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PROJECTIVE MAPPINGS AND DISTORTION THEOREMS

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1. Introduction

Distance- and volume-decreasing theorems have been investigated since Ahlfors [1] extended Schwarz's Lemma. In the complex domain, the results were distortion theorems for various holomorphic (see [9]) and even almost-complex mappings [5]. In the real domain, the theorems were obtained for certain classes of harmonic mappings, mainly by Chern [2], Goldberg [2], [6], [7], T. Ishihara [7], Petridis [7] and the present author [6], [8].

Although the notion of a projective change of a linear connection is classical, the notion of a projective mapping has not been investigated until recently. Two different notions were investigated, a weaker one by Yano and S. Ishihara [14] and a stronger by Kobayashi. The former, discussed in §2, requires the preservation of paths, while the latter, discussed in §4, requires, in addition, the preservation of the projective parameters of Whitehead [12].

In a recent paper [10], Kobayashi showed that *projective mappings* of an interval into a *Riemannian manifold whose Ricci curvature is negative and bounded away from zero are distance decreasing* up to a constant. This is generalized in §5 for mappings of a *complete Riemannian manifold whose Ricci curvature is bounded below*. In particular, this is valid for the hyperbolic open ball, which is the n-dimensional analog of Kobayashi's interval.

For projective mappings in the sense of Yano, we prove in §3 a volume-decreasing theorem, in the equidimensional case, under the same curvature requirements as above. We also show that the two notions of a projective mapping agree if the mapping is a diffeomorphism.

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2. Projective mappings and transformations

Let (M, ∇) and (M', ∇') be differential manifolds with symmetric linear connections. A curve $\gamma : I \to M$ with velocity vector $\dot{\gamma}$ is mapped by a

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smooth mapping $f : M \to M'$ to a curve $f \circ \gamma : I \to M'$ with velocity vector $f_* \dot{\gamma}$. γ is called a *path* in (M, ∇) if its acceleration vector $\nabla_D \dot{\gamma}$ is tangent to γ, that is, $\dot{\gamma}$ satisfies the differential equation $\nabla_D \dot{\gamma} = h\dot{\gamma}$ with a certain smooth function h on I, where D is the differentiation operator in $\mathbf{R}$. If an arbitrary path in (M, ∇) is mapped into a path in (M', ∇'), f is said to be a *projective mapping* (see [14]). If M' coincides with M (in the non-Riemannian case, ∇' does not coincide with ∇ necessarily), and f is a diffeomorphism, f is called a *projective transformation* of M. It is well known, (see [4]), that the identity transformation is projective if and only if there exists a smooth 1-form σ on M with the property that for any two vector fields, X, Y on M, $\nabla'_X Y - \nabla_X Y = \sigma(X)Y + \sigma(Y)X$. In this case, ∇' and ∇ are called *projectively related connections*. More generally, let M_f be the dense open submanifold of M on which rank f attains its maximum (if f has a constant rank, $M_f = M$). We prove:

Proposition 1. *Let $f : (M, \nabla) \to (M', \nabla')$ be a smooth mapping, the connections ∇, ∇' being symmetric. If f is projective, then there exists a smooth 1-form σ on M_f such that*

$$\nabla'_X f_* Y - f_* \nabla_X Y = \sigma(X) f_* Y + \sigma(Y) f_* X. \tag{1}$$

Conversely, if (1) *holds with σ defined on M, f is projective.*

($f_* Y$ is differentiated as a vector field along f, i.e., a section of the vector bundle $f^{-1}TM'$ with the connection induced from M'.)

Proof. Let $\overline{\nabla}$ be the covariant differentiation of tensor fields on M with values in the vector fields along f, i.e., the connection in the vector bundle $(\otimes TM) \otimes f^{-1}TM'$ induced from ∇ and ∇'. Consider f_* as a section of $(TM)^* \otimes f^{-1}TM'$, we have

$$\left(\overline{\nabla} f_*\right)(X, Y) = \left(\overline{\nabla}_X f_*\right) Y = \nabla'_X f_* Y - f_* \nabla_X Y.$$

If both connections are symmetric, $\overline{\nabla} f_*$ is a symmetric bilinear form on M (with values in the vector fields along f), and it is sufficient to show that

$$\left(\overline{\nabla} f_*\right)(X, X) = 2\sigma(X) f_* X,$$

or even

$$\left(\overline{\nabla} f_*\right)(\dot{\gamma}, \dot{\gamma}) = \nabla'_D f_* \dot{\gamma} - f_* \nabla_D \dot{\gamma} = 2\sigma(\dot{\gamma}) f_* \dot{\gamma}, \tag{2}$$

where γ is an arbitrary path in M, is equivalent to the projectiveness of f. Evidently, (2) implies the projectiveness. The converse is also obvious, except the linearity of σ. ($\sigma(\dot{\gamma})$ is not determined by (2) if $f_* \dot{\gamma} = 0$, a situation which does not happen if f is a transformation.)

