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A CONSTRUCTION OF METRICS OF NEGATIVE RICCI CURVATURE

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In this paper we will prove

Theorem 1. *Every compact* 3*-manifold admits a metric of negative Ricci curvature.*

This theorem was originally proved by Gao and Yau in [2]. The proof we give here, while based on [2], is substantially shorter, given signficant results about hyperbolic 3-manifolds, and is (to our taste) more constructive and conceptual. The simplicity comes from staying as long as possible in the category of hyperbolic metrics. In particular, we make strong use of the existence of tubular neighborhoods of specified width about short geodesics. This theory follows from Jorgensen's inequality [4] and was developed in [1] and [7].

Our construction is flexible enough to give

Theorem 2. *There exist positive constants a and b such that every compact* 3*-manifold admits a metric whose Ricci curvatures all lie between $-a$ and $-b$.*

Observe that if M admits a metric whose Ricci curvatures lie between $-a$ and $-b$, where $b/a < 2$, then this metric has negative sectional curvature. Our argument gives a ratio of b/a on the order of 1,000, but we do not compute it explicitly.

We also obtain results about higher-dimensional manifolds carrying metrics of negative Ricci curvature:

Theorem 3. *Let M be a hyperbolic orbifold of order k, where $k \geq 12$. Then M admits a metric of negative Ricci curvature.*

The terminology "hyperbolic orbifold of order k" is explained in §4. The proof of Theorem 3 was motivated by the paper [3], which has a number of points of contact with the present paper.

The plan of our argument is as follows.

Given a 3-manifold M, it follows from the Thurston theory [8] that there is a link L in M such that $M - L$ has a complete metric of constant curvature

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-1. Furthermore, for n sufficiently large, we may find a metric on M itself which is hyperbolic away from L, but which has an orbifold singularity about L of order n. This is discussed in detail in §1.

We then consider in §§2 and 3 the problem of smoothing out the metric along L. To do this, we argue in §2 that about each component γ of L, there is a tubular neighborhood $T_n(\gamma)$ of radius $r_n(\gamma)$, such that all the tubular neighborhoods are disjoint. We then provide estimates on the size of $r_n(\gamma)$.

Finally in §3 we show how to modify the metric in each tubular neighborhood to preserve negative curvature. We are led to a family of differential inequalities which must be met in order to retain negative Ricci curvature. We find that if the tubular neighborhood is sufficiently large, then this can be done. We also show that the tubular neighborhoods found in §2 are sufficiently large to carry out this construction. In §4, we then consider the case of dimensions greater than 3.

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1. The singular metrics

Let M be a compact 3-manifold. Then there exists a link $L \subset M$ such that $M - L$ has a complete hyperbolic structure of finite volume.

There are two ways of seeing this. One way is to appeal to a theorem of Thurston [8] that every 3-manifold can be obtained from the 3-sphere S^3 by doing Dehn surgery on a link whose complement has a hyperbolic structure. Another way is to appeal to the easier theorem [5] that every 3-manifold M arises as a branched covering $f: M \to S^3$ of D^3 whose branching locus on S^3 is the figure-eight knot. Since the figure-eight knot complement has a well-known hyperbolic structure, we may lift this to obtain a complete hyperbolic structure on $M - L$, where $L = f^{-1}$ (figure-eight knot) is the branching locus of f in M.

Let $\gamma_1, \cdots, \gamma_k$ be the components of L. Then for each k-tuple of pairs of integers $\prod_i (p_i, q_i)$, $i = 1, \cdots, k$, we may obtain a manifold $M^{\prod(p_i,q_i)}$ by removing a tubular neighborhood of γ_i and replacing it with a new identification of the boundary. Furthermore, when all the (p_i, q_i)'s are sufficiently large, $M^{\prod(p_i,q_i)}$, has a hyperbolic metric. This is Thurston's hyperbolic Dehn surgery [8]. We choose the ordering of (p_i, q_i) such that $M^{\prod(0,1)} = M$.

