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DISTALITY, COMPLETENESS, AND AFFINE STRUCTURES

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Abstract

Certain affine connections with distal holonomy on a closed manifold are proven geodesically complete. A closed affine manifold has distal holonomy iff it is finitely covered by a nilmanifold with a complete left invariant affine structure. Auslander's conjecture on translations in unipotent affine actions is false in general. Ergodic automorphisms of nilmanifolds are affine.

This paper contains a number of loosely connected results that are related to our work with Goldman and Hirsch on affine manifolds with nilpotent holonomy [6]. All these results involve geodesically complete connections and geometrical forms of "nilpotence."

The first result concerns a closed manifold M with an affine connection ∇ (not assumed flat). It is of course classical that ∇ is geodesically complete if it is Riemannian (i.e. the Levi-Cevita connection of a Riemannian metric) or even Euclidean (i.e. the holonomy preserves a Riemannian metric) [10]. In more intrinsic terms, ∇ is Euclidean if the image $H_v \subset TM$ of any vector $v \in TM$ under parallel transport is bounded in TM. A more general property is that H_v be bounded away from the zero section for any nonzero v: such a ∇ is called *distal*. We show in Theorems 1 and 2 that many distal (non-Euclidean) connections on a closed manifold are geodesically complete.

In the special case when ∇ is flat (the curvature and torsion tensors vanish) and the holonomy is unipotent this completeness was one of the main results in [6]. This is extended to the flat distal case in Theorem 2. We can then generalize the famous Bieberbach theorem about flat Riemannian manifolds and tori (Theorem 3). These results were announced in [5].

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Lest one imagine that distal affine structures on closed manifolds are easily classifiable, we exhibit such a structure that cannot be produced in the usual way from lower dimensional affine manifolds. This disproves a conjecture of L. Auslander that was supposedly proven by Scheuneman [4], [14]. Under a strong additional assumption we prove a variant of Auslander's conjecture in Theorem 4.

Finally we replace our holonomy hypotheses by dynamical ones. In Theorem 5 we show that an ergodic automorphism of a nilmanifold N is an affine map relative to a certain complete affine structure on N.

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1. Second order distality implies completeness

Before studying connections we will recall some definitions and algebra. An action of a group G on a uniform space Y is *distal* if given distinct points $y_1, y_2 \in Y$ there is a uniform neighborhood V of the diagonal $\Delta \subset Y \times Y$ so that $(gy_1, gy_2) \notin V$ for all $g \in G$. In the special case when Y is a finite-dimensional real vector space and the action of G is linear this means that nontrivial G orbits are bounded away from zero. One obvious type of distal linear action is an *orthopotent* action defined by a flag of subspaces $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_n = Y$ preserved by G such that the graded action of G on $\oplus Y_{i+1}/Y_i$ is bounded. (This generalizes the usual notion of a *unipotent* action which is the case when this graded action is trivial.) We will say the flag is *orthopotent for* G in this situation.

It is a nontrivial fact that a distal linear action has an orthopotent flag. This was shown independently by Conzes and Guivarch [3] and Sullivan [unpublished] after earlier results by Moore [11]. We thank Sullivan for informing us of this result.

Very briefly, the case $G = \mathbf{Z}$ is easy and the problem reduces to showing that an irreducible linear group G with bounded traces has bounded entries. This in turn follows from Burnside's theorem that an irreducible G spans $\operatorname{Hom} Y$.

We will call an affine connection ∇ on a smooth manifold M *distal* if the holonomy group $G_m \subset \operatorname{Hom}(T_m M)$ is a distal linear group for each $m \in M$. We call the smallest value of n, i.e., the shortest length of an orthopotent flag for the holonomy group, the *order* of ∇. Thus a Euclidean connection is the same as a connection of order 1. We will show

Theorem 1. *If M is a smooth closed manifold, any distal connection ∇ on M of order $\leqslant 2$ is geodesically complete.*

Proof. Clearly it suffices to consider the case when M is connected. Pick a basepoint $m \in M$ and let $0 = Y_0 \subset \cdots \subset Y_n = T_m M$ be an orthopotent flag for the holonomy group at m.

