

SCALAR CURVATURE AND CONFORMAL DEFORMATION OF RIEMANNIAN STRUCTURE

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1. Introduction

In this paper we consider the problem of describing the set of scalar curvature functions associated with Riemannian metrics on a given connected, but not necessarily orientable, manifold of dimension ≥ 3 . In recent work [10], [11] we have considered the analogous problem of Gaussian curvature on 2-manifolds. The key to our study of Gaussian curvatures was the Gauss-Bonnet theorem which imposes sign restrictions on the Gaussian curvatures of compact 2-manifolds depending on the Euler characteristic. There is also a topological implication of scalar curvature which provides an obstruction to positive scalar curvature for certain special manifolds.

Lichnerowicz has shown [13] that if the scalar curvature is nonnegative, but not identically zero, on a compact even-dimensional spin manifold, then there are no harmonic spinors. From this fact, using the Atiyah-Singer index theorem (see also [1]) he concluded that the Hirzebruch \hat{A} genus of such a manifold must be zero. Thus one cannot have a metric with nonnegative scalar curvature, except possibly identically zero, on a compact spin manifold whose \hat{A} genus is not zero. Examples of such manifolds arise in the theory of spin cobordism; see [14]. By a different use of the index theorem, N. Hitchin [8, Chap. 4, § 3] has recently shown that an exotic sphere which does not bound a spin manifold does not admit a metric of positive scalar curvature. We wish to thank I. M. Singer for bringing the Lichnerowicz result to our attention.

The above are the only connections known to us between scalar curvature and the topology of the underlying manifold. There is a result due to Yamabe, Trudinger, Eliasson, and Aubin which shows that there is no topological obstruction to constant negative scalar curvature. Yamabe in [20] attempted to show that any Riemannian structure on a compact manifold of dimension ≥ 3 could be pointwise conformally deformed to one of constant scalar curvature. Trudinger [19] pointed out a serious gap in Yamabe's proof, and the assertion is in doubt. However, Trudinger was able to obtain Yamabe's assertion under the additional assumption that one begins with a metric whose total scalar curvature (i.e., the integral of the scalar curvature) is nonpositive. The result-

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ing constant scalar curvature in this case is necessarily negative. More recently, Eliasson [7] and Aubin [2] have shown that every compact manifold of dimension ≥ 3 possesses a metric whose total scalar curvature is negative. This, together with Trudinger's result, shows that every compact manifold of dimension ≥ 3 admits a Riemannian metric with constant negative scalar curvature.

Our main theorem makes the much stronger assertion that there are no topological obstructions to scalar curvatures which may change sign as long as they are negative somewhere.

Theorem 1.1. *Let M be a compact manifold of dimension ≥ 3 . If $K \in C^\infty(M)$, and if K is negative somewhere on M , then there is a Riemannian structure on M with K as its scalar curvature.*

Except for scalar curvature identically zero, this result together with the results of Lichnerowicz and Hitchin provides a complete description of scalar curvatures on the special manifolds mentioned above. The existence or non-existence of the constant function 0 as a scalar curvature on these manifolds is still undetermined.

Theorem 1.2. *Let M be either a compact spin manifold with \hat{A} genus not zero or else an exotic sphere which does not bound a spin manifold. Then $K \not\equiv 0 \in C^\infty(M)$ is the scalar curvature of some Riemannian metric on M if and only if K is negative somewhere on M .*

An interesting open problem is to determine if there are other obstructions to positive scalar curvature on compact manifolds.

The next theorem concerns scalar curvature identically zero.

Theorem 1.3. *If a compact manifold of dimension ≥ 3 admits a metric of scalar curvature $k \geq 0$, then it admits a metric of scalar curvature identically zero.*

Since there are no known obstructions to zero scalar curvature, it is conceivable that every compact manifold ($\dim \geq 3$) admits a zero scalar curvature metric. This is an open question; see "Added in proof" at the end of the paper.

Much more can be said about scalar curvatures of open manifolds. In fact, for a large class of open manifolds every smooth function is a scalar curvature.

