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# THE UNORIENTED COBORDISM CLASSES OF COMPACT FLAT RIEMANNIAN MANIFOLDS

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This paper provides partial information on the conjecture that all compact flat riemannian manifolds are boundaries. The main result is

**Theorem 3.1.** *If $\phi$ is a finite 2-group whose elements of order two lie in its center, then any compact flat riemannian manifold with holonomy group $\phi$ is a boundary.*

Our approach is the study of certain canonical $Z_2^k$-actions called translational involutions. These involutions arise as the projections of translations of $R^n$. Theorem 3.1 is proved by first showing that the group of translational involutions has no stationary point and then appealing to the following well-known theorem of Conner and Floyd.

**Theorem** (*Conner & Floyd* [3, p. 76, Theorem 30.1]). *If $Z_2^k$ acts differentiably on the closed n-manifold $M^n$ without stationary points, then $M^n$ is an unoriented boundary.*

In §1 we prove Lemma 1.2 which relates the assumption that the group of translational involutions does have a stationary point to the two-rank of the holonomy group. Immediate corollaries provide, in §2 bounds for each finite 2-group $\phi$ on both the dimension and the first betti number of a nonbounding flat manifold with holonomy group $\phi$.

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## 1. Translational involutions

Recall that an $n$-dimensional compact flat riemannian manifold is the quotient of $R^n$ by the action of a torsion-free Bieberbach group (see Charlap [1]), i.e., by the action of a subgroup $\pi$ of the group of rigid motions of $R^n$ satisfying the following two defining conditions: (i) $\pi$ contains no elements of finite order, and (ii) there is an exact sequence

$$(1) \qquad 0 \to L \to \pi \xrightarrow{\rho} \phi \to 0$$

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in which $L$ is an $n$-dimensional lattice of pure translations of $R^n$ and where $\phi$ is a finite group which is isomorphic to the holonomy group of the quotient $R^n/\pi$. We note that the action of $\phi$ on $L$ induced by conjugation is faithful, a fact which will be crucial to our arguments. Henceforth $L$ will be viewed as a $\phi$-module via this action.

One knows (see Charlap and Vasques [2]) that for each torsion-free Bieberbach group [1], the set $(L/2L)^\phi$ of $\phi$-invariant elements of $L/2L$ is isomorphic to a subgroup of the group of affinities of $R^n/\pi$. Geometrically, this action is realized as the set of projections from $R^n$ to $R^n/\pi$ of the translations

$$x \to x + 1/2v \tag{2}$$

where $v$ ranges over all lifts to $L$ of elements of $(L/2L)^\phi$. Since $(L/2L)^\phi$ is elementary abelian, this action is a group of involutions.

**Definition 1.1.** The $Z_2^k$-action determined by $(L/2L)^\phi$ is called the group of translational involutions and is denoted by $I(\pi)$. For $W \in (L/2L)^\phi$, the corresponding involution is denoted by $I(W)$.

We adopt the convention that elements of $\pi$ are represented by pairs $(T, \sigma)$ where $T$ is a translation and $\sigma \in 0(n)$. For $x \in R^n$,

$$(T, \sigma) \cdot x = T + \sigma \cdot x.$$

The next lemma investigates the assumption that a subgroup of $I(\pi)$ has a stationary point.

**Lemma 1.1.** *Let $G$ be a nontrivial subgroup of $(L/2L)^\phi$, and suppose that the subgroup of $I(\pi)$ which $G$ determines has a stationary point $x$. Then there is an injective function $s$: $G \to \pi$, such that the composed map*

$$J = \rho \circ s : G \to \phi \tag{3}$$

*is an injective homomorphism. Further, the function $s$ may be chosen to be of the form,*

$$s(W) = (1/2v, \sigma_W), \tag{4}$$

*where $v$ is some lift of $W$ to $L$.*

*Proof.* By change of coordinates in $R^n$ we may assume that the origin covers the stationary point $x$. Choose a list $W_i$, $i = 1, p$, of the elements of $G$. Let $v_i \in L$ such that translation by $1/2v_i$ projects to the involution $I(W_i)$. $x$ stationary implies that for each $i$ there exists a unique element $(T_i, \sigma_i)$ of $\pi$ such that

$$0 + 1/2v_i = (T_i, \sigma_i) \cdot 0.$$

Note that $1/2v_i = T_i$ $\forall i$. One defines the map $s$ by $s(w_i) = (T_i, \sigma_i) \cdot s$ is injective because $1/2v_i \neq 1/2v_j$. We next show that $J$ is an injective homomorphism. Note that $J(W_i) = \rho((T_i, \sigma_i)) = \sigma_i$.