Given each Y_{i+1}/Y_i a G_m-invariant inner product. By parallel transport we extend the flag to a holonomy-invariant filtration of TM by subbundles E_i such that each E_{i+1}/E_i has a holonomy invariant inner product μ_{i+1}. We let g be a Riemannian metric on M compatible with ∇ in the sense that the natural map $E_i^{\perp} \cap E_{i+1} \to E_{i+1}/E_i$ is isometric. The infinitesimal holonomy of ∇ relative to the splitting $TM = \bigoplus (E_i^{\perp} \cap E_{i+1})$ has the block form

$$\begin{pmatrix} S_1 & B_{12} & B_{13} & \cdots & B_{1n} \\ 0 & S_2 & B_{23} & & \cdot \\ 0 & 0 & S_3 & & \cdot \\ & & & \ddots & \cdot \\ & & & & S_n \end{pmatrix},$$

where the blocks S_i are skew-symmetric and all entries are bounded by a multiple of the g-length $g(v, v)^{1/2}$ of the infinitesimal displacement v. We may say that the holonomy of ∇ is *uniformly orthopotent* relative to g.

We now drop our compactness assumption and finish the proof of showing

Lemma 1. *Let ∇ be a connection on a smooth manifold M whose holonomy preserves a filtration $E_0 = 0 \subset E_1 \subset E_2 = TM$. Suppose there is a complete Riemannian metric g on M such that ∇ is uniformly orthopotent relative to g. Then ∇ is geodesically complete.*

Proof. Let $\gamma(t)$, $t \in [0, \infty)$, be a ∇-geodesic reparametrized at unit g-speed. Choose a g-orthonormal frame at $\gamma(0)$ and propagate it along γ with ∇-parallel transport. Uniform orthopotence implies that the lengths of vectors in this moving frame grow no faster than a polynomial $p(t)$ of degree $\leqslant n - 1$.

Let $\delta(s)$ be the natural parametrization of this geodesic, $s \in [0, a)$. We want to show $a = \infty$. We have $\|\delta'(s)\| \leqslant p(t)$ by the preceding paragraph, where $t = t(s)$ is a diffeomorphism $[0, a)$, $[0, \infty)$. Then as γ has unit speed,

$$\frac{dt}{ds} = \left\| \frac{dt}{ds} \frac{d\gamma}{dt} \right\| = \| \delta'(s) \| \leqslant p(t).$$

Thus

$$a = \int_0^a ds \geqslant \int_0^\infty \frac{dt}{p(t)}.$$

Since $n \leqslant 2$, this last integral is infinite, so $a = \infty$ as desired. q.e.d.

It seems possible that a more global analysis could extend Theorem 1 to distal connections of higher order, at least if ∇ is assumed torsionfree so that the invariant filtration is integrable. We will show below that this works if ∇ is flat. Theorem 1 seems to be the first result where, without assuming flatness or bounded holonomy, one can show a class of connections on a closed manifold to be geodesically complete.

2. A Bieberbach theorem for affine manifolds

It is well known that a closed affine manifold (i.e. one equipped with a flat connection) may be geodesically incomplete. It is important to find conditions on the holonomy that assure completeness. In this section we will prove the most general result in this direction to date, namely

Theorem 2. *A closed affine manifold with distal holonomy is complete.*

The case of Euclidean (i.e. bounded, Riemannian, metric) holonomy is clear. The case of unipotent holonomy was treated in our work with Goldman and Hirsch [6]. Using the results and methods of that paper we can apply Theorem 2 to characterize closed distal affine manifolds in the same spirit as Bieberbach's characterization of closed Euclidean affine manifolds. The role of Euclidean space in the Bieberbach theorem is played by a nilpotent Lie group with a certain geometric structure.

Theorem 3. *Let M be a closed affine manifold. M has distal holonomy if and only if some finite cover M_0 of M is a nilmanifold $\Gamma \setminus N$ with the affine structure induced from a complete left invariant affine structure on the simply connected nilpotent Lie group N.*

Proof of Theorem 3. We first dispose of the converse. It was shown in [6] that the holonomy of M_0 is unipotent. This shows that the holonomy of M_0 is distal and M inherits this property.

So suppose M has distal holonomy. By Theorem 2 M is complete. Thus the affine holonomy Γ of M is a discrete subgroup of the affine group $\mathrm{Aff}(m)$, $m = \dim M$. The linear holonomy of M is the image of Γ in $\mathrm{Gl}(m)$ under the canonical map $\lambda : \mathrm{Aff}(m) \to \mathrm{Gl}(m)$. We know $\lambda(\Gamma)$ is distal, hence orthopotent. By viewing $\mathrm{Aff}(m)$ as a subgroup of $\mathrm{Gl}(m + 1)$ in the usual way we find that Γ is also an orthopotent linear group.