Theorem 1.4. *Let M be a noncompact manifold of dimension ≥ 3 diffeomorphic to an open submanifold of some compact manifold M_1 . Then every $K \in C^\infty(M)$ is the scalar curvature of some Riemannian metric on M .*

As a special case, any $K \in C^\infty(\mathbf{R}^n)$ is the scalar curvature of some metric on \mathbf{R}^n ($n \geq 3$). We believe that Theorem 1.4 is true for any noncompact manifold of dimension ≥ 3 ; however some additional technical work will be needed to avoid the assumption that M sits in a compact manifold.

Just as in our earlier work [10], [11], our approach to these problems is to attempt to realize the candidate function K as the scalar curvature of a metric which is either pointwise conformal or else just conformally equivalent to a prescribed metric. (We say the metrics g_1 and g are *pointwise conformal* if $g_1 = p(x)g$ for some positive function $p \in C^\infty(M)$, whereas we say that g_1 and

g are *conformally equivalent* if there are a diffeomorphism φ of M and a positive function $p \in C^\infty(M)$ such that pg is the metric obtained by pulling back g_1 under φ , i.e., $\varphi^*(g_1) = pg$. Pointwise conformal is the special case of conformal equivalence in which one demands that the diffeomorphism φ be the identity map.) Now if a given metric g on M , where $\dim M = n \geq 3$, has scalar curvature k , and we seek K as the scalar curvature of the metric $g_1 = u^{4/(n-2)}g$ pointwise conformal to g , then $u > 0$ must satisfy

$$(1.5) \quad \frac{4(n-1)}{n-2} \Delta u - ku + Ku^{(n+2)/(n-2)} = 0,$$

where Δ is the Laplacian in the g metric. Consequently, the problem of showing that K is the curvature of a metric g_1 conformally equivalent to g is precisely that of finding a diffeomorphism φ of M such that one can find a solution $u > 0$ of

$$(1.6) \quad \frac{4(n-1)}{n-2} \Delta u - ku + (K \circ \varphi)u^{(n+2)/(n-2)} = 0,$$

since then the metric $g_1 = (\varphi^{-1})^*(u^{4/(n-2)}g)$ will have scalar curvature K .

To summarize, we ask if a given function K is the scalar curvature of:

- (A) some metric on M ?
 - (B) a metric conformally equivalent to some prescribed metric g ?
 - (C) a metric pointwise conformal to some prescribed metric g ?
- (C) yes \Rightarrow (B) yes \Rightarrow (A) yes; of course, but it is a priori possible that for a given metric g , the answer to, say, (B) is "no" but the answer to (A) is "yes". In fact, we shall show that (A) yes $\not\Rightarrow$ (B) yes $\not\Rightarrow$ (C) yes.

Our results in this paper are proved by applying the techniques and results from [10] and [11] to obtain new existence and nonexistence theorems for (1.5). We shall assume the reader is familiar with those papers and shall refer to them freely. In addition, we will crucially use the existence of certain metrics (made explicit in § 3) which were established by Avez [3] for odd dimensional manifolds, and by Eliasson [7] and Aubin [2] in general.

It turns out that (1.5) is easier to analyze if we free it from geometry and consider instead

$$(1.7) \quad \Delta u - hu + Hu^a = 0, \quad u > 0,$$

where h and H are arbitrary functions, and $a > 1$ is a constant. Basic existence theorems for (1.7) are collected in § 2, where we use the method of upper and lower solutions.

In § 3 we apply the results of § 2 to prove Theorems 1.1 and 1.3. There we note that Theorem 1.1 extends to Hölder continuous K 's. Theorem 1.2 is an

obvious consequence of Theorem 1.1. Our proof of Theorem 1.1 shows that any K which is negative somewhere is the curvature of a metric which is conformally equivalent to, for example, any metric with negative scalar curvature. A precise statement is in Theorem 3.3. In carrying out our analysis of Question (B), the sign of the lowest eigenvalue $\lambda_1(g)$ of the linear part of (1.5), namely,

$$(1.8) \quad L\varphi = -\frac{4(n-1)}{n-2}\Delta\varphi + k\varphi = \lambda_1(g)\varphi,$$

plays a prominent part because *the sign of $\lambda_1(g)$ is a conformal invariant*, as we shall prove in Theorem 3.2. The sign of this eigenvalue plays the same role here as did the sign of the total curvature (which is a topological invariant by the Gauss-Bonnet theorem) in our investigation of Gaussian curvatures on compact 2-manifolds. There are topological obstructions to the existence of metrics with $\lambda_1(g) > 0$, but there are none to $\lambda_1(g) < 0$; see "Added in proof".