$J$ is injective: $\sigma_i = \sigma_j$, $i \neq j$, implies that $T_i - T_j$ is an element of $L$ because by (1), any two translations associated to an element of $\phi$ differ by a lattice point. Hence from the above, $1/2v_i - 1/2v_j$ is in $L$. But then, $I(W_i)$ equals $I(W_j)$. This is impossible because $W_i \neq W_j$. Thus $J$ is injective.

$J$ is a homomorphism: Since $1/2v_i = T_i$ for all $i$, we compute that

$$(T_i, \sigma_i) \cdot (T_j, \sigma_j) = (1/2v_i + 1/2v_j + l, \sigma_i\sigma_j)$$

for some $l \in L$ (since $v_j$ is $\phi$-invariant mod $2L$). Clearly, translation by $1/2(v_i + v_j)$ projects to $I(W_i + W_j)$. Since any two lifts of an element of $(L/2L)^\phi$ to $L$ differ by an element of $2L$, if $1/2v_k$ was the above previously chosen translation representing $W_i + W_j$, then $1/2v_k - 1/2(v_i + v_j)$ is in $L$. Hence $1/2v_k$ is a translation associated to $\sigma_i\sigma_j$. Further, $\sigma_i\sigma_j$ is the only element of $\phi$ with translation $1/2v_k$, because $\pi$ acts freely on $R^n$. Thus

$$J(W_i + W_j) = \sigma_i\sigma_j = J(W_i) \cdot J(W_j).$$

## 2. Two general theorems

In this section the following two theorems are proved.

**Theorem 2.1.** *Let $R^n/\pi$ be a $\phi$-manifold where $\phi$ is a finite 2-group. Suppose that the first betti number of $R^n/\pi$ exceeds the 2-rank of $\phi$. Then $R^n$ is a boundary.*

**Theorem 2.2.** *Let $R^n/\pi$ be a $\phi$-manifold where $\phi$ is a finite 2-group of order $2^p$ and of 2-rank $k$. Then if $n$ is greater than $k \cdot 2^p$, $R^n/\pi$ is a boundary.*

**Lemma 2.1.** *If the $Z_2$-dimension of $(L/2L)^\phi$ exceeds the 2-rank of $\phi$, then $R^n/\pi$ is a boundary.*

*Proof.* $\mathrm{Dim}_{Z_2}((L/2L)^\phi)$ greater than the 2-rank of $\phi$ implies that the homomorphism $J$ of Lemma 1.1 cannot be injective. Thus $I(\pi)$ has no stationary point, so the Lemma follows from Theorem 30.1 of Conner and Floyd [3].

*Proof of Theorem* 2.1. If the first betti number $\beta_1$ of $R^n/\pi$ equals $k$, then $L$ contains a $k$-dimensional trivial $\phi$-submodule (see Wolf [7, p. 81]). Hence the $Z_2$-dimension of $(L/2L)^\phi$ is no less than $\beta_1$, and so is strictly greater than the 2-rank of $\phi$. Now use Lemma 2.1.

*Proof of Theorem* 2.2. The theorem will follow from Lemma 2.1, once it has been shown that the $Z_2$-dimension of $(L/2L)^\phi$ is greater than or equal to $n/2^p$. Let $\sigma$ be a nontrivial element of order two in the center of $\phi$. As a $\sigma$-module, $L/2L$ is the direct sum of trivial factors and $Z_2$-group rings. Since the action of $Z_2$ on its $Z_2$-group ring has a unique fixed point, $\dim_{Z_2}(L/2L)^\sigma \geqslant n/2$. Now $(L/2L)^\sigma$ is a $\phi$-module because $\sigma$ is in the center of $\phi$. Thus we may repeat this argument for $\phi/\langle\sigma\rangle$ and so on.

