1}, \qquad b \neq 0.$$

We say that the embedded end F_j is of *catenoid type if* $n = 1$, and is a *flat end of order* $n-1$ *if* $n > 1$. In the latter case, the coordinate $x_3(q) = \operatorname{Re}\int^q gw$, $q \in D_j \cap M$, is bounded and the immersion approaches a plane parallel to the plane $x_3 = 0$.

3. Elliptic functions and minimal immersions of genus one

If $F = \{\tau = x + iy \in \mathbb{C}; x^2 + y^2 \ge 1,\ y > 0,\ -\frac{1}{2} \le x \le \frac{1}{2}\}$ and $L(1,\tau) = \{m + n\tau; m, n \in Z\}$ are the lattices of $\mathbb{C}$, where $\tau \in F$, then $\mathbb{C}/L(1,\tau)$ with complex structures induced by $\pi\colon \mathbb{C} \to C/L(1,\tau)$ are all Riemann surfaces of genus one. That is, in [6], we have the following theorem.

Theorem 3. *Let L be a lattice in $\mathbb{C}$ and let $\overline{M}_1 = \mathbb{C}/L$ equipped with the complex structure induced by L. Then there exists $\tau \in F$ such that $\mathbb{C}/L(1,\tau)$ is conformally equivalent to $\overline{M}_1$. Furthermore, if $\mathbb{C}/L(1,\tau)$ and $\mathbb{C}/L(1,\tilde{\tau})$, $\tau, \tilde{\tau} \in F$, are conformally equivalent, then τ, $\tilde{\tau} \in \partial F$.*

Remark 2. Henceforth, we shall consider only the compact Riemann surfaces $\mathbb{C}/L(1,\tau)$, where $L(1,\tau)$ is a lattice as in Theorem 3. To simplify, we set $T_\tau = \mathbb{C}/L(1,\tau)$.

Two points $z_1, z_2 \in \mathbb{C}$ are congruent, $z_1 \equiv z_2$, if $z_1 - z_2 \in L$. Otherwise, they are incongruent and we shall write $z_1 \not\equiv z_2$. A set $\{z_1, \cdots, z_n\}$ is incongruent if $z_i \not\equiv z_j$, $i, j = 1, \cdots, n$, $i \neq j$.

We define the middle-points of $L(1,\tau)$,

$$w_1 = \frac{1}{2}, \qquad w_2 = -\frac{1+\tau}{2}, \qquad w_3 = \frac{\tau}{2},$$

and the numbers

$$e_j = P(w_j), \qquad j = 1, 2, 3,$$

where P is the Weierstrass function of $L(1, \tau)$.

We also have the quasi-elliptic function $\zeta: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ of $L(1, \tau)$ and the numbers $\eta_j \in \mathbb{C}$, $j = 1, 2, 3$, such that

$$\zeta'(z) = -P(z), \quad \zeta(w_j) = \eta_j, \quad \zeta(z + 2w_j) = \zeta(z) + 2\eta_j.$$

We have the following additive property for ζ:

$$\zeta(z_1 \pm z_2) = \zeta(z_1) \pm \zeta(z_2) + \frac{P'(z_1) \mp P'(z_2)}{P(z_1) - P(z_2)},$$

and at $0 \in C$,

$$\zeta(z) = \frac{1}{z} + o(z)^3 \quad \text{and} \quad P(z) = \frac{1}{z^2} + o(z)^2.$$

We shall need the following proposition which was proved in [1].

Proposition 1. *In the lattice* $L(1, i)$ *we have* (a) $2\eta_1 = \pi$ *and* (b) $e_2 = 0$, $e_1 = -e_3 > 0$.

4. Proof of Theorem 1.

To prove our main result we shall need several propositions and lemmas. The first one is:

Proposition 2. *Let g, w be an elliptic function and an elliptical differential of a lattice $L(1, \tau)$, respectively, and let $z_1, z_2 \in \mathbb{C}$ such that*

(a) $z_1 + z_2 \not\equiv 0$,

(b) *g has order three, $\{z_1, z_2, w_s\}$ as an incongruent set of poles, where $s \in \{1, 2, 3\}$, and a zero in $0 \in \mathbb{C}$, and*

(c) $w = (P - e_s)\, dz$ *and* $\operatorname{Res}_{z_j} g^2 w = 0$, $j = 1, 2$.

Then

$$\frac{P'(z_1) + P'(z_2)}{P(z_1) - P(z_2)} = \frac{P'(z_1)}{P(z_1) - e_s} = -\frac{P'(z_2)}{P(z_2) - e_s},$$

and $z_1 \equiv z_2 + w_s$.

Proof. Figure 1 shows the poles and zeros of g and w in the case $s = 2$. Observe that we do not know the order of the zero of g at $0 \in \mathbb{C}$. We set

$$(1) \qquad P_j = P(z_j), \quad P_j' = P'(z_j), \qquad j = 1, 2.$$

From (a) and (b) we observe that

$$P_1 \neq P_2 \quad \text{and} \quad P_j - e_s \neq 0, \qquad j = 1, 2.$$

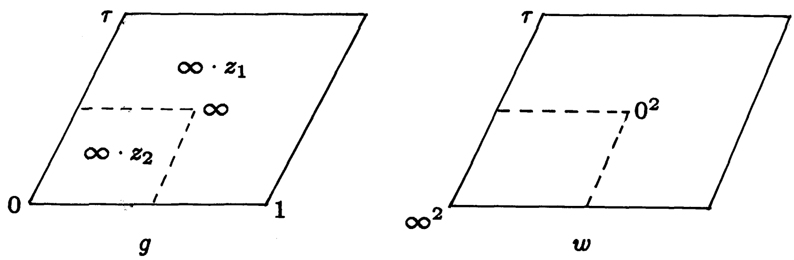

FIGURE 1

There exist $a, b, c \in \mathbb{C} - \{0\}$ such that

$$g(z) = a\zeta(z - w_s) + b\zeta(z - z_1) + c\zeta(z - z_2) + a\zeta(w_s) + b\zeta(z_1) + c\zeta(z_2),$$

$$a + b + c = 0. \tag{2}$$

By using the local expression for $\zeta(z)$ in a neighborhood of $0 \in \mathbb{C}$, we find the following expression for g in a neighborhood of z_1 (see §3):

$$\begin{aligned} g(z) = {} & \frac{b}{z - z_1} + a\zeta(z_1 - w_s) + c\zeta(z_1 - z_2) \\ & + a\zeta(w_s) + b\zeta(z_1) + c\zeta(z_2) + o(z - z_1)^1. \end{aligned}$$

Thus, by using the additive properties of ζ, (1) and (2) we have

$$g(z) = \frac{b}{z - z_1} + \frac{1}{2}\left[\frac{aP_1'}{P_1 - e_s} + \frac{c(P_1' + P_2')}{P_1 - P_2}\right] + o(z - z_1)^1.$$