A study of Question (C) is made in §4–§5. We do this by examining the existence of positive solutions to (1.7). The case $\lambda_1(g) < 0$ is in §4, while $\lambda_1(g) \geq 0$ is in §5. The significant item in §5 is a gradient obstruction to solving (1.7) on S^n with the standard metric. This obstruction shows, in particular, that there are no solutions of (1.5) if $K = \text{const.} + 1\text{st order spherical harmonic}$. An application of this sheds some further light on the gap in Yamabe's paper.

Theorem 1.4 is proved in §6, where remarks on Questions (B) and (C) for open manifolds are also made.

2. Preliminaries on $\Delta u - hu + Hu^a = 0$.

Let M be a compact connected n -dimensional manifold, which is not necessarily orientable and possesses a given Riemannian structure g . We denote the volume element of this metric by dV , the gradient by ∇ , and the associated Laplacian by Δ (we use the sign convention which gives $\Delta u = u_{xx} + u_{yy}$ for the standard metric on \mathbf{R}^2). The mean value of a function f on M is written \bar{f} , that is,

$$\bar{f} = \frac{1}{\text{vol}(M)} \int_M f dV.$$

We let $H_{s,p}(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in L_p . The norm on $H_{s,p}(M)$ will be denoted by $\|\cdot\|_{s,p}$. In the special case $s = 0$, $H_{s,p}(M)$ is just $L_p(M)$, and we denote the norm by $\|\cdot\|_p$. The usual $L_2(M)$ inner product will be written $\langle \cdot, \cdot \rangle$.

We recall the Sobolev inequality which asserts that for $p > n = \dim M$ there is a constant $\gamma > 0$ such that

$$(2.1) \quad \|u\|_\infty \leq \gamma \|u\|_{1,p} \quad \text{for all } u \in H_{1,p}(M) ,$$

and the obvious extension to bounding $\|\nabla u\|_\infty$ if $u \in H_{2,p}(M)$ [4, p. 221]. A consequence of (2.1) is that u and ∇u are continuous for all $u \in H_{2,p}(M)$. We will always assume that $p > n = \dim M$.

We consider the equation

$$(2.2) \quad -Lu \equiv \Delta u - hu = -Hu^a , \quad u > 0 ,$$

where here h and H are prescribed functions, and $a > 1$ is constant. We let λ_1 be the first eigenvalue of the operator L of (2.2). Thus, if φ is the corresponding eigenfunction, which we assume normalized by $\|\varphi\|_2 = 1$, then

$$(2.3) \quad L\varphi = \lambda_1\varphi .$$

Note that φ is never zero (the variational characterization of λ_1 —see Remark 2.4 below—shows that one can take $\varphi \geq 0$, while the strong maximum principle shows that then $\varphi > 0$). Thus the eigenspace has dimension 1, and we can—and do—assume that $\varphi > 0$.

Remark 2.4. Our main concern will be with the sign of λ_1 . In practice, one observes that $\lambda_1 < 0$ under the conditions $\bar{h} \leq 0$, but $h \not\equiv 0$. This follows from the variational characterization of the first eigenvalue of L , that is,

$$\lambda_1 = \min \frac{\langle Lv, v \rangle}{\|v\|_2^2} = \min \frac{\|\nabla v\|_2^2 + \langle v, hv \rangle}{\|v\|_2^2} ,$$

where the min is taken over, say, all $v \in H_{1,2}(M)$ with $v \not\equiv 0$. In particular, by letting $v \equiv 1$ we find $\lambda_1 \leq \bar{h} \leq 0$. To rule out $\lambda_1 = 0$, observe that then $v \equiv 1$ would minimize the functional and hence would be an eigenfunction, so that $-\Delta v + hv = \lambda_1 v = 0$. But since $h \not\equiv 0$, this is impossible.

It is obvious that if $h = 0$ then $\lambda_1 = 0$, since then $\varphi \equiv 1$ is an eigenfunction. It is almost as obvious that if $h \geq 0$ ($\not\equiv 0$), then $\lambda_1 > 0$. To see this, just integrate (2.3) over M and recall that $\varphi > 0$.