## 3. The main theorem

A finite 2-group all of whose elements of order two lie in its center will be called central. The main result is

**Theorem 3.1.** *If $R^n/\pi$ is a $\phi$-manifold where $\phi$ is a central 2-group, then $R^n/\pi$ is a boundary.*

**Corollary 3.1.** *If $R^n/\pi$ is a $\phi$-manifold where $\phi$ is a finite group whose Sylow 2-subgroups are central, then $R^n/\pi$ is a boundary.*

*Proof.* Let $\psi$ be a Sylow 2-subgroup of $\phi$, and let $\pi'$ be its inverse image in $\pi$. The holonomy group of $R^n/\pi'$ is $\psi$, so $R^n/\pi'$ bounds by Theorem 3.1. Since the index of $\pi'$ in $\pi$ is odd, $R^n/\pi'$ is an odd-fold covering space of $R^n/\pi$. But odd-fold covers preserve Stiefel-Whitney numbers, so $R^n/\pi$ bounds as well (see Milnor & Stasheff [6]). q.e.d.

The proof of Theorem 3.1 consists in using purely algebraic facts (Propositions 3.1 and 3.2 below) concerning the cohomology of central 2-groups with coefficients in a lattice to deduce that a certain subgroup of the group of translational involutions has no stationary point. Specifically it will be shown that the assumption that there exists a stationary point implies that $\pi$ contains an element of order two. The theorem then follows by contradiction and Theorem 30.1 of Conner and Floyd [3].

We turn next to the statements of Propositions 3.1 and 3.2, but postpone their proofs until these propositions have first been applied to proving Theorem 3.1.

A pure submodule $A$ of a $\phi$-module $M$ will be said to be maximal if there exists no $\phi$-module $B$ such that $A \oplus B$ is isomorphic to a pure submodule of $M$.

**Lemma 3.1.** *Every nonzero finitely generated $\phi$-module contains a nonzero maximal semisimple submodule.*

*Proof.* Obvious.

**Definition 3.1.** Let $A$ be a $k$-dimensional semi-simple $\phi$-module where $\phi$ is an elementary abelian 2-group of rank $k$. An extension

$$(5) \qquad 0 \to A \to G \to \phi \to 0$$

will be said to be proper if it can be obtained from the following construction. Let $J$ be an isomorphism from $A/2A$ onto $\phi$. Let $s$: $A/2A \to A$ be a section of $A/2A$ in $A$, i.e., $s$ is an injective function such that $r \circ s = \mathrm{id}_{A/2A}$ where $r$: $A \to A/2A$ is the projection homomorphism. Define $g$ in $\mathrm{Hom}_{\phi}(B_1(\phi); 1/2A)$, the group of one-dimensional $\phi$-cochains with coefficients in $1/2A$, by

$$g(a) = 1/2s(J^{-1}(a)). \tag{6}$$

Since the image of $\delta g$, the coboundary of $g$, is actually contained in $A$, $\delta g$ is an element of $Z^2(\phi; A)$, the group of inhomogeneous $\phi$-cocycles with coefficients in $A$. Thus $\delta g$ determines an extension of $A$ by $\phi$. (5) is proper if it may be obtained from some such $g$ in this way.

**Proposition 3.1.** *Let $L$ be a faithful $\phi$-module where $\phi$ is an elementary abelian 2-group of rank $k$. Let $0 \to A \to G \to \phi \to 0$ be a proper extension, and let $i$: $A \to L$ be an injective homomorphism of $\phi$-modules such that*: (i) *$i(A)$ is maximal, and* (ii) *$L/i(A)$ is semisimple. Let $0 \to L \to G' \to \phi \to 0$ be the extension determined by this change of coefficients. Then $G'$ contains an element of order two.*

**Proposition 3.2.** *Let $\phi$ be a finite central 2-group, and let $M$ be a finitely generated faithful $\phi$-module. Let $C$ be a nonzero maximal semisimple $\phi$-submodule, and let $C \approx \oplus S_i$, $i = 1, k$, be a decomposition of $C$ into $\phi$-irreducible direct factors. Then for each elementary abelian subgroup $G$ of $\phi$ of rank at least $k$ there exist a faithful $G$-submodule $L$ of $M$ and a semisimple $G$-submodule $A$ of $L$ such that*:

(a) *$A$ is generated (over $Z$) by precisely those elements of $C - 2C$ which are $\phi$-invariant* mod $2M$;

(b) *$C$ splits as a $G$-direct sum, $A \oplus B$*;

(c) *If $L'$ and $A'$ are the projections of $L$ and $A$ into $M/B$, then $A'$ is maximal in $L'$ and $L'/A'$ is semisimple*;

(d) $(L \cap B) \subset 2B$.

*Proof of Theorem* 3.1. Let $0 \to M \to \pi \to \phi \to 0$ be a torsion free Bieberbach group with $\phi$ central. Choose a nonzero $\phi$-module $C$ in $M$ as in Proposition 3.2. By condition (a) of Proposition 3.2 the corresponding module $A/2A$ represents a subgroup of $I(\pi)$. Assuming that this subgroup has a stationary point $x$ we conclude from Lemma 1.1 that there is an elementary abelian subgroup $\phi'$ of $\phi$ which is isomorphic to $A/2A$ via the homomorphism $J$. Assuming that the origin of $R^n$ covers the fixed point, $J$ may be written as $\rho \circ s$ where $s$ is defined by (4). Setting $g$: $\phi' \to 1/2A$ by (6), we obtain a proper extension,

$$0 \to A \to G \to \phi' \to 0.$$

Clearly the induced extension

$$0 \to A' \to G' \to \phi' \to 0$$

is proper. Hence by (c) of Proposition 3.2 and Proposition 3.1, the extension

$$0 \to L' \to G'' \to \phi' \to 0$$

contains an element of order two. Writing $G''$ as the cartesian product $L' \times \phi'$ with multiplication given by

$$(w, \sigma) \cdot (w', \tau) = (w + \sigma \cdot w' + \delta g(\sigma, \tau), \sigma\tau), \tag{7}$$

we see that the element of order two is of the form

$$(w, \sigma) \quad \text{where } \sigma \neq \text{id}.$$

We have from (7)

$$(0, \text{id}) = (w, \sigma)^2 = (w + \sigma \cdot w + \delta g(\sigma, \sigma), \text{id}).$$

Hence $\sigma \cdot w + w = -\delta g(\sigma, \sigma)$. By condition 4) of Proposition 3.2, in $L$ we have

$$\sigma \cdot w + w = -\delta g(\sigma, \sigma) + 2b \quad \text{where } b \text{ is in } B.$$

Since $\sigma$ acts trivially on both $\sigma \cdot w + w$ and $\delta g(\sigma, \sigma)$, it acts trivially on $b$ as well. This gives

$$\sigma \cdot (w - b) + (w - b) = -\delta g(\sigma, \sigma).$$

Thus $(w - b, \sigma)$ is of order two. From this contradiction we see that $I(\pi)$ has no stationary point and hence that $R^n/\pi$ is a boundary. This completes the proof.

*Proof of Proposition* 3.1. The aim here is to locate an element $\gamma$ in $\phi$ and an element $x$ in $L$ such that $\gamma \cdot x + x = \delta g(\gamma, \gamma)$. For then, viewing $G'$ as the cartesian product $L \times \phi$ with multiplication given by

$$(v, \sigma) \cdot (w, \tau) = (v + \sigma \cdot w + \delta g(\sigma, \tau), \sigma\tau),$$

one sees that $(-x, \gamma)$ is of order two.

Since $L/i(A)$ is semisimple for all $x$ in $L$, $\gamma$ in $\phi$, the projection $r(\gamma \cdot x + x)$ of $\gamma \cdot x + x$ to $L/2L$ may be viewed as an element of $i(A)/2i(A)$. Hence there is a homomorphism

$$P : L \to \text{Hom}(\phi, \phi)$$

defined by

$$P(x)(\gamma) = J(r(\gamma \cdot x + x)). \tag{8}$$

**Lemma 3.2.** *In order to prove Proposition* 3.1 *it is sufficient to show that there exist $x$ in $L$ and $\gamma$ in $\phi$ such that*

$$P(x)(\gamma) = J(r(\delta g(\gamma, \gamma))). \tag{9}$$

*Proof.* It follows from (9) that

$$\gamma \cdot x + x = \delta g(\gamma, \gamma) + 2w \quad \text{for some } w \text{ in } L,$$

so that

$$\gamma \cdot (x - w) + (x - w) = \delta g(\gamma, \gamma),$$

since $\gamma$ acts trivially on $\delta g(\gamma, \gamma)$.