When p_i and q_i are relatively prime, the hyperbolic metric on $M^{\prod(p_i,q_i)}$ is smooth. However, if $(p_i, q_i) = n(p_i', q_i')$, with p_i' and q_i' relatively prime, then the metric on $M^{\prod(p_i,q_i)}$ is branched about γ_i of order n—in a neighborhood of γ_i the metric looks like the quotient of hyperbolic space by rotation through angle $2\pi/n$ about a geodesic line.

We may compute the metric about a γ_i as follows: let γ be a geodesic line in hyperbolic space. We may describe the metric in a tubular neighborhood of γ in Fermi coordinates by

$$ds^2 = (\cosh^2(r))\,dt^2 + dr^2 + (\sinh^2(r))\,d\theta^2,$$

where t is the coordinate along the geodesic, r is the radial coordinate, and θ the angular coordinate. Quotienting out by a rotation of $2\pi/n$ gives the metric

$$ds^2 = (\cosh^2(r))\,dt^2 + (dr)^2 + \left(\frac{\sinh(r)}{n}\right)^2 d\theta^2.$$

Observe that a metric of the form

$$ds^2 = g^2(r)\,dt^2 + (dr)^2 + f^2(r)\,d\theta^2$$

will be smooth at $r = 0$ provided

$$f(0) = 0, \quad f'(0) = 1, \qquad g(0) > 0, \quad g'(0) = 0.$$

In conclusion, we have:

Lemma 1. *Given M, and a link L such that $M - L$ has a hyperbolic structure, and suitably large integers n_i, there is a metric on $M = M^{\prod(0,n_i)}$ which is hyperbolic away from the link, and which is of the form*

$$ds^2 = (\cosh^2(r))\,dt^2 + (dr)^2 + \left(\frac{\sinh(r)}{n_i}\right)^2 d\theta^2$$

in Fermi coordinates about each component γ_i of L.

2. Tubular neighborhoods

Let M be a hyperbolic 3-manifold, and γ a closed geodesic in M. Then there is an element A in $\mathrm{PSL}(2, \mathbf{C})$ whose axis is γ, and whose complex translation length τ is defined by

$$\mathrm{tr}(A) = \pm 2\cosh(\tau/2).$$

In [1], it is shown that, if $|\sinh(\tau/2)| < 1/\sqrt{6}$, then there is a tubular neighborhood $T(\gamma)$ of radius r, where

$$(*) \qquad \sinh^2(r) = \frac{1}{4|\sinh(\tau/2)|^2} - \frac{3}{2},$$

and, furthermore, corresponding tubular neighborhoods about disjoint short ($|\sinh(\tau/2)| < 1/\sqrt{6}$) geodesics are disjoint. Slightly sharper estimates appear in [7] (see also §4 below), but the estimate (*) suffices for our purposes.

We apply these considerations to the branched coverings of our manifold M which unwind the singularity about γ_i. In this manifold, γ_i is the axis of rotation by $2\pi/n$, and one sees easily that a tubular neighborhood in the covering descends to a tubular neighborhood of γ_i in M. We therefore may apply the above considerations to $\tau/2 = i\pi/n$ to see that there is a tubular neighborhood about γ_i in M of radius r_i where

$$\sinh^2(r_i) = \frac{1}{4\sin^2(\pi/n_i)} - \frac{3}{2} \sim C \cdot n_i^2,$$

where $C \sim 1/2\pi$.

In particular, we observe that $r_i \to \infty$ as $n_i \to \infty$, and $\sinh(r_i)/n_i \to 1/2\pi > 0$ as $n_i \to \infty$.

3. Negative Ricci curvature

Let us begin with a tubular neighborhood of a geodesic in which the metric is of the form

$$ds^2 = g^2\, dt^2 + dr^2 + f^2\, d\theta^2.$$

This will describe a metric of negative Ricci curvature provided

(a) $g''/g + f''/f > 0$,
(b) $g''/g + (g'/g) \cdot (f'/f) > 0$,
(c) $f''/f + (g'/g) \cdot (f'/f) > 0$.