Then, at z_1,

$$g^2(z) = \frac{b^2}{(z - z_1)^2} + \left[\frac{aP_1'}{P_1 - e_s} + \frac{c(P_1' + P_2')}{P_1 - P_2}\right]\frac{b}{z - z_1} + o(z - z_1)^0,$$

and from (c)

$$w(z) = [P_1 - e_s + P_1'(z - z_1) + o(z - z_1)^2]\,dz.$$

Hence,

$$\operatorname*{Res}_{z_1} g^2 w = b^2 P_1' + b(P_1 - e_s)\left[\frac{aP_1'}{P_1 - e_s} + \frac{c(P_1' + P_2')}{P_1 - P_2}\right].$$

From (c) and (2) it follows that

$$\frac{P_1'}{P_1 - e_s} = \frac{P_1' + P_2'}{P_1 - P_2}.$$

In the same way, by using the local expressions for g^2 and w at z_2, we obtain

$$\frac{P_2'}{P_2 - e_s} = \frac{P_2' + P_1'}{P_2 - P_1},$$

and therefore

$$\frac{P_1'}{P_1 - e_s} = -\frac{P_2'}{P_2 - e_s} = \frac{P_1' + P_2'}{P_1 - P_2}. \tag{3}$$

On the other hand, let h be the elliptic function

$$h(z) = \frac{P'(z)}{P(z) - e_s} - \frac{P_1'}{P_1 - e_s}.$$

Since h is of order two and has simple poles at 0 and w_s, P is an even function and P' is an odd function. Thus by (3) we have

$$h(z_1) = h(-z_2) = 0,$$

and therefore $z_1 \equiv -z_2$ or $z_1 - z_2 \equiv w_s$. But from (a) it follows that $z_1 + z_2 \not\equiv 0$. Hence $z_1 \equiv z_2 + w_s$, which completes the proof.

By using Proposition 2, we shall prove the following.

Lemma 1. *There does not exist a complete minimal immersion in* $\mathbb{R}^3$ *of genus one and three ends with the following properties*:

(a) *The ends are embedded and parallel.*

(b) *Two ends are of catenoid type and one end is flat of order* 1.

Proof. By contradiction, we will suppose that there exists an immersion $x: M = T_\tau - \{q_1, q_2, q_3\} \to \mathbb{R}^3$ with the properties of the lemma. Let (g, w) be the Weierstrass representation of x. Since the ends are embedded, the total curvature $c(M) = 12\Pi$ and g is a meromorphic function in T_τ (that is, an elliptic function in $L(1, \tau)$) of order three. Let F_j be the ends of x associated to the points q_j, $j = 1, 2, 3$, where F_3 is the flat end of order 1. We must consider two cases:

Case 1°: $g(q_1) = g(q_2) \neq g(q_3)$ and *Case* 2°: $g(q_1) \neq g(q_2) = g(q_3)$.

In the first case, after a rotation of x in $\mathbb{R}^3$ we may suppose that $g(q_1) = g(q_2) = \infty$ and $g(q_3) = 0$, where q_3 is a double zero of g. Since F_1 and F_2 are ends of catenoid type, q_1 and q_2 are simple poles of g. Therefore there exist $\overline{q} \in M$, a simple pole of g, and $q' \in M$, a simple zero of g. Since F_3 is embedded, w has a double pole at q_3. From (b) of Theorem 2, w has a double zero at $\overline{q} \in M$. Thus, after a homothety, a rotation of the immersion around the x_3-axis in $\mathbb{R}^3$ and a translation of coordinates in $\mathbb{C}$, we have

$$w = (P - e_s)\,dz, \quad s \in \{1, 2, 3\}, \qquad \pi(0) = q_3, \qquad \pi(w_s) = \overline{q},$$

where $\pi: C \to T_\tau = \mathbb{C}/L(1,\tau)$ is the canonical projection. We define $z_1, z_2, z' \in \mathbb{C}$ such that

$$\pi(z') = q', \qquad \pi(z_j) = q_j, \quad j = 1, 2,$$

and remark that

(4) $$\{0, w_s, z', z_1, z_2\} \text{ is } L(1,\tau)\text{-incongruent.}$$

Figure 2 shows g and w for $s = 2$.

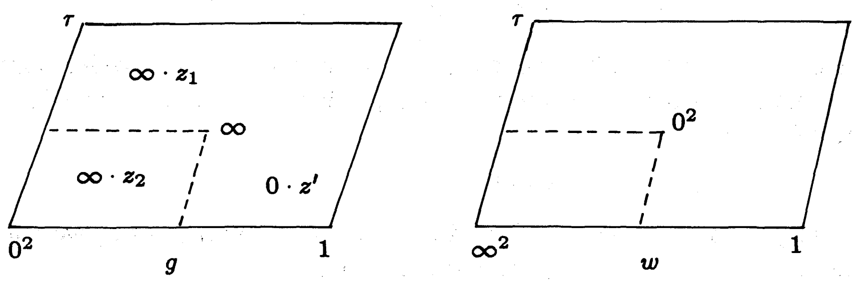

FIGURE 2

We have

(5) $$z_1 + z_2 + w_s \equiv z',$$

which together with (4) implies that $z_1 + z_2 \not\equiv 0$. We also have $\operatorname{Res}_{z_j} w = 0$, $j = 1, 2$. Thus from (c) of Theorem 2 it follows that

$$\operatorname*{Res}_{z_j} g^2 w = 0, \qquad j = 1, 2,$$

so that g and w satisfy the hypothesis of Proposition 2. Therefore by using the notation (1) we obtain

(6) $$z_1 \equiv z_2 + w_s, \qquad \frac{P_1' + P_2'}{P_1 - P_2} = \frac{P_1'}{P_1 - e_s} = \frac{-P_2'}{P_2 - e_s}.$$

On the other hand, the elliptic differential gw has simple poles at z_1, z_2 and simple zeros at w_s, z'. Then there exists $\tilde{a} \in \mathbb{C}$, $\tilde{a} \neq 0$, such that

$$gw = \tilde{a}[\zeta(z - z_1) - \zeta(z - z_2) - \zeta(w_s - z_1) + \zeta(w_s - z_2)]\, dz.$$

From (c) of Theorem 2,

(7) $$\operatorname*{Res}_{z_1} gw = \tilde{a} \in \mathbb{R}.$$

By using the additive properties for ζ and the notation (1), we have

$$gw = \frac{\tilde{a}}{2}\left[\frac{P' + P_1'}{P - P_1} - \frac{P' + P_2'}{P - P_2} + \frac{P_1'}{P_1 - e_s} - \frac{P_2'}{P_2 - e_s}\right] dz.$$