**Lemma 3.3.** *Let $M$ be a faithful semisimple $\phi$-module where $\phi$ is an elementary abelian 2-group of rank $k$. Then there exist vectors $x_i$, $i = 1, k$, in $M$ and a minimal set of generators $\sigma_i$, $i = 1, k$, of $\phi$ such that*

$$(10) \qquad \begin{aligned} \sigma_i \cdot x_j &= x_j, \quad i \neq j, \\ &= -x_j, \quad i = j. \end{aligned}$$

*Proof.* We omit the details. One proceeds by induction on $k$. q.e.d.

Since $\phi$ is elementary abelian, there are direct sum decompositions $L/i(A) \approx \bigoplus 1_i$, $i(A) \approx \bigoplus t_j$, where the $1_i$'s and $t_j$'s are one-dimensional. Since $L$ is faithful, $L/i(A) \oplus i(A)$ is faithful as well. Hence by reindexing if necessary, we may assume that $\bigoplus_{1,s} 1_i \oplus i(A)$ is a faithful $\phi$-module where $s$ equals the rank of the subgroup $\psi$ of $\phi$ of elements which act trivially on $i(A)$. Apply Lemma 3.3 to $\psi$ to get vectors $x_1, \cdots, x_s$ in $\bigoplus 1_i$ and a minimal set of generators $\sigma_1, \cdots, \sigma_s$ of $\psi$ such that $\sigma_i$ is "dual" to $x_i$ in the sense of (10). Split $\phi$ as $\psi \times K$ where $K$ is the kernel of the action of $\phi$ on $\bigoplus 1_i$. Next apply Lemma 3.3 to $K$ to get vectors $e_j$, $j = 1, k - s$, in $i(A)$ and a minimal set of generators $\tau_j$, $j = 1, k - s$, of $K$ such that $\tau_j$ is "dual" to $e_j$ in the sense of (10). Note that $\sigma_i \cdot t_j = t_j$ and $\tau_j \cdot x_i = x_i$, $\forall i, j$. We intend to choose lifts $u_i$ of the $x_i$'s to $L$ such that the homomorphisms $P(u_i)$ have a particularly nice form.

**Lemma 3.4.** *There exists lifts $u_i$ of the $x_i$'s to $L$ such that $\sigma_j \cdot u_i = u_i$ for $i \neq j$, $-u_i + w_i$ for $i = j$ for some $w_i \in i(A) - 2i(A)$ and such that $\tau_j \cdot u_i \equiv u_i$,* mod $i(A)$ *for all $i, j$.*

*Proof.* Let $T_i$ denote the $\phi$-module generated by $A$ together with an arbitrary lift of $x_i$ to $L$. $L/i(A)$ semisimple implies

$$T_i \otimes Q \approx L_i \oplus (i(A) \otimes Q),$$

where $1_i$ is one-dimensional. Let $v_i$ be a $Z$-generator of $T_i \cap L_i$. Clearly, $\sigma_i \cdot v_i = -v_i$. Since $i(A)$ is maximal, $(L_i \cap T_i) \oplus i(A)$ is not pure. Thus there exist $u_i \in T_i$, $a_i \in i(A)$, and $m \in Z - \{0, \pm 1\}$ such that $m \cdot u_i = v_i + a_i$ and $u_i \notin (L_i \cap T_i) \oplus i(A)$. Now

$$\sigma_i \cdot u_i = -1/m v_i + 1/m\, a_i = -u_i + 2/m\, a_i.$$

Note that $2/m\, a_i \in A$. In fact, $2/m\, a_i \in A - 2A$ for otherwise, $1/m\, a_i \in A$ which implies that $1/m\, v_i$ is in $T_i$, but this contradicts the fact that $v_i$ is a $Z$-generator of $(L_i \cap T_i)$. Setting $w_i = 2/m\, a_i$ takes care of the case where

$i = j$. For $i \neq j$, there is a similar computation: $\tau_j \cdot u_i \equiv u_i$, mod $i(A)$, by construction. q.e.d.