Fix an orthopotent flag for Γ and an inner product on $\mathbf{R}^{n+1}$ such that the graded action of Γ is isometric. Then the group G of *all* linear transformations that preserve this flag and that have isometric graded action is the extension of a simply connected nilpotent Lie group N_0 (those transformations with trivial graded action) by a compact Lie group. But Zassenhaus showed in his studies of the usual Bieberbach theorem [15] that $\Gamma_0 = \Gamma \cap N_0$ has finite index in Γ.

Let M_0 be the finite cover of M corresponding to $\Gamma_0 \subset \Gamma$. Then M_0 is a complete oriented closed affine manifold with nilpotent holonomy and so by [6] M_0 is a nilmanifold with the affine structure described in the theorem. We mention that N is the Malcev completion of Γ_0 in N_0 under this construction.

Proof of Theorem 2. As in the proof of Theorem 1, we fix an invariant filtration $0 = E_0 \subset \cdots \subset E_n = TM$, an invariant inner product μ_{i+1} on each E_{i+1}/E_i, and a metric g on M so that $E_i^{\perp} \cap E_{i+1} \to E_{i+1}/E_i$ is an isometry for all i. Let $\mathscr{F}_i$ be the parallel foliation of M obtained by integrating E_i. Let $\tilde{\mathscr{F}}_i$ be the lifted foliation in the universal cover $\tilde{M}$. We will show by induction on i that the development map $D: \tilde{M} \to A$ maps each leaf of $\tilde{\mathscr{F}}_i$ bijectively to an affine subspace of affine space A: for $i = n$ we will have that D is a homeomorphism, hence M is complete. The case $i = 0$ being trivial we need only attend to the induction step.

Choose a leaf L of $\tilde{\mathscr{F}}_{i+1}$ and let Λ be the affine subspace of A such that $D_0 = D|L$ maps L to Λ by a local diffeomorphism. Then L and Λ each have parallel Riemannian foliations corresponding to $\tilde{\mathscr{F}}_i$ and E_i, respectively. These foliations and their transverse metrics correspond under D_0.

Now collapse all the fibers parallel to E_i to get a local diffeomorphism of leaf spaces $\hat{D}_0: \hat{L} \to \hat{\Lambda}$. We must show that $\hat{D}_0$ is a diffeomorphism. Since $\hat{\Lambda}$ is an affine space, it is simply connected and so it suffices to show that $\hat{D}_0$ has path lifting. Since $\hat{D}_0$ is an isometry and g is a complete metric this is immediate. q.e.d.

We return to the question of when a closed affine manifold M is complete. In all currently known examples, completeness follows from the existence of a parallel volume form. This seems to us to be unlikely, however, and we even propose

Problem. Find a closed affine manifold M with a parallel Lorentz metric that is geodesically incomplete.

It is known by work of Goldman and Hirsch that in such an example M is not finitely covered by a solvmanifold [8]. Such an M would be an interesting spacetime where incompleteness could not be blamed on curvature or lack of compactness.

3. Translations in unipotent affine actions

A left invariant complete affine structure on a simply connected Lie group N is the same as a simply transitive affine action of N on affine space A. L. Auslander conjectured that such an action contains a nonzero translation if N is nilpotent [1], [14], i.e. the restriction to $N \subset \mathrm{Aff}(A)$ of the homomorphism

$\lambda: \mathrm{Aff}(A) \to \mathrm{Gl}(A)$ is not injective. When this is so there is also a translation in the center of N and factoring by the central translations in N gives a simply transitive affine action of a lower dimensional group. Thus one could hope to construct all these actions by an inductive procedure. Unfortunately the conjecture is false.

Indeed, let $\mathscr{N}$ be the Lie algebra of infinitesimal affine motions of $\mathbf{R}^4$ of the following form ($w, x, y, z \in \mathbf{R}$):

$$\left(\begin{array}{cccc|c} 0 & -w & x & -y & z \\ 0 & 0 & z & 0 & y \\ 0 & 0 & 0 & z & x \\ 0 & 0 & 0 & 0 & w \end{array}\right).$$

Denoting this matrix by $\langle w, x, y, z\rangle$ one can check that

$$[\langle w, x, y, z\rangle, \langle w', x', y', z'\rangle] = \langle 0, w'z - wz', x'z - xz', 0\rangle.$$