Suppose f is projective; $f_* : M_p \to M'_{f(p)}$, $p \in M_f$, induces a splitting $M_p = \ker f_* \oplus N_p$, where N is a smooth distribution on M_f. Define $\sigma : N_p \to \mathbf{R}$ by

$$(3) \qquad \left(\overline{\nabla} f_*\right)(v, v) = 2\sigma(v) f_* v$$

for $v \in N_p$. If $v, w \in N_p$ are linearly independent, so are $f_* v$ and $f_* w$, thus

$$\begin{aligned} 2\sigma(v) f_* v + 2\sigma(w) f_* w &= \left(\overline{\nabla} f_*\right)(v, v) + \left(\overline{\nabla} f_*\right)(w, w) \\ &= \tfrac{1}{2}\left(\overline{\nabla} f_*\right)(v + w, v + w) + \tfrac{1}{2}\left(\overline{\nabla} f_*\right)(v - w, v - w) \\ &= \sigma(v + w)\left(f_* v + f_* w\right) + \sigma(v - w)\left(f_* v - f_* w\right) \\ &= (\sigma(v + w) + \sigma(v - w)) f_* v \\ &\quad + (\sigma(v + w) - \sigma(v - w)) f_* w, \end{aligned}$$

which yields $\sigma(v \pm w) = \sigma(v) \pm \sigma(w)$, and σ is linear on N_p ($\sigma(av) = a\sigma(v)$ evidently). Now, extend σ to M_p linearly by setting $\sigma|_{\ker f_*} = 0$. As N is smooth, σ is a smooth 1-form on M_f. To show that (3) holds for all $v \in M_p$, set $v = v_1 + v_0$ with $v_1 \in N_p, f_* v_0 = 0$. Then the symmetry of $\overline{\nabla} f_*$ implies

$$\begin{aligned} \left(\overline{\nabla} f_*\right)(v, v) &= \left(\overline{\nabla} f_*\right)(v_1, v_1) + 2\left(\overline{\nabla} f_*\right)(v_0, v_1) + \left(\overline{\nabla} f_*\right)(v_0, v_0) \\ &= 2\sigma(v_1) f_* v_1 = 2\sigma(v) f_* v, \end{aligned}$$

where $(\overline{\nabla} f_*)(v_0, w) = 0$ for any $w \in M_p$. ($f_* v_0 = 0$ implies $\nabla'_{v_0} f_* Y = 0$ for any vector field Y on M, because $\nabla'_{v_0}(Y' \circ f) = \nabla'_{f_* v_0} Y' = 0$ for any field Y' on M', and $f_* Y$ is locally a combination of vector fields along f with the form $Y' \circ f$. Also, a proper extension Y of w may be chosen so that $\nabla_{v_0} Y = 0$.) q.e.d.

Let R be the curvature tensor on (M, ∇), defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Then a straightforward computation together with a use of (1) shows (cf. [13, Chapter 1, Formula 4.6]; R' is defined similarly on M')

$$(4) \qquad \begin{aligned} R'\left(f_* Y, f_* Y\right) f_* Z = f_*\{ &R(X, Y)Z + (d\sigma)(X, Y)Z \\ &+ (\Box\sigma)(X, Z)Y - (\Box\sigma)(Y, Z)X\} \end{aligned}$$

on M_f, where

$$(5) \qquad (\Box\sigma)(X, Y) = (\nabla\sigma - \sigma \otimes \sigma)(X, Y) = (\nabla_X \sigma)(Y) - \sigma(X)\sigma(Y),$$

and

$$(d\sigma)(X, Y) = (\nabla\sigma)(X, Y) - (\nabla\sigma)(Y, X) = (\nabla_X \sigma)(Y) - (\nabla_Y \sigma)(X)$$

as ∇ is symmetric. If f is a projective transformation, we have

$$f_*^{-1}R'(f_*X, f_*Y)f_*Z = R(X, Y)Z + (d\sigma)(X, Y)Z + (\Box\sigma)(X, Z)Y - (\Box\sigma)(Y, Z)X.$$