Theorem 1 will be proved once we show:

Theorem 4. *For n sufficiently large, there are functions $g(r)$, $f(r)$ such that*

(i) *conditions* (a)–(c) *are fulfilled,*

(ii) *For $r > r_n$, where $\sinh(r_n) = C \cdot n$, $C < 1/2\pi$, we have $g(r) \equiv \cosh(r)$ and $f(r) \equiv \sinh(r)/n$,*

(iii) *$f(0) = 0$, $f'(0) = 1$, $g(0) > 0$, and $g'(0) = 0$.*

Proof. We will construct f and g in the following way. First, we will describe f and g piecewise so as to satisfy (i)–(iii). Then we will show how to smooth out the singularities to obtain smooth f and g satisfying (a)–(c).

To smooth out the singularities, we will use the following two elementary lemmas. Lemma 2 was proved in [2].

Lemma 2 [2]. *Let f_1, f_2 be positive strictly increasing functions with f_1'', $f_2'' > 0$.*

Suppose that $f_1(a) = f_2(a)$ and $f_1'(a) < f_2'(a)$. Then for $b < a < c$, there exists a smooth function f such that f is strictly increasing, $f'' > 0$, and $f(r) = f_1(r)$ for $r < b$, $f(r) = f_2(r)$ for $r > c$.

Lemma 3. *Let f_1, f_2 and g be positive strictly increasing, $f_2'' > 0$, $g'' > 0$, and (f_1, g) and (f_2, g) satisfy* (a)–(c). *Suppose that $f_1(a) = f_2(a)$ and $f_1'(a) < f_2'(a)$. Then for $b < a < c$, there exists a smooth function f such that (f, g) satisfy* (a)–(c), *and $f(r) = f_1(r)$ for $r < b$ and $f(r) = f_2(r)$ for $r > c$.*

Proof. If $f_1''(a) > 0$, this is just Lemma 2. So we may assume $f_1''(a) < 0$.

Let us set $\varepsilon = r - a$, and choose ε_0 such that if $c_{\varepsilon 0} = f_2(r + \varepsilon_0)$, then,

(a′) $g''/g + f_1''/c_{\varepsilon 0} > 0$ for $|r - a| < \varepsilon_0$,
(b′) $f_1''/c_{\varepsilon 0} + (g'/g)f_1'/c_{\varepsilon 0} > 0$ for $|r - a| < \varepsilon_0$,
(c′) $g''/g + (g'/g)f_1'/c_{\varepsilon 0} > 0$ for $|r - a| < \varepsilon_0$.

Now choose $c > \sup|f_1''(\varepsilon)|$ for $|\varepsilon| < \varepsilon_0$. Given δ, consider the function h_δ which is 0 for $\varepsilon < -\delta/c$, and which is c for $\varepsilon > -\delta/c$, and define f_3 by

$$f_3(\varepsilon) = f_1(\varepsilon) \quad \text{for } \varepsilon < -\delta/c, \qquad f_3'' = f_1'' + h_\delta$$

so that

$$\begin{aligned} f_3' &= f_1' + c^\varepsilon + \delta \quad \text{for } \varepsilon \geq -\frac{\delta}{c}, \\ f_3' &= f_1' \quad \text{for } \varepsilon \leq -\frac{\delta}{c}, \\ f_3 &= f_1 + \frac{\varepsilon^2 c}{2} + \delta\varepsilon + \frac{\delta^2}{2c} \quad \text{for } \varepsilon \geq -\frac{\delta}{c} \\ &= f_1 \quad \text{for } \varepsilon \leq -\frac{\delta}{c}. \end{aligned}$$