Thus $\mathscr{N}$ is a nilpotent Lie algebra. The associated Lie group is the elements of Aff(4) of form

$$g = \exp\langle w, x, y, z\rangle$$

$$= \left(\begin{array}{cccc|c} 1 & -w & x - \frac{1}{2}wz & -y + \frac{1}{2}xz - \frac{1}{6}wz^2 & z + \frac{1}{2}x^2 - wy - \frac{1}{24}w^2z^2 \\ 0 & 1 & z & \frac{1}{2}z^2 & y + \frac{1}{2}xz + \frac{1}{6}wz^2 \\ 0 & 0 & 1 & z & x + \frac{1}{2}wz \\ 0 & 0 & 0 & 1 & w \end{array}\right).$$

This action on $\mathbf{R}^4$ is simply transitive since the rightmost column $g(0)$ equals $(a, b, c, d)^t \Leftrightarrow w = d,\ z = a + bd - \frac{1}{2}c^2,\ x = c - \frac{1}{2}wz,\ y = b - \frac{1}{2}xz - \frac{1}{6}wz^2$. There are no nontrivial translations in N since $\lambda(g) = I \Rightarrow 0 = w = z = x = y$. These explicit computations could be avoided by using known properties of unipotent actions, but we feel that the confusion over this conjecture warrants a detailed example.

In the given basis, $\mathscr{N}$ has integral structure constants. Thus N has a cocompact discrete subgroup Γ [12]. The nilmanifold $M = \Gamma \backslash N$ is a closed distal affine manifold. M is *not* constructible as a circle bundle over a lower dimensional affine nilmanifold in such a way that the linear holonomy of the circle fiber is trivial.

Auslander's conjecture is true in dimensions < 4 [4]. This was helpful in classifying the affine crystallographic groups in these dimensions. It also holds under the strong added assumption that N is abelian, as in 4b below.

Theorem 4a. *Let $\mathscr{A}$ be an abelian Lie algebra of nilpotent infinitesimal affine transformations of $\mathbf{R}^n$. If the linear map $\mathscr{A} \to \mathbf{R}^n$ given by evaluation at the origin is surjective, then it is bijective. In this case every vector v in $\mathbf{R}^n$*

annihilated by the linear Lie algebra $\lambda(\mathscr{A}) \subset \mathrm{Gl}(n)$ determines an infinitesimal translation in $\mathscr{A}$.

Theorem 4b. *Let G be a connected abelian Lie group of unipotent affine transformations of $\mathbf{R}^n$. If G is transitive, then it is simply transitive. In this case there is a translation in G corresponding to each invariant parallel vector field on $\mathbf{R}^n$.*

Proof. It is easy to deduce 4b from 4a.

To prove 4a, begin with $v\mathbf{R}$ and build a $\lambda(\mathscr{A})$ invariant maximal flag on $\mathbf{R}^n$. With respect to a compatible basis $v_1 = v,\ v_2, \cdots, v_n$, $\mathscr{A}$ is strictly upper triangular. We let x denote an element of $\mathscr{A}$ and let x_{ij}, $i = 1, \cdots, n$, $j = 1, \cdots, n,\ n+1$, be its entries in this coordinate system.

Claim. If $x_{i,n+1} = 0$ for $i = n,\ n-1, \cdots, j+1$, then $x_{j,k} = 0$ for $k \leqslant n$.

To see this, suppose the contrary. Let j be the largest value for which it fails and for this j, x let $k \leqslant n$ be the smallest k with $x_{j,k} \neq 0$. By the surjectivity hypothesis there is an $x' \in \mathscr{A}$ with $x'_{i,n+1} = 0$ for $i > k$, $x'_{k,n+1} = 1$. As $k > j$ we have (by our choice of $j,\ k$) that $x'_{i,l} = 0$ for $i > k$ or for $i = k,\ l \leqslant n$. Then the equation $xx' = x'x$ gives, in the $(j, n+1)$ entry, $x_{jk}x'_{k,n+1} = 0$ which is a contradiction. This proves the claim.

It follows that the only $x \in \mathscr{A}$ for which $x_{i,n+1} = 0$ for $i = 2, \cdots, n$ and $x_{1,n+1} = 1$ is the infinitesimal translation along v.

Finally the claim shows that the map $x \mapsto (x_{n,n+1}, \cdots, x_{1,n+1})$ is injective. But this is just the projection $\mathscr{A} \to \mathbf{R}^n$. Thus this projection is bijective. q.e.d.

Incidentally, the attempted proof of Auslander's conjecture in [14] fails by an unjustifiable simplification of the coordinates in its induction step.