Let Ric be the Ricci tensor on (M, ∇), defined by $\mathrm{Ric}(Y, Z) = \mathrm{tr}(X \to R(X, Y)Z)$, then we get (Ric$'$ is defined similarly on M')

$$\mathrm{Ric}'(f_*Y, f_*Z) = \mathrm{Ric}(Y, Z) + (d\sigma)(Y, Z) + (\Box\sigma)(Y, Z) - n(\Box\sigma)(Y, Z)$$

on $M_f = M$, or

(6) $$f^* \mathrm{Ric}' = \mathrm{Ric} - d\sigma - (n - 1)\Box\sigma.$$

The relation is equally true for a projective mapping of equidimensional manifolds, except at the singularities of f_*, i.e., at the points where f is *degenerate*.

3. A volume-decreasing theorem

Let f: $M \to M'$ be a projective mapping of equidimensional Riemannian manifolds, with the metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ and the Levi-Civita connections ∇ and ∇' respectively. Let V be the unit frame field of $\Lambda^n TM$, dual to the volume n-form on M, and set

$$u = \langle f_*V, f_*V\rangle',$$

where f_* and $\langle\ ,\ \rangle'$ are naturally extended to $\Lambda^n TM$ and $f^{-1}\Lambda^n TM'$ respectively. f is volume decreasing (up to a constant C) if and only if $u \leqslant 1$ ($\leqslant C^2$ respectively), and f is degenerate at p if and only if $u(p) = 0$. (Note that u is globally defined even if M is nonorientable.)

Let γ be a *geodesic* in M_f, and $(X_i)_{i=1}^n$ a parallel frame field along γ such that $V \circ \gamma = X_1 \wedge \cdots \wedge X_n$ and $\dot{\gamma} = X_1$. As f is projective,

$$\nabla'_D f_* X_i = \sigma(\dot{\gamma})f_*X_i + \sigma(X_i)f_*\dot{\gamma},$$

so

$$\begin{aligned}\nabla'_D f_* V \circ \gamma &= \sum_{i=1}^n f_*X_1 \wedge \cdots \wedge \nabla'_D f_* X_i \wedge \cdots \wedge f_*X_n \\ &= \sum_{i=1}^n \sigma(\dot{\gamma})f_*V \circ \gamma + \delta_{1i}\sigma(\dot{\gamma})f_*V \circ \gamma \\ &= (n + 1)\sigma(\dot{\gamma})f_*V \circ \gamma.\end{aligned}$$

Thus

$$du(\dot{\gamma}) = D(u \circ \gamma) = 2\langle f_*V \circ \gamma, \nabla'_D f_*V \circ \gamma\rangle' = 2(n + 1)\sigma(\dot{\gamma})(u \circ \gamma),$$

or

$$du = 2(n+1)u\sigma.$$

Hence, at all the points where f is nondegenerate,

$$\sigma = \frac{du}{2(n+1)u}. \tag{7}$$

(As a result we find that if f is an immersion, σ is exact.)

We now substitute σ as given by (7) in (6). We have

$$\nabla\sigma = \frac{1}{2(n+1)}\left(\frac{\nabla^2 u}{u} - \frac{du \otimes du}{u^2}\right),$$

so

$$\Box\sigma = \frac{1}{2(n+1)}\frac{\nabla^2 u}{u} - \frac{2n+3}{4(n+1)^2}\frac{du \otimes du}{u^2},$$

and also

$$d\sigma = 0.$$

Thus, at the points where $u \neq 0$,

$$f^* \operatorname{Ric}' = \operatorname{Ric} - \frac{n-1}{2(n+1)}\frac{\nabla^2 u}{u} + \frac{(n-1)(2n+3)}{4(n+1)^2}\frac{du \otimes du}{u^2}.$$

Taking the trace of both sides with respect to $\langle\ ,\ \rangle$, we obtain

$$S' = S - \frac{n-1}{2(n+1)}\frac{\Delta u}{u} + \frac{(n-1)(2n+3)}{4(n+1)^2}\frac{\langle du, du\rangle}{u^2},$$

where Δ is the Laplacian on M, S is the scalar curvature of M, and S' is the trace of $f^* \operatorname{Ric}'$. We have locally

$$S' = \sum_{i=1}^{n} (f^* \operatorname{Ric}')(E_i, E_i)$$

with (E_i) an arbitrary orthonormal frame field in M.