By choosing ε_1 and δ small, we have $f_3' < f_2'$ for $|\varepsilon| < \varepsilon_1$. Note that

$$f_3(0) = f_1(0) + \frac{\delta^2}{2c} > f_2(0).$$

We now want to arrange it so that $f_3(\varepsilon_1) < f_2(\varepsilon_1)$. Since $f_2''(a) > 0$, we have $f_2(\varepsilon) > f_2(0) + f_2'(0) \cdot \varepsilon$ and since $f_1''(a) < 0$, we have

$$f_3(\varepsilon) < f_1(0) + f_1'(0) \cdot \varepsilon + \frac{\varepsilon^2 c}{2} + \delta\varepsilon + \frac{\delta^2}{2c}$$

and from $f_1(0) = f_2(0)$, we must show that

$$f_1'(0) \cdot \varepsilon_1 + \frac{\varepsilon_1^2 c}{2} + \delta\varepsilon_1 + \frac{\delta^2}{2c} < f_2'(0) \cdot \varepsilon_1.$$

We now choose ε_1 sufficiently small so that

$$(f_2'(0) - f_1'(0))\varepsilon_1 - \frac{\varepsilon_1^2 c}{2} > 0,$$

and then δ sufficiently small so that

$$\delta\varepsilon_1 - \frac{\delta^2}{2c} < (f_2'(0) - f_1'(0))\varepsilon_1 - \frac{\varepsilon_1^2 c}{2}.$$

It follows that there is a point x between a and $a+\varepsilon$, at which $f_3(x) = f_2(x)$, and we have already shown that $f_3'(x) < f_2'(x)$. Furthermore, $f_3''(x) > 0$ by construction.

To see that (f_3, g) satisfies (a)–(c), we observe that $f_3(r) < f_2(r+\varepsilon_0)$, since f_3 is increasing and $f_3(x) = f_2(x)$. Furthermore, $f_3'' \geq f_1''$, $f_3' \geq f_1'$, so the fact that f_3 satisfies (a)–(c) follows from the fact that this is true for f_1.

f_3 is not smooth at $-\delta/c$, but by approximating h_δ by a smooth function k_δ, and defining f_4 by $f_4 = f_1$ for $\varepsilon < -\delta/c$, $f_4'' = f_1'' + k_\delta$, we see that f_4 possesses the same properties we have demonstrated for f_3, and that f_4 is smooth.

Finally, we apply Lemma 2 to f_4 and f_2 at x (where $f_4'' > 0$) to obtain a smooth function f which satisfies the conclusion of the lemma.

To prove Theorem 4, we first apply Lemma 3 to

$$f_1 = \frac{1}{\lambda_1}\arctan(\lambda_1 r), \quad f_2 = \frac{\sinh(r)}{n}, \quad g = \alpha(\cosh(\lambda_2 r)).$$

Notice that arctan is an increasing function whose second derivative is negative, and that $\arctan(x) < \pi/2$ for all x. It follows that, given $D > 0$, for λ_1 sufficiently large, there is a point r_0 such that

$$f_1(r_0) = f_2(r_0) < D \quad \text{and} \quad f_1'(r_0) < f_2'(r_0).$$

We will fix D later.

f_2 and g are visibly positive increasing functions whose second derivative is positive. Note also that $f_1(0) = 0$, $f_1'(0) = 1$.

We now must show f_1 and g satisfy (a)–(c). Condition (a) then becomes

$$\lambda_2^2 - \lambda_1^2 \frac{\sinh(\lambda_1 r)}{\cosh^2(\lambda_1 r)\arctan(\lambda_1 r)} > 0$$

or $\lambda_2^2 \arctan(\lambda_1 r) > \lambda_1^2 \sinh(\lambda_1 r)/\cosh^2(\lambda_1 r)$.