Letting $\phi^*$ denote the dual space of $\phi$ (considered as a $Z_2$-vector space) and identifying $\phi^* \otimes \phi$ with $\mathrm{Hom}(\phi, \phi)$ in the usual way, define $h_i$, $g_j$ in $\mathrm{Hom}(\phi, \phi)$ by

$$h_i = \sigma_i^* \otimes P(u_i)(\sigma_i), \quad g_j = \tau_j^* \otimes J(z_j), \tag{11}$$

where $z_j$ is the projection to $i(A)/2i(A)$ of $e_j$.

Let $F$ be the additive subgroup of $\mathrm{Hom}(\phi, \phi)$ generated by the $h_i$'s and the $g_j$'s. Note that $F$ is generated by homomorphisms which are dual to the generators $\sigma_i$ and $\tau_j$ of $\phi$. The importance of Lemma 3.4 is that none of the $h_i$'s are the zero homomorphism.

**Lemma 3.5.** *There exist $f$ in $F$ and $\gamma$ in $\phi - 0$ such that $f(\gamma) = \gamma$.*

*Proof.* The lemma is a special case of the more general fact that if $x_1, \cdots, x_k$ is a basis of a $Z_2$-vector space $V$, and if $f_1, \cdots, f_k$ are elements of $\mathrm{Hom}(V, V)$ such that $f_i(x_i) \neq 0$ for all $i$ and $f_i(x_j) = 0$ for all $i \neq j$, then for some linear combination of the $f$'s fixes a nonzero element $a$ in $V$. We omit the details. q.e.d.

Using Lemma 3.5, let $f \in F$, $\gamma \in \phi - 0$ such that $f(\gamma) = \gamma$. Factor $f$ as

$$f = \sum h_i + \sum g_j, \tag{12}$$

and remove any of the $h$'s and $g$'s which send $\gamma$ to zero. Assuming that this has already been done we see that $\gamma$ contains the factor

$$\sum \sigma_i + \sum \tau_j$$

which we write simply as $\sigma\tau$. From Lemma 3.4 and equations (8) and (11) it follows that

$$f(\gamma) = P\left(\sum u_i\right)(\sigma) + J\left(r\left(\sum e_j\right)\right),$$

where the indices $i$ and $j$ correspond to the indices in (12). This last equation is written more simply as

$$f(\gamma) = P(x)(\sigma) + J(r(e)). \tag{13}$$

By Lemma 3.2, the proof is completed with

**Lemma 3.6.** $P(x)(\gamma) = J(r(\delta g)(\gamma, \gamma))$.

*Proof.* From (6), (8), and (13), the equation $f(\gamma) = \gamma$, and the equation $s \cdot r = \mathrm{id}_A$, mod $2A$, it follows that

$$2g(\gamma) = \sigma \cdot x + x + e + 2w, \quad \text{where } w \in i(A). \tag{14}$$

Since $\gamma \cdot e = -e$ we get

$$\delta g(\gamma, \gamma) = 1/2(\tau \cdot (\sigma \cdot x + x) + \sigma \cdot x + x + 2(\gamma \cdot w + w)). \tag{15}$$

Since $A$ is a direct sum of one-dimensional $\gamma$-modules, $A$ splits as a direct sum $A^+ \oplus A^-$, where $\gamma$ acts trivially on $A^+$ and by negation on $A^-$. Hence from (15) it follows that

$$\delta g(\gamma, \gamma) = (\sigma \cdot x + x)^+, \bmod 2i(A). \tag{16}$$

The lemma now follows from (16) and the easily checked fact that $\gamma \cdot x + x = (\sigma \cdot x + x)^+$. q.e.d.

*Proof of Proposition* 3.2. The following fact will be needed.