We define the paths $\alpha, \beta\colon [0,1] \to \mathbb{C}$ as

$$\alpha(t) = t + v_0\tau, \qquad \beta(t) = u_0 + t\tau,$$

where $0 < u_0 < 1$, $0 < v_0 < 1$ and such that $\pi \circ \alpha \subset M$, $\pi \circ \beta \subset M$. Then, from [11, vol. 3, p. 61 and vol. 4, p. 109] we have

$$\int_\delta \frac{P_j'}{P - P_j}\,dz = \left[\log\frac{\sigma(z - z_j)}{\sigma(z + z_j)} + 2z\zeta(z_j)\right]_\delta,$$

$$\operatorname{Re}\int_\delta \frac{P'}{P - P_j}\,dz = \left[\operatorname{Re}\log|P - P_j|\right]_\delta = 0,$$

where $j = 1, 2$, $\delta \in \{\alpha, \beta\}$, and σ is the classical σ-function of Weierstrass. We remark that σ satisfies the following properties of addition in $L(1, \tau)$ [11, vol. 1, p. 61]:

$$\sigma(z + 2w_j) = -e^{2\eta_j(z + w_j)}\sigma(z), \qquad j = 1, 2, 3.$$

By using (c) of Theorem 2, (7) and the additive properties of σ, we conclude that $\operatorname{Re}\int_\alpha gw = \operatorname{Re}\int_\beta gw = 0$ if and only if

$$\operatorname{Re}[2\eta_1(z_2 - z_1) + \zeta(z_1) - \zeta(z_2)] + \frac{1}{2}\operatorname{Re}\left(\frac{P_1'}{P_1 - e_s} - \frac{P_2'}{P_2 - e_s}\right) = 0,$$

$$\operatorname{Re}[2\eta_3(z_2 - z_1) + \tau\zeta(z_1) - \tau\zeta(z_2)] + \frac{1}{2}\operatorname{Re}\left(\frac{\tau P_1'}{P_1 - e_s} - \frac{\tau P_2'}{P_2 - e_s}\right) = 0.$$

Also, from (6) and the additive properties of ζ it follows that

$$\zeta(z_1) - \zeta(z_2) + \frac{1}{2}\frac{P_1'}{P_1 - e_s} = \zeta(z_1 - z_2).$$

The equations above imply that

$$\operatorname{Re}\left[2\eta_1(z_2 - z_1) + \zeta(z_1 - z_2) - \frac{1}{2}\frac{P_2'}{P_2 - e_s}\right] = 0, \tag{8}$$

$$\operatorname{Re}\left[2\eta_3(z_2 - z_1) + \tau\zeta(z_1 - z_2) - \frac{\tau}{2}\frac{P_2'}{P_2 - e_s}\right] = 0. \tag{9}$$

On the other hand, from (6) we have

$$z_2 - z_1 = m + n\tau + w_s, \qquad m, n \in Z.$$

Hence,

$$\zeta(z_1 - z_2) = -2m\eta_1 - 2n\eta_3 - \eta_s.$$

This last expression, the Legendre relation, $\eta_3 = \tau\eta_1 - \pi i$, and the additive relation $\sum_{j=1}^3 \eta_j = 0$ imply that

$$\operatorname{Re}[2\eta_j(z_2 - z_1) + 2w_j\zeta(z_1 - z_2)] = 0, \qquad j = 1, 3,\ s = 1, 2, 3.$$

Finally, using the expression above, (8) and (9) we obtain

$$\frac{P_2'}{P_2 - e_s} = 0.$$

But from (4) it follows that $P_2 - e_s \neq 0$. Thus $P_2' = P'(z_2) = 0$ and $z_2 \equiv w_t$, $t \in \{1,2,3\}$, $t \neq s$. This fact combined with (5) and (6) implies that $z_1 \equiv w_k$, where (s,t,k) is a permutation of $(1,2,3)$, and $z' \equiv 0$. Since this is a contradiction to (4), the first case cannot occur.

In the second case, after a rotation of x in $\mathbb{R}^3$ we can suppose that $g(q_2) = g(q_3) = \infty$ and $g(q_1) = 0$. Since F_3 is a flat end of order 1 and F_1,F_2 are ends of catenoid type, q_3 is a double pole of g, q_2 is a simple pole of g and q_1 is a simple zero of g. Thus g is holomorphic in M, and from (b) of Theorem 2, w is holomorphic in M and $w(q) \neq 0$, for all $q \in M$. On the other hand, since the ends are embedded and F_3 is flat, w has a double pole at q_1 and a double zero at q_3. So, after a homothety of the immersion in $\mathbb{R}^3$, a rotation around the x_3-axis and a translation of coordinates in $\mathbb{C}$, we have

$$w = (P - e_s)\,dz, \quad s \in \{1,2,3\}, \qquad \pi(0) = q_1, \qquad \pi(q_3) = w_s.$$

We also define $z_2 \in \mathbb{C}$, such that $\pi(z_2) = q_2$. Figure 3 shows g and w for $s = 2$.

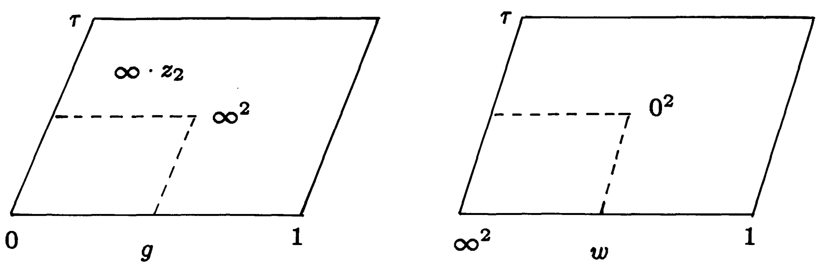

FIGURE 3

So, there exist $a, b \in \mathbb{C} - \{0\}$ and $d \in \mathbb{C}$ such that

$$g = a\zeta(z - z_2) + bP(z - w_s) - a\zeta(z - w_s) + d.$$

At w_s, we have,

$$g(z) = \frac{b}{(z - w_s)^2} - \frac{a}{(z - w_s)} + o(z - w_s)^0,$$

$$w(z) = \left[\frac{P''(w_s)}{2}(z - w_s)^2 + o(z - w_s)^4\right] dz.$$

Also, since w has a zero of order exactly two at w_s, $P''(w_s) \neq 0$. Furthermore [11, vol. 1 p. 175], we have

$$P''(w_s) = 2(e_k - e_s)\cdot(e_t - e_s) \neq 0. \tag{10}$$

Thus,

$$\operatorname*{Res}_{w_s} g^2 w = -abP''(w_s) \neq 0.$$

But, from (c) of Theorem 2,

$$2\pi i \operatorname*{Res}_{w_s} g^2 w = -2\pi i \overline{\operatorname*{Res}_{w_s} w} = 0.$$

This is a contradiction, so we conclude the proof of Lemma 1.