Our first lemma gives an elementary sign condition on H which is necessary for a positive solution of (2.2) to exist.

Lemma 2.5. *If λ_1 and $\varphi > 0$ are as in (2.3), and u is a solution of (2.2), then*

$$\lambda_1 \langle \varphi, u \rangle = \langle \varphi, Hu^a \rangle .$$

In particular, if $u > 0$ and if H is never zero, then necessarily λ_1 and H must have the same sign. If H and h are never zero, they must have the same sign.

Proof. If u is a solution of (2.2), then

$$\lambda_1 \langle \varphi, u \rangle = \langle L\varphi, u \rangle = \langle \varphi, Lu \rangle = \langle \varphi, Hu^a \rangle .$$

Remark 2.4 and the fact that $\varphi > 0$ and $u > 0$ yield the sign conditions.

q.e.d.

If both $\lambda_1 < 0$ and $H < 0$, or else if $\lambda_1 = 0$ and $H = 0$, this necessary sign condition turns out to be sufficient (Theorem 2.11). However it is not sufficient in certain cases where both $\lambda_1 > 0$ and $H > 0$ (Theorem 5.17).

Except for Theorem 2.11, the remainder of this section concerns existence for (2.2) in the case $\lambda_1 < 0$.

Lemma 2.6. *Let $h, H \in L_p(M)$ for some $p > n = \dim M$. If there exist functions $u_+, u_- \in H_{2,p}(M)$ such that*

$$(2.7) \quad Lu_+ \geq Hu_+^a, \quad Lu_- \leq Hu_-^a,$$

with $0 < u_- \leq u_+$, then there is a $u \in H_{2,p}(M)$ satisfying (2.2). Moreover, u is $C^{j+2+\alpha}$ in any open set in which h and H are $C^{j+\alpha}$; in particular, u is C^∞ in any open set in which h and H are C^∞ .

Here $C^{j+\alpha}$ is the set of functions whose j th derivatives are continuous and satisfy a Hölder condition with exponent $0 < \alpha \leq 1$. u_+ and u_- are called *upper* and *lower* (or *super* and *sub*) solutions respectively. The L_p assumptions in this lemma will be needed to obtain the results (Theorem 1.4) on scalar curvatures on open manifolds. For scalar curvatures on compact manifolds (Theorem 1.1) we will only need the lemma with $h, H \in C^\infty$. This lemma is proved in detail in [10] for a slightly different equation. Since the proof is exactly the same (in fact it works for a fairly general class of equations) we shall not present it here. The basic idea of the proof is a standard iteration argument as follows (for simplicity assume that h and H are smooth). One lets

$$f(x, u) = hu - Hu^a, \quad k = \underset{\substack{x \in M \\ u_-(x) \leq u \leq u_+(x)}}{\text{l.u.b.}} f_u(x, u)$$

and if necessary, add a positive constant to k to insure that $k > 0$. Set $u_0 = u_+$, and then define u_{j+1} inductively as the unique solution on M of

$$\Delta u_{j+1} - ku_{j+1} = f(x, u_j) - ku_j .$$

One uses the maximum principle to show that

$$u_- \leq u_{j+1} \leq u_j \leq \dots \leq u_+ .$$

A standard argument (as in [6, pp. 370–371] for example) shows that the u_j converge to a solution u of the desired equation. Since $0 < u_- \leq u \leq u_+$, one

has $u > 0$ too. For the details, especially the treatment of the case with L_p data, we refer the reader to [10, § 9].

We will now show that if $\lambda_1 < 0$, then one can always find a positive lower solution of (2.2) less than any positive continuous function. This reduces the existence problem for (2.2) to that of finding upper solutions.