Noting that one gets $0 = 0$ when $r = 0$, and taking derivatives, one sees that (a) is fulfilled when $\lambda_2 > \lambda_1$. Condition (b) is trivially fulfilled, since g, g', g'', and f' are all positive. Condition (c) becomes

$$-\lambda_1^2 \frac{\sinh(\lambda_1 r)}{\cosh^2(\lambda_1 r)\arctan(\lambda_1 r)} + \left(\lambda_2 \frac{\sinh(\lambda_2 r)}{\cosh(\lambda_2 r)}\right)\left(\lambda_1 \frac{1}{\cosh(\lambda_1 r)\arctan(\lambda_1 r)}\right).$$

Since tanh is an increasing function, this will again be fulfilled when $\lambda_2 > \lambda_1$.

We now would like to join up $\alpha\cosh(\lambda_2 r)$ and $\cosh(r)$, but it is easy to see that, for λ_2 large, for the value of α where $\alpha\cosh(\lambda_2 r)$ meets $\cosh(r)$, the derivative of $\alpha\cosh(\lambda_2 r)$ is greater, so that Lemma 2 does not apply directly. We therefore proceed in two steps. Let r_n be the value of n such that $\sinh(r_n)/n = C$.

Let $g_3 = \cosh(r)$ and let $g_2 = \cosh(r_n) + k(r - r_n)$, where k is close to but less than $\sinh(r_n)$. Then $g_2(r_n) = g_3(r_n)$, $g_2'(r_n) < g_3'(r_n)$ and Lemma 2 applies with $f = \sinh(r)/n$, since g_2, g_3, and f all have nonnegative first and second derivatives.

Now observe that g_2 meets the x-axis at a point $> r_n - \beta$, for $\beta = \coth(r_n)$ slightly larger than 1. We now may choose α sufficiently small so that $g_1(r) = \alpha\cosh(\lambda_2 r)$ meets g_2 at a point between $r_n - \beta$ and r_n where $g_1' < g_2' = k$. We now apply Lemma 2 twice to obtain a function g which agrees with $\alpha\cosh(\lambda_2 r)$ for $r < r_n - \beta$, and with $\cosh(r)$ for $r > r_n$.

As soon as $r_n - \beta > 0$, we may choose λ_1 sufficiently large so that $(\lambda_1)^{-1}\arctan(\lambda_1 r)$ never exceeds $(\sinh(r_n) - \beta)/n$, and λ_2 accordingly. This will happen as soon as $r_n \geq \coth(r_n)$, that is, $r_n \geq 1.19968$. From $(*)$ of §2, this will happen when $n \geq 14$.

From the equality

$$\sinh(r - \beta) = \sinh(r)\cosh(\beta) - \cosh(r)\sinh(\beta)$$

and $\sinh(r_n)/n = C$, we see that for n large (so that $\sinh(r_n)$ is close to $\cosh(r_n)$),

$$\frac{\sinh(r_n - \beta)}{n} >\sim C(\cosh(\beta) - \sinh(\beta)).$$

We may now take $D = C(\cosh(\beta) - \sinh(\beta))$. Noting that β, and hence D, do not depend on λ_2 we may now choose λ_1 so that $(\lambda_1)^{-1}\arctan$ never exceeds D; hence for large n we may choose λ_1 and λ_2 independent of n.

This proves Theorem 4, and hence Theorem 1. To prove Theorem 2, we make use of the following refinement of [5].

Theorem ([**6**]): *Let L be the Borromean rings link in S^3. Then any 3-manifold M arises as a branched covering of $f\colon M \to S^3$, whose branch locus is L, such that the order of branching is 1, 2, or 4 on each component of $f^{-1}(L)$.*

To prove Theorem 2, choose n_0 divisible by 4, sufficiently large so that doing $(0, n_0)$-Dehn surgery on each component of the Boormean rings gives a hyperbolic metric on S^3, branched of order n_0 along L, and such that $n_0/4 \geq 14$.

It follows that, for every 3-manifold M, there is a link L such that M has a hyperbolic metric which is branched of order n_0, $n_0/2$, or $n_0/4$ along each component of L. Therefore, in our proof of Theorem 4, we need only take n to be one of these three values. The existence of a and b as in Theorem 2 is now immediate.