Lemma 2. *Let $x\colon M = T_\tau - \{q_1, q_2, q_3\} \to \mathbb{R}^3$ be a complete minimal immersion of genus one and three ends with the following properties*:

(a) *The total curvature $c(M) = 12\Pi$.*

(b) *The ends are parallel.*

(c) *Two ends of the immersion are of catenoid type, and one end is flat of order two.*

Then, after a rigid motion and a homothety of the immersion in $\mathbb{R}^3$, the Weierstrass representation (g, w) of x is

$$g = \frac{2B(e_k - e_t)}{P'}, \qquad w = (P - e_s)\,dz,$$

where (s, k, t) is a permutation of $(1, 2, 3)$, $B \in \mathbb{R} - \{0\}$, $\pi(0) = q_3$, $\pi(w_k) = q_1$ and $\pi(w_t) = q_2$. Furthermore, in the lattice $L(1, \tau)$, $\tau = x + iy \in F$, we have

$$\overline{(2\eta_1 + e_s)} = B^2\left(\frac{2\eta_1 + e_k}{e_k - e_s} + \frac{2\eta_1 + e_t}{e_t - e_s}\right), \tag{I}$$

$$y\cdot\overline{(2\eta_1 + e_s)} - \pi = \pi B^2\left(\frac{1}{e_k - e_s} + \frac{1}{e_t - e_s}\right). \tag{II}$$

Proof. From (a) we have that g is a meromorphic function of order 3 in T_τ. Then, after a rotation of the immersion in $\mathbb{R}^3$, g has a zero of order 3 at q_3 and simple poles at q_1 and q_2. So, there exists $q' \in M$ such that q' is a simple pole of g. Since the ends are embedded, w has double pole at q_3. From (b) of Theorem 2 w has double zero at q'. Then, after a homothety and a rotation of the immersion around the x_3-axis in $\mathbb{R}^3$ and a translation of coordinates of $\mathbb{C}$, we arrive at

$$w = (P - e_s)\,dz, \quad s \in \{1, 2, 3\}, \qquad \pi(0) = q_3, \qquad \pi(w_s) = q'. \tag{11}$$

We define $z_1, z_2 \in \mathbb{C}$ such that

$$\pi(z_1) = q_1, \qquad \pi(z_2) = q_2. \tag{12}$$

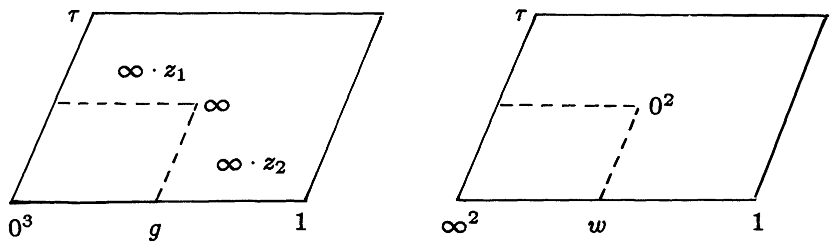

FIGURE 4

Then $\{0, z_1, z_2, w_s\}$ is $L(1,\tau)$-incongruent.

Figure 4 shows g and w for $s = 2$.

Abel's theorem implies that

$$z_1 + z_2 + w_s \equiv 0. \tag{13}$$

Also, from (c) of Theorem 2 and (11), we have

$$\operatorname*{Res}_{z_j} g^2 w = -\overline{\operatorname*{Res}_{z_j} w} = 0, \qquad j = 1, 2. \tag{14}$$

Using (11), (13), (14) and considering the positions of the poles and zeros of g, we conclude that g and w satisfy the hypothesis of Proposition 2. Thus $z_1 \equiv z_2 + w_s$, which together with (13) implies that $2z_j \equiv 0$, $j = 1, 2$. Hence,

$$z_1 \equiv w_k, \qquad z_2 \equiv w_t, \tag{15}$$

where (k, s, t) is a permutation of (1,2,3).

From (11), (12) and (15) we see that there exists $B \in \mathbb{C}$, $B \neq 0$, such that the Weierstrass representation of x is

$$g = 2B(e_k - e_t)\frac{1}{P'}, \qquad w = (P - e_s)\, dz. \tag{16}$$

Since

$$(P')^2 = 4\prod_{j=1}^{3}(P - e_j) \quad \text{and} \quad \frac{(e_r - e_k)(e_r - e_l)}{P - e_r} = P(z - w_r) - e_r, \tag{17}$$

where (r, h, l) is a arbitrary permutation of (1,2,3), we arrive at

$$gw = \frac{B}{2}\left(\frac{P'}{P - e_k} - \frac{P'}{P - e_t}\right) dz,$$

$$g^2 w = B^2 \left[\frac{P(z - w_k)e - k}{e_k - e_s} + \frac{P(z - w_t) - e_t}{e_t - e_s}\right] dz. \tag{18}$$

Since at w_s,

$$P(z-w_s)=\frac{1}{(z-w_s)^2}+o(z-w_s)^2,$$

from (10), (17) and (c) of Theorem 2 we obtain

$$\operatorname*{Res}_{w_k} gw=\frac{B}{2}\operatorname{Res} P'\left[\frac{P(z-w_k)-e_k}{(e_t-e_k)(e_s-e_k)}\right]=B\in\mathbb{R}-\{0\},$$

which together with (16) proves the first part of Lemma 2.

Now, we define the paths $\alpha, \beta\colon [0,1]\to\mathbb{C}$,

$$\alpha(t)=\frac{\tau}{3}+t,\qquad \beta(t)=\frac{1}{3}+t\tau.$$

Then $\pi\circ\alpha$ and $\pi\circ\beta$ generate the homology of T_τ. From (18), (16) and (c) of Theorem 2, we have

$$\overline{-\int_\alpha w}=\overline{(2\eta_1+e_s)}=-\int_\alpha g^2w=B^2\left[\frac{2\eta_1+e_k}{e_k-e_s}+\frac{2\eta_1+e_t}{e_t-e_s}\right],$$

$$\overline{-\int_\beta w}=\overline{(2\eta_3+\tau e_s)}=-\int_\beta g^2w=B^2\left[\frac{2\eta_3+e_k\tau}{e_k-e_s}+\frac{2\eta_3+e_t\tau}{e_t-e_s}\right].$$

Hence, Legendre's relation, $\eta_3=\tau\eta_1-\pi i$, implies that the equations above are equivalent to equations (I) and (II). This concludes the proof.