Theorem 1. *Let $f: M \to M'$ be a projective mapping of n-dimensional Riemannian manifolds, M being complete. If the Ricci curvature of M is bounded below by a constant $-A$, and the Ricci curvature of M' is bounded above by a constant $-B < 0$, then either f is totally degenerate, or $A > 0$ and f is volume decreasing up to a constant $(A/B)^{n/2}$.*

Proof. By the curvature assumption we have

$$\begin{aligned} S &= \sum_{n=1}^{n} \operatorname{Ric}(E_i, E_i) \geqslant -nA, \\ S' &= \sum_{i=1}^{n} \operatorname{Ric}'(f_* E_i, f_* E_i) \\ &\leqslant -B \sum_{i=1}^{n} \langle f_* E_i, f_* E_i \rangle' \\ &\leqslant -nB\big(\langle f_* E_1 \wedge \cdots \wedge f_* E_n, f_* E_1 \wedge \cdots \wedge f_* E_n \rangle'\big)^{1/n} \\ &= -nBu^{1/n}. \end{aligned}$$

Thus

$$-nBu^{1/n} \geqslant -nA - \frac{n-1}{2(n+1)} \frac{\Delta u}{u},$$

or $(B > 0)$

$$u\left(u^{1/n} - \frac{A}{B}\right) \leqslant \frac{n-1}{2n(n+1)B} \Delta u,$$

wherever $u \neq 0$. The proof is concluded by Omori-Yau maximum principle (see Lemma below), which provides a sequence of points (p_ν) in M with the properties

$$\lim_{\nu \to \infty} u(p_\nu) = \sup u (\leqslant \infty), \quad \lim_{\nu \to \infty} \frac{(\Delta u)(p_\nu)}{(u(p_\nu) + \delta)^{1+2\alpha}} \leqslant 0$$

with α, δ arbitrary positive numbers. Hence

$$\lim_{\nu \to \infty} \frac{u(p_\nu)\big((u(p_\nu))^{1/n} - (A/B)\big)}{(u(p_\nu) + \delta)^{1+2\alpha}} \leqslant 0.$$

Choose $0 < \alpha < \frac{1}{2n}$. Then the degree of the denominator is lower than the degree of the numerator, thus $\sup u$ is finite, and either $u \equiv 0$ or $0 < \sup u \leqslant (A/B)^n$. q.e.d.

The above proof uses the following version of the maximum principle, which is proved in [6].

Lemma. *Let M be a complete Riemannian manifold with Ricci curvature bounded below, and let u be a C^2 function on M. Then, for any $\alpha > 0$ and $\delta > -\sup u$, there exists a sequence (p_ν) in M such that*

$$\lim_{\nu \to \infty} u(p_\nu) = \sup u, \quad \lim_{\nu \to \infty} \frac{\|du(p_\nu)\|}{|u(p_\nu) + \delta|^{1+\alpha}} = 0, \quad \lim_{\nu \to \infty} \frac{(\Delta u)(p_\nu)}{|u(p_\nu) + \delta|^{1+2\alpha}} \leqslant 0.$$

4. Strongly projective mappings

The discussion in §3 assumed the validity of (6), which is not true in the general situation. We shall now show that a similar formula can be proven even if $\dim M \neq \dim M'$, for a restricted class of projective mappings.

We first discuss the classical situation, in which ∇' and ∇ are projectively related connections in M, i.e., the identity transformation id: $(M, \nabla) \to (M, \nabla')$ is projective. If $\gamma: I \to M$ is a ∇-geodesic, we have

$$(\nabla\sigma)(\dot\gamma, \dot\gamma) = (\nabla_{\dot\gamma}\sigma)(\dot\gamma) = \nabla_D(\sigma \circ \gamma)(\dot\gamma) = D(\sigma(\dot\gamma)) - \sigma(\nabla_D\dot\gamma) = D(\sigma(\dot\gamma)),$$

or

$$(\Box\sigma)(\dot\gamma, \dot\gamma) = D(\sigma(\dot\gamma)) - (\sigma(\dot\gamma))^2. \tag{8}$$