We remark that one could probably greatly improve on this n_0, giving better estimates for a and b, but doing so would take us out of the scope of this paper.

4. Manifolds of higher dimension

Let M be a manifold of dimension $n > 3$. We will say that M is a hyperbolic orbifold of order k if there is a codimension 2 hypersurface X in M such that M carries a hyperbolic metric which is smooth and of constant curvature -1 away from X, and in a neighborhood of X looks like the quotient of hyperbolic space by an element of order k fixing a geodesic codimension 2 hyperbolic space:

In this section, we will show:

Theorem 3. *Let M be a hyperbolic orbifold of order k, where $k \geq 12$. Then M admits a metric of negative Ricci curvature.*

Proof. We proceed as before. We may choose "Fermi coordinates" $x_1, \cdots, x_{n-2}, r$, and θ about x, where $x_1, \cdots, x_{n-2}$ are coordinates for the fixed-point set, r the distance from the fixed-point set, and θ the angular coordinate.

Let us compute the Ricci curvature of a metric in a tubular neighborhood of x of the form

$$ds^2 = g^2(r)\,dx^2 + f^2(r)\,d\theta^2.$$

The Ricci curvatures are

$$-\left((n-3) + \frac{g''}{g} + \frac{g'}{g}\cdot\frac{f'}{f}\right),$$
$$-\left((n-2)\frac{g''}{g} + \frac{f''}{f}\right),$$
$$-\left((n-2)\frac{g'}{g}\cdot\frac{f'}{f} + \frac{f''}{f}\right).$$

We seek functions f and g satisfying these equations, and also satisfying

$$f(0) = 0, \qquad f'(0) = 1,$$
$$f(r) = \frac{\sinh(r)}{k} \quad \text{for } r > r_0,$$
$$g(r) = \cosh(r) \quad \text{for } r > r_0.$$

We now observe that in our solutions to these questions when $n = 3$, our solutions f and g satisfied that f, f', g, g' and g'' were all positive. Therefore, these solutions are also solutions to the present equations, so that we have solved these equations provided that X has a tubular neighborhood of radius r_0 such that $r_0 > \coth(r_0)$, i.e., such that $r_0 > 1.19968$.

We now want to estimate r_0 in terms of k, but at first sight the results of [1] are unavailable to us.

A generalization of [1], valid for elliptic elements A fixing a geodesic hypersurface of codimension 2, was given in [3]. Since we will want sharper constants than those given there, we will repeat the argument here.

Lemma 4 [3]. *Let X and M be as above. Then X has a tubular neighborhood of radius r_0, where*

$$2\sin(\pi/k)\cosh(r_0) = 1.$$

Proof. In hyperbolic space, let γ be a geodesic of minimal length joining two distinct lifts of X, and let $\lambda = \text{length}(\gamma)$. Then $A(\gamma)$ is another such, and makes angle $2\pi/k$ with γ. The geodesic joining the other ends of γ and $A(\gamma)$ then forms an isosceles triangle in hyperbolic space, with sides of length λ and angle $2\pi/k$. By elementary hyperbolic trigonometry, the opposite side has length $2x$, where

$$\sinh(x)/\sinh(\lambda) = \sin(\pi/k).$$

We must have $2x \geq \lambda$ by construction, so

$$\sinh(\lambda/2)/\sinh(\lambda) \leq \sin(\pi/k)$$

or $2\cosh(\lambda/2)\sin(\pi/k) \geq 1$.

Since the width of the tubular neighborhood is $\lambda/2$, this gives the lemma.

It follows that we get a tubular neighborhood of radius $r_0 = 1.19968$ when k is at least 12.

We remark that it is not too difficult to construct such M and X by arithmetic methods, giving interesting examples of manifolds with negative Ricci curvature, but at present we have little feeling for the topology of such M.

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