Our next goal is to show that equations (I) and (II) of Lemma 2 are satisfied if and only if $L(1,\tau)=L(1,i)$, $B=\sqrt{\pi/2}$ and $s=2$. This result will be an immediate consequence of Lemmas 3 and 4 below. In order to prove this fact we will need several propositions. First a little remark: If $\tau=x+iy\in F$, then

$$y\geq\sqrt{3}/2,\quad e^{\pi y}>15\quad\text{and}\quad e^{\pi}>23. \tag{19}$$

Proposition 3. *For every lattice $L(1,\tau)$, $\tau=x+iy\in F$, we have*

(a) $y\operatorname{Re}(2\eta_1+e_1)>2\pi$ *and* $\operatorname{Re}(e_j-e_1)<0$, $j=2,3$,

(b) $(-1)^j\operatorname{Im}(e_j-e_1)>0$ *if* $x>0$, *and* $(-1)^j\operatorname{Im}(e_j-e_1)<0$ *if* $x<0$, $j=2,3$.

Proof. To prove (a) we shall use [11, vol. 3, p. 138]

$$2\eta_1+e_1=\pi^2-8\pi^2\sum_{n=1}^{\infty}(-1)^n\frac{nq^{2n}}{1-q^{2n}},\qquad q=e^{i\pi\tau}. \tag{20}$$

Thus,

$$\operatorname{Re}(2\eta_1+e_1)=\pi^2-8\pi^2\sum_{n=1}^{\infty}(-1)^n\frac{ne^{-2\pi ny}(\cos 2\pi nx-e^{-2\pi ny})}{1-2e^{-2\pi ny}\cos 2\pi nx+e^{-4\pi ny}},$$

which implies that

$$\operatorname{Re}(2\eta_1 + e_1) \geq \pi^2 - \frac{8\pi^2(1+e^{-2\pi y})}{(1-e^{-2\pi y})^2} \sum_{n=1}^{\infty} n e^{-2\pi n y},$$

and therefore that, in consequence of (19),

$$y \operatorname{Re}(2\eta_1 + e_1) \geq y \left[\pi^2 - \frac{8\pi^2(1+e^{-2\pi y})e^{-2\pi y}}{(1-e^{-2\pi y})^4}\right] > 2\pi.$$

Also, we have [11, vol. 2, p. 27]

(21) $$e_2 - e_1 = -\pi^2 \gamma_{1,2}^4, \qquad e_3 - e_1 = -\pi^2 \gamma_{1,3}^4,$$

where

$$\gamma_{1,2} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}, \qquad \gamma_{1,3} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

Thus, by (19) we otain, for $j = 2, 3$,

$$|\operatorname{Re} \gamma_{1,j} - 1| \leq 2\sum_{n=1}^{\infty} e^{-\pi n y} < \frac{1}{7}, \qquad |\operatorname{Im} \gamma_{1,j}| < \frac{1}{7},$$

and

(22) $$|\operatorname{Arg} \gamma_{1,j}| < \pi/8, \qquad j = 2, 3,$$

which together with (21) implies that

$$\operatorname{Re}(e_j - e_1) < 0, \qquad j = 2, 3.$$

Hence (a) is proved. To prove (b) for $j = 2$, we notice that

$$\operatorname{Im} \gamma_{1,2}(x) = 2\sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi y} \sin n^2 \pi x.$$

If $\frac{1}{6} \leq x \leq \frac{1}{2}$, then

$$\operatorname{Im} \gamma_{1,2}(x) \leq -2e^{-\pi y} \sin\frac{\pi}{6} + 2\sum_{n=4}^{\infty} e^{-n\pi y} < 0.$$

If $0 < x \leq \frac{1}{6}$, we obtain

$$\begin{aligned}(\operatorname{Im} \gamma_{1,2})'(x) &= 2\pi \sum_{n=1}^{\infty} (-1)^n n^2 e^{-\pi n^2 y} \cos \pi n^2 x \\ &\leq -\pi\sqrt{3} e^{-\pi y} + 2\pi \sum_{n=4}^{\infty} n e^{-\pi n y} < 0.\end{aligned}$$

Since $\operatorname{Im}\gamma_{1,2}(0) = 0$ and $\operatorname{Im}\gamma_{1,2}(x)$ is an odd function of x, the results above imply that

$$\operatorname{Im}\gamma_{1,2}(x) < 0, \quad x > 0 \qquad \text{and} \qquad \operatorname{Im}\gamma_{1,2}(x) > 0, \quad x < 0.$$

This fact together with (21) and (22) prove (b) for the case $j = 2$. In the same way, we can prove (b) for the case $j = 3$. Hence the proof of Proposition 3 is complete.

Proposition 4. *Let $L(1, \tau)$ be a lattice where $\tau = x + iy \in F$. Then the following hold*:

(a) $\operatorname{Re}(2\eta_1 + e_3) = 2\eta_1 + e_3 < 0$ *if* $x = 0$.

(b) $(-1)^j \operatorname{Im}(2\eta_1 + e_j) > 0$ *if* $x > 0$, *and* $(-1)^j \operatorname{Im}(2\eta_1 + e_j) < 0$, *if* $x < 0$, $j = 2, 3$.

(c) $\operatorname{Im}(e_2 - e_3) > 0$ *if* $x > 0$, *and* $\operatorname{Im}(e_2 - e_3) < 0$ *if* $x < 0$.

(d) $\operatorname{Re}(e_2 - e_3) = (e_2 - e_3) > 0$ *if* $x = 0$.

Proof. We have [11, vol. 3, p. 138]

$$(23) \quad 2\eta_1 + e_3 = -8\pi^2 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}}, \qquad 2\eta_1 + e_2 = 8\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} nq^n}{1 - q^{2n}},$$

where $q = e^{i\pi\tau}$. If $x = 0$, then

$$\operatorname{Re}(2\eta_1 + e_3) = -8\pi^2 \sum_{n=1}^{\infty} \frac{ne^{-\pi ny}}{1 - e^{-2\pi ny}} < 0,$$

which proves (a).

In order to prove (b), let $y \geq \sqrt{3}/2$ be fixed. Then

$$\operatorname{Im}(2\eta_1 + e_3) = -8\pi^2 \sum_{n=1}^{\infty} L_n(x),$$

$$\operatorname{Im}(2\eta_1 + e_2) = -8\pi^2 \sum_{n=1}^{\infty} (-1)^n L_n(x),$$

where

$$L_n(x) = \frac{1}{D_n(x)} [n(1 + e^{-2\pi ny}) e^{-\pi ny} \sin \pi nx],$$

$$D_n(x) = 1 - 2e^{-2\pi ny} \cos 2\pi nx + e^{-4\pi ny}.$$

Also

$$L_n'(x) = \frac{1}{D_n^2(x)} [\pi n^2 e^{-\pi ny} (1 + e^{-2\pi ny})(D_n(x) - 8e^{-2\pi ny} \sin^2 nx) \cos \pi nx].$$

Since $\operatorname{Im}(2\eta_1 + e_j)$ is an odd function of x, it is sufficient to prove (b) for $x > 0$.