**Lemma 3.7.** *Let $\phi$ be a central group acting on the $Q$-vector space $V$. Suppose that $V$ is $\phi$-irreducible. If $\sigma \in \phi$ is of order* 2, *then $\sigma$ either acts trivially on $v$ or $\sigma \cdot v = -v$ for all $v \in V$.*

*Proof.* Every vector space of over $Q$ which is a $Z_2$-module is isomorphic to a direct sum $V^+ \oplus V^-$, of submodules such that $v \in V^+ \Leftrightarrow \sigma \cdot v = v$, and $v \in V^- \Leftrightarrow \sigma \cdot v = -v$. Since $\sigma$ is in the center of $\phi$, $V^+$ and $V^-$ are $\phi$-modules. But $V$ is irreducible so $V = V^+$ or $V = V^-$. q.e.d.

We next construct the module $A$. $(C/2C)^\phi \approx \bigoplus (S_i/2S_i)^\phi$ where by assumption $C = \bigoplus S_i \cdot S_i$ is irreducible, so by the previous lemma each element of $G$ either acts trivially or by negation on it. Thus, if we choose a complete set of representatives of $(C/2C)^\phi$ in $C$ as generated by the union of complete sets of representatives of each $(S_i/2S_i)^\phi$, the resulting $G$-module $A$ which they generate is a direct sum of one-dimensional $G$-submodules and projects surjectively onto $(C/2C)^\phi$.

Split $M \otimes_z Q$ as $D \oplus (C \otimes Q)$. Lemma 3.7 permits us to reason exactly as in Lemma 3.3 to find a $Z_2$-basis $\sigma_i$, $i = 1, m$, of the subgroup $H$ of $G$ of elements which act trivially on $A$ and $\phi$-irreducible direct summands $D_i$, $i = 1, m$, of $D$ such that $\sigma_i$ acts trivially on each $D_j \cap M$ excepting $D_i \cap M$ upon which it acts by negation. By our assumptions each $(D_i \cap M) \oplus C$ is not pure, so by the same arguments as in Lemma 3.3 there exists $u_j \in (D_i \oplus (C \otimes Q)) \cap M$ such that

$$\sigma_i \cdot u_j = \begin{cases} u_j, & j \neq i, \\ -u_j + w_j, & i = j, \end{cases}$$

where $w_j \in C - 2C$. Further, the vectors $\sigma_1, \cdots, \sigma_m$ may be extended to a $Z_2$-basis $\sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_\rho$ where

$$\tau_i \cdot u_j \equiv u_j, \quad \bmod C.$$

Now $w_j \in C - 2C$ implies that the projection $v_j$ of $w_j$ to $C/2C$ is not zero. Since $\phi$ is a 2-group, the $\phi$-module generated by $v_j$ has nontrivial intersection with $(C/2C)^\phi - 0$. Since this module is just all $\phi$-linear combinations of $v_j$ there exists $f_j \in Z[\phi]$ such that $f_j(v_j)$ is in $(C/2C)^\phi - 0$. Set $y_j = f_j(u_j)$ and let

$L = \text{Span}_{Z[G]}(A \cup \{y_j\text{'s}\})$. Since $\phi$ is central, $\sigma_j$ commutes with $f$. Hence

$$(17) \qquad \sigma_j \cdot y_j = f_j(\sigma_j \cdot u_j) = -f_j(u_j) + f_j(w_j) = -y_j + f_j(w_j).$$

Since $f_j(w_j)$ is a lift of $f_j(v_j)$ to $C$, it is a nonzero element of $A$, mod $2C$. The same type arguments give the following equations:

$$(18) \qquad \sigma_i \cdot y_j = y_j,\ i \neq j;\ \tau_i \cdot y_j = y_j + z_{ij},$$

where $z_{ij}$ is also an element of $A$ mod $2C$. Since each element of $G$ acts either trivially or by pure negation on each $S_i$, $S_i$ splits as $(A \cap S_i) \oplus B_i$ where $B_i$ is any $Z$-complement of $A \cap S_i$. Clearly $L \cap B \subset 2B$. That $A'$ is maximal in $L'$ follows from (17). The semisimplicity of $L'/A'$ follows from (17) and (18) together.

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