Lemma 2.8 (Existence of lower solutions). *Let $h, H \in L_p$ with $p > \dim M$. If $\lambda_1 < 0$, then given any positive continuous function u on M , there is a function $u_- \in H_{2,p}$ with $0 < u_- \leq u$ satisfying $Lu_- \leq Hu_-^a$.*

Proof. If H is bounded from below, it is easy to verify that we can let $u_- = \alpha\varphi$, where φ is the first normalized eigenfunction, and $\alpha > 0$ is a sufficiently small constant. For $H \in L_p$, let $H_1(x) = \min(-1, H(x))$, and let w be a solution of

$$(2.9) \quad Lw - \lambda_1 w = H_1 - \langle H_1, \varphi \rangle \varphi,$$

which exists since the right side is orthogonal to φ . Let $\beta > 0$ and $\gamma > 0$ be chosen so large that $w + \beta\varphi$ is positive and $\lambda_1\gamma \leq \langle H_1, \varphi \rangle$. Then $z = w + (\beta + \gamma)\varphi$ is a positive solution of (2.9). Now let the constant $\alpha > 0$ be so small that $\alpha^{a-1}z^a \leq 1$ and that $\alpha z \leq u$, where u is from the statement of the lemma. We claim that $u_- = \alpha z$ is the desired positive lower solution, which follows since

$$\begin{aligned} Lu_- - Hu_-^a &\leq Lu_- - H_1 u_-^a \\ &= \alpha[\lambda_1 z + H_1 - \langle H_1, \varphi \rangle \varphi - H_1 \alpha^{a-1} z^a] \\ &= \alpha[\lambda_1(w + \beta\varphi) + (\lambda_1\gamma - \langle H_1, \varphi \rangle)\varphi + H_1(1 - \alpha^{a-1} z^a)] \\ &\leq 0. \end{aligned}$$

Remark 2.10. One can give a more elementary proof of Lemma 2.8 if the hypothesis $\lambda_1 < 0$ is replaced by $\tilde{h} < 0$. Since this is all that is needed for most geometric applications, we sketch the proof. With $H_1 = \min(-1, H)$, choose $\alpha > 0$ so that $\langle h - \alpha H_1, 1 \rangle = 0$, and let $w \in H_{2,p}$ be a solution of $\Delta w = h - \alpha H_1$. It is straightforward to verify that $u_- = \exp(w - \beta)$ is a positive lower solution for any sufficiently large positive constant β . This can be made less than any given positive function u by choosing β still larger.

Our task now is to find upper solutions for (2.2) under hypotheses suitable for our geometric applications. This is particularly easy if $H < 0$, the result in this case constituting a proof of the sufficiency of the sign condition in Lemma 2.5.

Theorem 2.11. (a) *Assume $H \equiv 0$. Then a positive solution of (2.2) exists if and only if $\lambda_1 = 0$.*
 (b) *Assume $H \in L_p(M)$ and $H < 0$. Then a positive solution of (2.2) exists if and only if $\lambda_1 < 0$.*

Proof. (a) If $H \equiv 0$ and a positive solution exists, then $\lambda_1 = 0$ by Lemma 2.5. If $\lambda_1 = 0$, then the positive eigenfunction φ is a positive solution of (2.2) with $H \equiv 0$.

(b) With $H < 0$, if a positive solution exists, then $\lambda_1 < 0$ by Lemma 2.5. Conversely, if $\lambda_1 < 0$, then $u_+ \equiv \text{large constant}$ is an upper solution, while Lemma 2.8 guarantees the existence of a positive lower solution $u_- \leq u_+$. Thus there is a solution u by Lemma 2.6.

Remark 2.12. A useful substitute for Theorem 2.11 in the case $\lambda_1 > 0$ is: *there is some $H \in C^\infty(M)$ with $H > 0$ such that one can find a positive solution of (2.2) if and only if $\lambda_1 > 0$.* The necessity that $\lambda_1 > 0$ follows from Lemma 2.11. To prove the sufficiency, let $\varphi > 0$ be an eigenfunction of (2.3). Then

$$L\varphi = \lambda_1\varphi = (\lambda_1\varphi^{1-a})\varphi^a = H\varphi^a ,$$

where we have defined $H = \lambda_1\varphi^{1-a} > 0$.

In order to prove Theorem 1.1, we need to prove existence for (2.2) for a large class of functions H , which are negative somewhere but are permitted to change sign. *If, however, H changes sign then equation (2.2) may not have a solution* (see Prop. 4.11). This accounts for the technical assumption of the following lemma, which simply asserts that there exist upper solutions (and hence solutions by Lemma 2.8) if $h \equiv \text{const} < 0$ and H is “mostly negative”. At the cost of slightly further complication one can replace the assumption $h \equiv \text{const} < 0$ by $\lambda_1 < 0$. This extension is not needed for our purpose, so we relegate it to the reader.