If $0 < x \le \frac{1}{4}$, then, from (19),

$$L_1'(x) \ge \frac{\pi e^{-\pi y}(1 - 10e^{-2\pi y})}{(1 + e^{-2\pi y})^4} \cos\tfrac{\pi}{4} > \frac{\pi e^{-\pi y}}{2},$$

and, for $n \ge 2$,

$$|L_n'(x)| \le \frac{\pi(1 + e^{-4\pi y})^3}{(1 - e^{-4\pi y})^4} n^2 e^{-\pi n y} < \frac{6\pi}{5} n^2 e^{-\pi n y}.$$

Furthermore, by using (19), we obtain

$$\sum_{n=2}^{\infty} n^2 e^{-\pi n y} = \frac{4e^{-2\pi y} - 3e^{-3\pi y} + e^{-4\pi y}}{(1 - e^{-\pi y})^3} \le \frac{2}{5} e^{-\pi y}.$$

So, if $0 < x \le \frac{1}{4}$, the expressions above yield that

$$\sum_{n=1}^{\infty} L_n'(x) \ge L_1'(x) - \sum_{n=2}^{\infty} |L_n'(x)| > 0,$$

$$\sum_{n=1}^{\infty} (-1)^n L_n'(x) \le -L_1'(x) + \sum_{n=2}^{\infty} |L_n'(x)| < 0,$$

which further imply that $(-1)^j \operatorname{Im}(2\eta_1 + e_j)$ is an increasing function of x, $0 < x \le \frac{1}{4}$, $j = 1, 2$. Since $\operatorname{Im}(2\eta_1 + e_j) = 0$, if $x = 0$, $j = 2, 3$, we conclude that

$$(-1)^j \operatorname{Im}(2\eta_1 + e_j) > 0, \qquad 0 < x \le \tfrac{1}{4},\ j = 2, 3.$$

On the other hand, if $\frac{1}{4} \le x \le \frac{1}{2}$ then, from (19),

$$L_1(x) \ge \frac{e^{-\pi y}}{(1 + e^{-2\pi y})^2} \sin\frac{\pi}{4} \ge \frac{3}{5} e^{-\pi y},$$

and, for $n \ge 2$,

$$|L_n(x)| \le \frac{1 + e^{-4\pi y}}{(1 - e^{-4\pi y})^2} n e^{-\pi n y} \le 2n e^{-\pi y}.$$

Furthermore, by using (19), we obtain

$$\sum_{n=2}^{\infty} n e^{-\pi n y} = \frac{2e^{-2\pi y} - e^{-3\pi y}}{(1 - e^{-\pi y})^2} \le \frac{e^{-\pi y}}{5}.$$

So, if $\frac{1}{4} \le x \le \frac{1}{2}$, then

$$\sum_{n=1}^{\infty} L_n(x) \ge L_1(x) - \sum_{n=2}^{\infty} |L_n(x)| > 0,$$

$$\sum_{n=1}^{\infty} (-1)^n L_n(x) \le -L_1(x) + \sum_{n=2}^{\infty} |L_n(x)| < 0.$$

Thus,

$$(-1)^j \operatorname{Im}(2\eta_1 + e_j) > 0, \qquad \tfrac{1}{4} \le x \le \tfrac{1}{2},\ j = 2, 3,$$

which completes the proof of (b).

In order to prove (c), let $y \ge \sqrt{3}/2$ fixed. Then [11, vol. 2, 27]

$$e_2 - e_3 = 16\pi^2 q \left[1 + \sum_{n=1}^{\infty} q^{(n^2+n)}\right]^4 = 16\pi^2 q \gamma_{2,3}^4(x), \tag{24}$$

where

$$\operatorname{Re} \gamma_{2,3}(x) = 1 + \sum_{n=1}^{\infty} e^{-\pi(n^2+n)y} \cos \pi(n^2+n)x,$$

$$\operatorname{Im} \gamma_{2,3}(x) = \sum_{n=1}^{\infty} e^{-\pi(n^2+n)y} \sin \pi(n^2+n)x.$$

Thus, from (19) we have

$$|\operatorname{Re} \gamma_{2,3}(x) - 1| \le \sum_{n=2}^{\infty} e^{-\pi n y} < 10^{-2}, \qquad |\operatorname{Im} \gamma_{2,3}(x)| < 10^{-2},$$

and therefore

$$|\operatorname{Arg} \gamma_{2,3}^4(x)| < \pi/6. \tag{25}$$

Now, if $0 < x \le \frac{1}{6}$, then, by using (19),

$$\begin{aligned}(\operatorname{Im} \gamma_{2,3})'(x) &= \pi \sum_{n=1}^{\infty} (n^2+n) e^{-\pi(n^2+n)y} \cos \pi(n^2+n)x \\ &\ge \pi e^{-2\pi y} - \pi \sum_{n=6}^{\infty} n e^{-\pi n y} > 0.\end{aligned}$$

Furthermore, $\operatorname{Im} \gamma_{2,3}(0) = 0$. Thus

$$\operatorname{Im} \gamma_{2,3}(x) > 0, \qquad 0 < x \le \tfrac{1}{6}.$$

Also, if $0 < x \le \frac{1}{6}$, then $0 < \operatorname{Arg} q \le \pi/6$, and from (24), (25) and the inequality above it follows that

$$0 < \operatorname{Arg}(e_2 - e_3) < \pi/3, \qquad 0 < x \le \tfrac{1}{6}.$$

On the other hand, if $\frac{1}{6} \le x \le \frac{1}{2}$, then $\pi/6 \le \operatorname{Arg} q \le \pi/2$, and from (24) and (25) it follows that

$$0 < \operatorname{Arg}(e_2 - e_3) < 2\pi/3, \qquad \tfrac{1}{6} \le x \le \tfrac{1}{2}.$$

These results show that

$$\operatorname{Im}(e_2 - e_3) > 0, \qquad x > 0.$$

Furthermore, since $\operatorname{Im}(e_2 - e_3)$ is an odd function of x,

$$\operatorname{Im}(e_2 - e_3) < 0, \qquad x < 0.$$

(c) is proved.

Part (d) of the proposition follows immediately from (24). Hence the proof of the proposition is complete.

Lemma 3. *Let $L(1,\tau)$ be a lattice, where $\tau = x + iy \in F$, and let $B \in \mathbf{R} - \{0\}$ such that equation* (II) *of Lemma* 2 *is satisfied. Then $s = 2$ and $x = 0$.*

Proof. By Propositions 3 and 4, if $|x| \leq \frac{1}{2}$ and $s = 1$ or $x = 0$ and $s = 3$, the real parts of the first and second members, respectively, of equation (II) are of opposite sign. Furthermore, if $x \neq 0$ and $s = 3$ or $x \neq 0$ and $s = 2$, the imaginary parts of the first and second members respectively, of equation (II) are of opposite sign. Hence the lemma is proved.