Let $\phi: I \to \tilde I$ be a reparameterization of γ, such that $\tilde\gamma = \gamma \circ \phi^{-1} : \tilde I \to M$ is a ∇'-geodesic. (ϕ is called an *affine parameter* with respect to ∇'.) Then

$$\dot{\tilde\gamma} = (D\phi^{-1})\dot\gamma \circ \phi^{-1} = \frac{\dot\gamma}{D\phi} \circ \phi^{-1}$$

implies

$$\nabla'_D\dot{\tilde\gamma} = \left[\frac{\nabla'_D\dot\gamma}{(D\phi)^2} - \frac{(D^2\phi)\dot\gamma}{(D\phi)^3}\right] \circ \phi^{-1} = 0,$$

or

$$\nabla'_D\dot\gamma = \frac{D^2\phi}{D\phi}\dot\gamma. \tag{9}$$

Thus, by (2), if γ is not constant, we get $2\sigma(\dot\gamma) = (D^2\phi)/(D\phi)$, and

$$(\Box\sigma)(\dot\gamma, \dot\gamma) = \frac{1}{2}\left(D\left(\frac{D^2\phi}{D\phi}\right) - \frac{1}{2}\left(\frac{D^2\phi}{D\phi}\right)^2\right) = \frac{1}{2}\mathcal{S}\phi, \tag{10}$$

where $\mathcal{S}$ is the *Schwarzian differentiation operator*. We reparametrize γ and $\tilde\gamma$ using the classical *projective parameters* [12], i.e., the solutions $p: I \to \mathbf{R}$ and $\tilde p$: $\tilde I \to \mathbf{R}$ of the differential equation

$$\mathcal{S}p = \frac{2}{n-1}\mathrm{Ric}(\dot\gamma, \dot\gamma),$$

$$\mathcal{S}\tilde p = \frac{2}{n-1}\mathrm{Ric}'(\dot{\tilde\gamma}, \dot{\tilde\gamma}) = \left[\frac{2}{n-1}\frac{1}{(D\phi)^2}\mathrm{Ric}'(\dot\gamma, \dot\gamma)\right] \circ \phi^{-1}.$$

Then, using the chain rule for $\mathcal{S}$ as well as (6) and (10), we get

$$\begin{aligned}\mathcal{S}(\tilde p \circ \phi) &= (D\phi)^2(\mathcal{S}\tilde p) \circ \phi + \mathcal{S}\phi \\ &= \frac{2}{n-1}\left(\mathrm{Ric}'(\dot\gamma, \dot\gamma) + \frac{n-1}{2}\mathcal{S}\phi\right) = \frac{2}{n-1}\mathrm{Ric}(\dot\gamma, \dot\gamma).\end{aligned}$$

This means that if $\tilde{p}$ is a projective parameter for the ∇'-geodesic $\tilde{\gamma}$, then $\tilde{p} \circ \phi$ is a projective parameter for the ∇-geodesic $\gamma = \tilde{\gamma} \circ \phi$, and if p is any other projective parameter for γ, then $\mathbb{S}(\tilde{p} \circ \phi) = \mathbb{S}p$ implies that $\tilde{p} \circ \phi = (C_1 p + C_2)/(C_3 p + C_4)$. This is the classical statement that a projective change of the symmetric affine connection preserves both paths and their projective parameters.

Definition. A smooth mapping f: $M \to M'$ is said to be *strongly projective* if it maps each path in M into a path in M', preserving the projective parameters.

By (6), a *projective transformation is strongly projective*. In the general situation, we prove

Proposition 2. *Let f: $M \to M'$ be a strongly projective mapping of manifolds with symmetric linear connection. Then for each $v \in TM$ with $f_* v \neq 0$,*

$$(f^* \operatorname{Ric}')(v, v) = \operatorname{Ric}(v, v) - \frac{n-1}{2} \mathbb{S}\phi|_{\circ},$$

where ϕ is an affine parameter for the path $t \mapsto (f \circ \exp)(tv)$. In particular, if $v \in TM_f$,

$$(f^* \operatorname{Ric}')(v, v) = \operatorname{Ric}(v, v) - (n-1)(\Box\sigma)(v, v).$$

Proof. Let γ be the geodesic in M with $\dot{\gamma}(0) = v$. Then $\tilde{\gamma} = f \circ \gamma \circ \phi^{-1}$ is a geodesic in M'. Let p, $\tilde{p}$ be projective parameters for γ and $\tilde{\gamma}$ respectively. Then

$$\mathbb{S}p = \frac{2}{n-1} \operatorname{Ric}(\dot{\gamma}, \dot{\gamma}),$$

$$\mathbb{S}(\tilde{p} \circ \phi) = \frac{2}{n-1}\left(\operatorname{Ric}'(f_* \dot{\gamma}, f_* \dot{\gamma}) + \frac{n-1}{2} \mathbb{S}\phi\right).$$