For this lemma we introduce the change of variable $v = u^{1-a}$ used to linearize the classical Bernoulli equation $u' - hu = Hu^a$. Thus, let

$$(2.13) \quad v = u^{1-a} , \quad b = \frac{1}{a-1} > 0 .$$

Then $v > 0$ satisfies

$$(2.14) \quad \Delta v + \frac{h}{b}v = \frac{H}{b} + (1+b)\frac{|\nabla v|^2}{v} ,$$

which is essentially identical to one version of the equation which we studied in our earlier work (see (10.3) in [10]).

Lemma 2.15. *Let $H \in L_p(M)$ for some $p > \dim M$, and let $a > 1$ and $r < 0$ be constants. Then there is a constant $\eta > 0$ such that if $\|H + 1\|_p < \eta$ then there is a positive solution $u \in H_{2,p}(M)$ of*

$$(2.16) \quad \Delta u - ru = -Hu^a ,$$

with u smooth on any open set where H is smooth.

Proof. Let $c = -b + r/b < 0$, where $b = (a - 1)^{-1}$. By Remark 10.11 of [10], there is a constant $\eta > 0$ depending on c but independent of H such that if $\|H + 1\|_p < \eta$, then there exists a function $v \in H_{2,p}(M)$ having the properties $v > 0$, $|\nabla v|/v < 1$, and

$$T(v) \equiv \Delta v + cv - \frac{H}{b} - \frac{|\nabla v|^2}{v} \geq 0 .$$

Consequently,

$$\Delta v + \frac{r}{b}v - \frac{H}{b} - (1 + b)\frac{|\nabla v|^2}{v} = T(v) + \left(\frac{r}{b} - c - b\frac{|\nabla v|^2}{v^2}\right)v \geq 0 .$$

In view of (2.13) and (2.14), the function $u_+ = v^{-b} > 0$ is an upper solution of (2.16). But by Remark 2.10 or Lemma 2.8 there is a lower solution $0 < u_- \leq u_+$. Therefore Lemma 2.5 proves there exists a positive solution $u \in H_{2,p}$ of (2.16) with $u \in C^\infty$ on any open set where $H \in C^\infty$.

3. Scalar curvature on compact manifolds

In this section M will be a compact connected n -dimensional manifold ($n \geq 3$) which is not necessarily orientable. If M has a Riemannian metric g with scalar curvature k , we will write

$$(3.1) \quad Lu = -\frac{4(n-1)}{n-2}\Delta u + ku ,$$

where Δ is the Laplacian of the metric, and we will write $\lambda_1(g)$ —or sometimes just λ_1 —for the first eigenvalue of L . The integral $\int_M kdv$ will be called the *total scalar curvature* of the metric.

We first observe that the sign of $\lambda_1(g)$ is a conformal invariant.

Theorem 3.2. *If the metrics g_1 and g_2 on M are conformally equivalent, then $\lambda_1(g_1)$ and $\lambda_1(g_2)$ have the same sign or are both zero.*

Proof. Since g_1 and g_2 are conformally equivalent, then $g_2 = \varphi^*(pg_1)$ for some diffeomorphism φ and some positive function $p \in C^\infty(M)$. Because φ is an isometry of the metrics g_2 and pg_1 , we know that $\lambda_1(g_2) = \lambda_1(pg_1)$. Thus it is sufficient to show that $\lambda_1(pg_1)$ and $\lambda_1(g_1)$ have the same sign, that is, to show that the sign of λ_1 is a pointwise conformal invariant.

Consider (1.5). By Theorem 2.11 in the case $\lambda_1(g_1) < 0$ or the case $\lambda_1(g_1) = 0$, and Remark 2.12 in the case $\lambda_1(g_1) > 0$, we find that g_1 is pointwise conformal to a metric g_0 whose scalar curvature K_0 has the same sign as $\lambda_1(g_1)$. Therefore the metric g_0 is pointwise conformal to pg_1 . Theorem 2.11 and Remark 2.12 then show that $\lambda_1(pg_1)$ has the same sign as K_0 , which has the same sign as $\lambda_1(g_1)$.