4. Ergodic automorphisms are affine

The simplest dynamical systems are Anosov diffeomorphisms. The simplest Anosov diffeomorphisms are the hyperbolic toral automorphisms. Closely related to these are the hyperbolic nilmanifold automorphisms induced on a nilmanifold $\Gamma \backslash N$ by an automorphism α of N that preserves Γ and whose linearization α_* is a hyperbolic automorphism of the Lie algebra $\mathscr{N}$ of N. We will show that the algebraic structure of α determines an underlying affine structure, generalizing the obvious fact that toral automorphisms are affine. In fact we will be able to work with the wider class of ergodic automorphisms of $\Gamma \backslash N$, where α_* has no roots of unity as eigenvalues but possibly has some eigenvalues of unit length.

Theorem 5. *Let G be a simply connected Lie group and $\alpha : G \to G$ an automorphism of G such that $\alpha_* : T_eG \to T_eG$ has no roots of unity as eigenvalues.*

Then G admits a complete left invariant affine structure for which α is an affine map.

Proof. Let $\mathcal{G} = T_eG \otimes \mathbf{C}$ and let $V_\lambda \subset \mathcal{G}$ be the generalized λ-eigenspace of α_*. By [9, p. 19], $[V_\lambda, V_\nu] \subset V_{\lambda\mu}$. Let $\pi_\lambda : \mathcal{G} \to V_\lambda$ be the spectral projection. Note that $\pi_\lambda + \pi_{\bar\lambda}$ and $\lambda\pi_\lambda + \bar\lambda\pi_{\bar\lambda}$ are the complexifications of linear transformations of T_eG.

Choose N a nonzero integer so that $\mathrm{Re}(\lambda^N) > 0$ for all eigenvalues λ of α_*. Let $D = \Sigma(\log \lambda^N)\pi_\lambda$, where we choose the usual branch of log on $\mathrm{Re} > 0$.

If λ, μ, and $\lambda\mu$ are eigenvalues of α_*, then our choice of N implies $\log \lambda^N + \log \mu^N = \log(\lambda\mu)^N$. It follows immediately that D is a derivation of $\mathcal{G}$. Moreover $\overline{\log \lambda^N} = \log(\bar\lambda)^N$ so D is a real linear combination of operators of the form $\pi_\lambda + \pi_{\bar\lambda}$, $\lambda\pi_\lambda + \bar\lambda\pi_{\bar\lambda}$. Hence D is the complexification of a linear operator on T_eG that we again denote by D. D is a derivation of the real Lie algebra T_eG. As no $\lambda^N = 1$, D is a nonsingular linear transformation of T_eG.

It follows from [13] that the injection $\phi : G \to \mathrm{Aff}(T_eG)$ defined by $\phi(\exp v) = \exp(\mathrm{ad}_v | D_v)$ is the affine holonomy of a complete left invariant affine structure on G. Under ϕ, the automorphism α corresponds to an affine map, namely $\alpha_* \in \mathrm{Gl}(T_eG) \subset \mathrm{Aff}(T_eG)$. Thus α preserves this affine structure on G. q.e.d.

Before passing to nilmanifolds, we make a few remarks about the preceding argument. As we were somewhat free to choose N, the affine structure we constructed is not canonical. If we make the stronger assumption that α_* is hyperbolic, then the derivation $D' = \Sigma \log|\lambda|\pi_\lambda$ is nonsingular and may be used instead of D to give a natural affine structure. This case of the theorem seems to be folklore.

The argument above could also be used whenever the spectrum S of α_* may be mapped into $\{\mathrm{Re}\, z > 0,\ z \neq 1\}$ by a map f satisfying (i) $f(\bar w) = \overline{f(w)}$ and (ii) $f(xy) = f(x)f(y)$ whenever $x, y, xy \in S$. Such a hypothesis on α is strictly weaker than that of the theorem but rather less interesting.

We draw two conclusions of a dynamical nature from Theorem 5. They apply to the usual algebraic examples of Anosov diffeomorphisms and expanding maps.

Corollary 1. *An ergodic endomorphism A of a compact nilmanifold M preserves a complete affine structure on M.*

Proof. Lift A to an automorphism α of the universal cover G. Apply Theorem 5. The affine structure it gives G is left invariant and preserved by α. Hence it induces a complete affine structure on $M = \Gamma \setminus G$ that is preserved by A. q.e.d.

The following result was first proven by Bowen by different methods [2].

Corollary 2. *The entropy conjecture holds for an ergodic endomorphism of a nilmanifold.*

Proof. The entropy conjecture has a simple proof for an affine map [7].

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