Remark 3. In accordance with Lemma 3, if there exist $B \in \mathbb{R} - \{0\}$ and a lattice $L(1,\tau)$, $\tau \in F$, such that equations (I) and (II) of Lemma 2 are satisfied, then $L(1,\tau) = L(1,iy)$, $y \geq 1$, and $s = 2$. In this case $2\eta_1 + e_j \in \mathbb{R}$, $e_k - e_j \in \mathbb{R}$, $k, j = 1,2,3$, and equations (I) and (II) are respectively equivalent to

$$\text{(I}'\text{)} \qquad 2\eta_1 + e_2 = B^2\left(\frac{2\eta_1 + e_1}{e_1 - e_2} + \frac{2\eta_1 + e_3}{e_3 - e_2}\right),$$

$$\text{(II}'\text{)} \qquad y(2\eta_1 + e_2) - \pi = \pi B^2\left(\frac{1}{e_1 - e_2} + \frac{1}{e_3 - e_2}\right).$$

We shall prove that the unique solution of equations (I′) and (II′) is $y = 1$ and $B = \sqrt{\pi/2}$. For this purpose we need Propositions 5–8:

Proposition 5. *Let $L(1,iy)$ be the lattices of $\mathbb{C}$, where $y \geq 1$. Then*:

(a) $(2\eta_1 + e_1)/(e_1 - e_2) + (2\eta_1 + e_3)/(e_3 - e_2) \geq \frac{3}{2}(1 - e^{-2\pi y})^3$,

(b) *the function $f(y) = y(2\eta_1 + e_2) - \pi$ satisfies*

$$f'(y) \leq \frac{-12\pi^2 e^{-\pi y}}{(1 - e^{-2\pi y})^2} \quad \textit{and} \quad f(1) = 0,$$

(c) *the function $h(y) = \pi e^{-\pi y}/(e_1 - e_2) + \pi e^{-\pi y}/(e_3 - e_2)$ satisfies*

$$0 > h'(y) \geq \frac{-2(1 + e^{-\pi y})e^{-\pi y}}{(1 - e^{-2\pi y})^2} \quad \textit{and} \quad h(1) = 0.$$

Proof. From (20) and (21) we have

$$2\eta_1 + e_1 = \pi^2 + 8\pi^2 \sum_{n=1}^{\infty} S_n(y), \qquad S_n(y) = \frac{(-1)^{n+1} n e^{-2\pi n y}}{1 - e^{2\pi n y}},$$

$$e_1 - e_2 = \pi^2\Big(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 y}\Big)^4.$$

By using (19), we obtain for n odd,

$$S_n(y) + S_{n+1}(y) > \left(1 - \frac{2e^{-2\pi y}}{1 - e^{-4\pi y}}\right) n e^{-2\pi n y} > 0,$$

$$-e^{-\pi n^2 y} + e^{-\pi(n+1)^2 y} < 0,$$

so, $2\eta_1 + e_1 \geq \pi^2$ and $\pi^2 \geq e_1 - e_2 > 0$. Hence

$$\frac{2\eta_1 + e_1}{e_1 - e_2} \geq (1 - e^{-2\pi y})^3. \tag{26}$$

Also, from (23) and (24) we have

$$0 > e_3 - e_2 \geq -16\pi^2 e^{-\pi y}\left(1 + \sum_{n=1}^{\infty} e^{-2\pi n y}\right)^4 = -\frac{16\pi^2 e^{-\pi y}}{(1 - e^{-2\pi y})^4},$$

$$2\eta_1 + e_3 \leq \frac{-8\pi^2 e^{-\pi y}}{1 - e^{-2\pi y}}.$$

Thus

$$\frac{2\eta_1 + e_3}{e_3 - e_2} \geq \frac{1}{2}(1 - e^{-2\pi y})^3,$$

which together with (26) proves (a).

In order to prove (b), from Proposition 1 we notice that $f(1) = 0$. Also

$$f'(y) = 2\eta_1 + e_2 + y(2\eta_1 + e_2)'(y). \tag{27}$$

Using (23) we obtain

$$2\eta_1 + e_2 = 8\pi^2 \sum_{n=1}^{\infty} R_n(y), \qquad R_n(y) = (-1)^{n+1}\frac{n e^{-\pi n y}}{1 - e^{-2\pi n y}}. \tag{28}$$

Thus, for $k = 1, 2, \cdots$, from (19) we have

$$R_{2k} + R_{2k+1} \leq -2k e^{-2k\pi y}\left(1 - \frac{3e^{-\pi y}}{2(1 - e^{-6\pi y})}\right) < 0,$$

$$2\eta_1 + e_2 = 8\pi^2\left[\frac{e^{-\pi y}}{1 - e^{-2\pi y}} + \sum_{k=1}^{\infty}(R_{2k} + R_{2k+1})\right] \leq \frac{8\pi^2 e^{-\pi y}}{(1 - e^{-2\pi y})^2}. \tag{29}$$

Also,

$$R'_n(y) = (-1)^n \frac{\pi n^2 (1 + e^{-2\pi n y}) e^{-\pi n y}}{(1 - e^{-2\pi n y})^2},$$

and for $k = 1, 2, \ldots$

$$R'_{2k-1}(y) + R'_{2k}(y) \le \frac{-\pi(2k-1)^2}{e^{\pi(2k-1)y}} + \frac{4\pi k^2(1+e^{-4\pi})}{(1-e^{-4\pi})^2 e^{2\pi k y}}.$$

So,

$$R'_{2k-1}(y) + R'_{2k}(y) \le -\left(1 - \frac{4(1+e^{-4\pi})e^{-\pi}}{(1-e^{-4\pi})^2}\right)\frac{\pi(2k-1)^2}{e^{\pi(2k-1)y}}. \tag{30}$$

But,

$$\sum_{k=1}^{\infty}(2k-1)^2 e^{-\pi(2k-1)y} > \sum_{k=1}^{\infty} k e^{-\pi(2k-1)y} = \frac{e^{-\pi y}}{(1-e^{-2\pi y})^2}. \tag{31}$$

Also, by (19) we obtain

$$1 - \frac{4(1+e^{-4\pi})e^{-\pi}}{(1-e^{-4\pi})^2} \ge \frac{4}{5}. \tag{32}$$

Thus, from (30), (31) and (32) it follows that

$$\sum_{n=1}^{\infty} R'_n(y) = \sum_{k=1}^{\infty} R'_{2k-1}(y) + R'_{2k}(y) < -\frac{4\pi e^{-\pi y}}{5(1-e^{-2\pi y})^2},$$

which together with (27), (28) and (29) implies that

$$f'(y) \le \frac{-8\pi^2(4\pi y - 5)e^{-\pi y}}{5(1-e^{-2\pi y})^2} \le \frac{-12\pi^2 e^{-\pi y}}{(1-e^{-2\pi y})^2}. \tag{33}$$

Hence (b) is proved.