Remark. By being more explicit in the above discussion, one can show that

$$\lambda_1(pg_1) = \lambda_1(u^{a-1}g_1) = \frac{\langle \psi, u\varphi \rangle_1}{\langle \psi, u^a\varphi \rangle_1} \lambda_1(g_1) ,$$

where we have written $p = u^{a-1}$ with $a = (n + 2)/(n - 2)$, and let $\varphi > 0$ (respectively $\psi > 0$) be the first eigenfunction of L in the g_1 (respectively, pg_1) metric. The inner product \langle , \rangle_1 is in the g_1 metric. Since the quotient on the right is positive, this evidently shows that $\lambda_1(pg_1)$ and $\lambda_1(g_1)$ have the same sign.

The next theorem, concerning conformally equivalent metrics in the case $\lambda_1 < 0$, is the main step in proving Theorem 1.1.

Theorem 3.3. *Let M have a Riemannian metric g . If the first eigenvalue $\lambda_1(g)$ of L is negative, then a function $K \in C^\infty(M)$ is the scalar curvature of a metric \tilde{g} conformally equivalent to g if and only if K is negative somewhere on M .*

Remark 3.4. In view of Remark 2.4, λ_1 will be negative if the total scalar curvature of g is negative.

Proof of Theorem 3.3. The necessity that K be negative somewhere follows from Lemma 2.5.

Sufficiency. Since $\lambda_1 < 0$, by Theorem 2.11 the function $K_1 \equiv -1$ is the scalar curvature of a metric g_1 which is pointwise conformal to the given metric g . Thus we need only to show that K is the scalar curvature of a metric g_2 conformally equivalent to g_1 , that is, that for some diffeomorphism φ of M one can find a positive solution of

$$(3.5) \quad \frac{4(n - 1)}{n - 2} \Delta_1 u + u = -(K \circ \varphi) u^{(n+2)/(n-2)} ,$$

where Δ_1 is the Laplacian in the g_1 metric. For convenience write $s = (n - 2)/(4n - 4)$.

Let φ be a diffeomorphism of M , which makes $K \circ \varphi$ nearly (in $L_p(M)$) equal to a negative constant. Then there is a constant $\alpha > 0$ such that $\alpha s(K \circ \varphi)$ is nearly equal to -1 in L_p on M . More precisely, given $\varepsilon > 0$ we can choose the above diffeomorphism φ and constant $\alpha > 0$ so that

$$\| \alpha s(K \circ \varphi) + 1 \|_p < \varepsilon$$

for some $p > \dim M$. This enables us to apply Lemma 2.15 with $r = -s$ and $H = \alpha s(K \circ \varphi)$ to conclude that there is a diffeomorphism φ such that a positive solution v of

$$\frac{4(n - 1)}{n - 2} \Delta_1 v + v = -\alpha(K \circ \varphi)v^{(n+2)/(n-2)}$$

exists. Therefore $u = \alpha^{\frac{1}{2}(n-2)}v$ is the desired positive solution of (3.5). (See Theorem 11.6 of [10] for more details of a similar proof.) q.e.d.

In order for Theorem 3.3 to be useful, it is important to know the following.

Lemma 3.6 (cf. [7, § 1]). *On any compact connected manifold of dimension ≥ 3 there exists a Riemannian metric g whose total scalar curvature is negative, and hence having $\lambda_1(g) < 0$.*

Elíasson actually proves that the total scalar curvature can assume any real value, even if one also requires that the total volume is 1. He does this by fixing a metric g_0 on M and seeking the desired metric g in the form $g = \theta(\psi g_0 + d\varphi \otimes d\varphi)$, where θ , ψ and φ are smooth functions with $\theta > 0$ and $\psi > 0$. By an explicit computation he shows that there are functions θ , ψ , φ so that the total scalar curvature has the desired value.

Our main result, Theorem 1.1, is now quite easily proved.

Proof of Theorem 1.1. Given M and $K \in C^\infty(M)$ with K negative somewhere, let g be a metric having $\lambda_1(g) < 0$. Such a metric exists by Lemma 3.6. Then Theorem 3.3 shows that K is the scalar curvature of a metric conformally equivalent to g .

Remark 3.7. By appealing to the full regularity assertion of Lemma 2.5, it is easy to see that if in Theorem 1.1 one only assumes K is Hölder continuous, then there is a metric with Hölder continuous second derivatives having K as its scalar curvature.