If f is strongly projective, $\mathbb{S}(\tilde{p} \circ \phi) = \mathbb{S}p$ implies

$$\operatorname{Ric}'(f_* \gamma, f_* \dot{\gamma}) = \operatorname{Ric}(\dot{\gamma}, \dot{\gamma}) - \frac{n-1}{2} \mathbb{S}\phi,$$

and the assertion follows from (10) for the path $f \circ \gamma$.

5. A distance-decreasing theorem

In this section we restrict ourselves again to the Riemannian case. We shall show that under the curvature conditions already discussed in §3, a strongly projective mapping is distance decreasing up to a constant.

Theorem 2. *Let $f : M \to M'$ be a strongly projective mapping of Riemannian manifolds, M being complete. If the Ricci curvature of M is bounded below by a constant $-A$, and the Ricci curvature of M' is bounded above by a constant*

$-B < 0$, *then either f is constant, or $A > 0$ and f is distance decreasing up to a constant* $(A/B)^{1/2}$.

Proof. We show that for each $v \in TM$ with $\|v\| = 1$, $\|f_* v\|^2 \leqslant A/B$. Let γ be the unit-speed geodesic with $\dot\gamma(0) = v$. Set $u = \|f_*\dot\gamma\|^2$. As $f \circ \gamma$ is a path, either it is constant and $u \equiv 0$, or else u is nowhere zero, in which case we show that $u \leqslant A/B$ along γ. By (9) we have, for an affine parameter ϕ for $f \circ \gamma$,

$$\frac{D^2\phi}{D\phi} = \frac{\langle \nabla'_D f_*\dot\gamma, f_*\dot\gamma\rangle'}{\langle f_*\dot\gamma, f_*\dot\gamma\rangle'} = \frac{Du}{2u}.$$

Thus

$$\frac{1}{2}\mathbb{S}\phi = \frac{1}{4}\frac{D^2u}{u} - \frac{5}{16}\left(\frac{Du}{u}\right)^2,$$

and by Proposition 2 (as $f_*\dot\gamma \neq 0$)

$$(f^* \operatorname{Ric})(\dot\gamma, \dot\gamma) = \operatorname{Ric}(\dot\gamma, \dot\gamma) - \frac{n-1}{4}\frac{D^2u}{u} + \frac{5(n-1)}{16}\left(\frac{Du}{u}\right)^2.$$

Since

$$\operatorname{Ric}(\dot\gamma, \dot\gamma) \geqslant -A\|\dot\gamma\|^2 = -A, \quad (f^* \operatorname{Ric}')(\dot\gamma, \dot\gamma) \leqslant -B\|f_*\gamma\|^2 = -Bu,$$

we have

$$-Bu \geqslant -A - \frac{n-1}{4}\frac{D^2u}{u},$$

or

$$u\left(u - \frac{A}{B}\right) \leqslant \frac{n-1}{4}D^2u.$$

Finally, we take a sequence of real numbers (t_ν) with the properties

$$\lim_{\nu\to\infty} u(t_\nu) = \sup u (\leqslant \infty), \quad \lim_{\nu\to\infty} \frac{(D^2u)(t_\nu)}{(u(t_\nu) + \delta)^{1+2\alpha}} \leqslant 0,$$

where α, δ are arbitrary positive numbers. This follows from the Lemma, or from a similar statement about $u \in C^2(-\infty, \infty)$. Hence

$$\lim_{\nu\to\infty} \frac{u(t_\nu)(u(t_\nu) - (A/B))}{(u(t_\nu) + \delta)^{1+2\alpha}} \leqslant 0,$$

and since by our assumption $u \not\equiv 0$, we obtain, for $\alpha < \frac{1}{2}$, that $0 < \sup u \leqslant A/B$.

Since a projective transformation and its inverse are strongly projective, we get

Corollary. *A projective transformation of a negatively curved complete Einstein manifold is an isometry.*

This generalizes a result of Couty [3] for infinitesimal projective transformations.

We thank Professor Kobayaski for the following remark: Since an affine transformation of an Einstein manifold is necessarily an isometry (as it preserves the Ricci tensor) the corollary follows from the main theorem of Tanaka [11].

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