In order to prove (c) we have $h(1) = 0$ from Proposition 1, and from (21) and (24) it follows that

$$h'(y) = -\frac{(1+2\sum_{n=1}^{\infty}(-1)^n(1-4n^2)e^{-\pi n^2 y})e^{-\pi y}}{(1+2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi y})^5} - \frac{\sum_{n=1}^{\infty}(n^2+n)e^{-(n^2+n)\pi y}}{4(1+\sum_{n=1}^{\infty} e^{-(n^2+n)\pi y})^5}.$$

So, $h'(y) < 0$ and

$$h'(y) \ge \frac{-(1+6e^{-\pi})e^{-\pi y}}{(1-2e^{-\pi})^5} - \frac{1}{2}\sum_{n=1}^{\infty} n e^{-2\pi n y}.$$

Thus, by using (19) we have

$$h'(y) \ge -2e^{-\pi y} - \frac{e^{-2\pi y}}{2(1-e^{-2\pi y})^2} \ge \frac{-2(1+e^{-\pi y})e^{-\pi y}}{(1-e^{-2\pi y})^2},$$

which finishes the proof.

Remark 4. Proposition 5 allows us to define C^∞-functions $B_j\colon (1,\infty) \to \mathbb{R}$, $j = 1, 2$, such that, in the lattices $L(1, iy)$,

$$\text{(I}''\text{)} \qquad \frac{2\eta_1 + e_2}{e^{-\pi y}} = B_1(y)\left(\frac{2\eta_1 + e_1}{e_1 - e_2} + \frac{2\eta_1 + e_3}{e_3 - e_2}\right),$$

$$\text{(II}''\text{)} \qquad y(2\eta_1 + e_2) - \pi = \pi B_2(y)\left(\frac{1}{e_1 - e_2} + \frac{1}{e_3 - e_2}\right)e^{-\pi y}.$$

Then there exist $B \in \mathbb{R} - \{0\}$ and a lattice $L(1, iy)$, $y > 1$, such that equations (I′) and (II′) are satisfied if and only if there exists $y > 1$ such that $B_1(y) = B_2(y) > 0$.

Proposition 6. $B_1(y) \le \frac{16\pi^2}{3(1-e^{-2\pi})^5}$.

Proof. The proof follows from (29) and (a) of Proposition 5.

To evaluate $B_2(y)$ we need the following elementary proposition.

Proposition 7. *Let $f, h, K\colon (1-\varepsilon, \infty) \to \mathbb{R}$ be C^∞-functions such that* (a) $f(1) = h(1) = 0$, (b) $h'(y) < 0$, $K'(y) \le 0$ *and* $f'(y)/h'(y) > K(y) > 0$, $y > 1 - \varepsilon$. *Then the function $f(y)/h(y)$ is defined in $[1, \infty)$ and $f(y)/h(y) > K(y)$.*

Proposition 8. $B_2(y) \ge 6\pi^2/(1 + e^{-\pi})$.

Proof. Let $f, h\colon [1, \infty] \to \mathbb{R}$ be defined by

$$f(y) = y(2\eta_1 + e_2) - \pi, \qquad h(y) = \pi\left(\frac{1}{e_1 - e_2} + \frac{1}{e_3 - e_2}\right)e^{-\pi y}.$$

From Proposition 5 we have $f(1) = h(1) = 0$, $h'(y) < 0$ and

$$\frac{f'(y)}{h'(y)} \ge \frac{6\pi^2}{1 + e^{-\pi y}} \ge \frac{6\pi^2}{1 + e^{-\pi}}.$$

Thus, by using Proposition 7 we obtain

$$B_2(y) = \frac{f(y)}{h(y)} > \frac{6\pi^2}{1 + e^{-\pi}},$$

which completes the proof of Proposition 8.

Lemma 4. *Let $L(1, iy)$ be a lattice where $y \ge 1$, and let $B \in \mathbb{R} - \{0\}$ such that equations* (I) *and* (II) *of Lemma* 2 *are satisfied for $s = 2$. Then $y = 1$.*

Proof. It follows from Propositions 6 and 8 that $B_2(y) > B_1(y)$. Then Remarks 3 and 4 prove the lemma.

Using all the results above, we can finally proceed to the proof of Theorem 1 and its corollary.

Proof of Theorem 1. Let $x\colon M = T_\tau - \{q_1, q_2, q_3\} \to \mathbb{R}^3$ be a minimal immersion as in Theorem 1, where $T_\tau = C/L(1, \tau)$, $\tau \in F$. Then, from Lemmas 1 and 2 we have the following:

(1) After a rigid motion, a homothety of the immersion in $\mathbb{R}^3$ and a translation of coordinates of $\mathbb{C}$, the Weierstrass representation (g, w) of x is given by

$$g = 2B(e_k - e_t)/P', \quad w = (P - e_s)\,dz, \quad B \in \mathbb{R} - \{0\},$$
$$\pi(0) = q_3, \quad \pi(w_k) = q_2, \quad \pi(w_t) = q_1,$$

where (s, k, t) is a permutation of (1,2,3), $B \in \mathbb{R} - \{0\}$, and $\pi\colon \mathbb{C} \to T_\tau$ is the canonical projection, and the end associated to the point q_3 is the flat end.

(2) Equations (I) and (II) of Lemma 2 are satisfied in the lattice $L(1, \tau)$.

But, from Lemmas 3 and 4, if equations (I) and (II) are satisfied in $L(1, \tau)$, then $\tau = i$ and $s = 2$.

On the other hand, if $B = \sqrt{\pi/2}$, by Proposition 1 we find that equations (I) and (II) are satisfied in $L(1, i)$. That is, under these conditions the immersion obtained is the same one constructed in [1]. This finishes the proof.

Proof of Corollary 1. Let $x\colon M \to \mathbb{R}^3$ be an embedded, complete minimal surface with $c(M) = 12\pi$. Then M is conformally equivalent to a compact Riemann surface, $\overline{M}_\gamma$, of genus γ, punctured at the points $q_1, \cdots, q_N$. Thus we only need to show that $\gamma = 1$ and $N = 3$. Since the ends of M are embedded, it follows from [5] that

$$12\pi = -2\pi[2 - 2\gamma - 2N],$$

and therefore

$$\gamma + N = 4.$$

However, M cannot be embedded in $\mathbb{R}^3$ as a complete minimal surface of finite total curvature in the three following cases:

(1) $M \simeq S^2 - \{q_1, \cdots, q_N\}$, $3 \le N \le 5$,

(2) $M \simeq \overline{M}_\gamma - \{q_1, q_2\}$, $\gamma > 0$,

(3) $M \simeq \overline{M}_\gamma - \{q_1\}$, $\gamma > 0$.

Cases (1) and (2) follow from [5] and [10] respectively. Case (3) follows from the fact that the coordinates of a minimal surface are harmonic functions. Hence the proof is complete.

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