Lemma 3.6 shows that on any M there is always a metric g with $\lambda_1(g) < 0$. It is unknown if there always exist metrics with $\lambda_1(g) = 0$; see "Added in proof". However, there are topological obstructions to having a metric with $\lambda_1(g) > 0$.

Proposition 3.8. *M admits a metric g with $\lambda_1(g) > 0$ if and only if M admits a metric having strictly positive scalar curvature.*

Thus the manifolds of Theorem 1.2 do not admit a metric g with $\lambda_1(g) > 0$.

Proof. This is immediate from Remark 2.12 applied to (1.5). It shows that if $\lambda_1(g) > 0$, then g is pointwise conformal to a metric having strictly positive scalar curvature. q.e.d.

The next theorem is a partial answer to the existence of a metric with $\lambda_1(g) = 0$.

Theorem 3.9. *If M admits a metric g_1 with $\lambda_1(g_1) > 0$, that is, if M admits a metric having positive scalar curvature, then it admits a metric g with $\lambda_1(g) = 0$.*

Proof. By Lemma 3.6 there is a metric g_0 with $\lambda_1(g_0) < 0$. Let $g_t = tg_1 + (1-t)g_0$ for $0 \leq t \leq 1$, and let L_t be the operator of (3.1) corresponding to the metric g_t . Since L_t depends continuously—even analytically—on t , the first eigenvalue $\lambda_1(g_t)$ depends continuously on t [9, VII, § 6, esp. § 6.5]. However $\lambda_1(g_0) < 0$ and $\lambda_1(g_1) > 0$, so $\lambda_1(g_\tau) = 0$ for some $0 < \tau < 1$. q.e.d.

The proof of Theorem 1.3 is almost an immediate consequence.

Proof of Theorem 1.3. Combine Theorem 3.9 with part (a) of Theorem

2.11 applied to (1.5), to obtain the desired metric. q.e.d.

The analogue of Theorem 3.3 on Question (B) is open for the case of a metric g with either $\lambda_1(g) = 0$ or $\lambda_1(g) > 0$; see "Added in proof".

4. Pointwise conformal deformation of scalar curvature: The case $\lambda_1(g) < 0$

Let M be as in § 3. Question (C) concerning pointwise conformation deformation of scalar curvature is the subject of this section. Thus we are discussing existence and nonexistence of positive solutions for (1.5), where k is prescribed in advance. We wish to determine the conditions on K which guarantee existence Lemma 2.5 shows that K must satisfy an elementary sign condition depending on the first eigenvalue λ_1 of (3.1). If K satisfies this sign condition, can one solve (1.5)? We will show that the answer to this is "no" in general.

This section will consider the case $\lambda_1 < 0$, while the cases $\lambda_1 = 0$ and $\lambda_1 > 0$ —for which our results are rather fragmentary—are in § 5. Our proofs are adapted from similar results in §§ 10 and 11 of [10].

Given a smooth metric g on M , let $PC(g)$ denote the set of functions $K \in C^\infty(M)$ which are the scalar curvatures of metrics pointwise conformal to g ; in other words, $PC(g)$ is the set of functions for which one can find a positive solution of (1.5).

Theorem 4.1. *Assume $K < 0$. Then $K \in PC(g)$ if and only if $\lambda_1(g) < 0$.*

Proof. An immediate consequence of Theorem 2.11.

Theorem 4.2 (Uniqueness). *If $K \leq 0$ ($\neq 0$), then K is the scalar curvature of at most one metric pointwise conformal to the given metric g .*

Proof. We must show that (1.5) has at most one positive solution. By the change of variable $u = \exp(bw)$, where $b = (a - 1)^{-1}$, w is a solution of

$$\Delta w + b|\nabla w|^2 - \frac{k}{b} + \frac{K}{b}e^w = 0.$$

If $K \leq 0$ ($\neq 0$), then there is at most one solution of this equation by a standard maximum principle argument. See [6, pp. 322–323] or [4, pp. 283–284].

q.e.d.

If one still assumes that $\lambda_1 < 0$, but allows K to be positive occasionally, then it is more involved to determine if $K \in PC(g)$. Lemma 2.15 shows that if K is not "too positive too often", then indeed $K \in PC(g)$. For the remainder of this section, we will investigate the border between existence and nonexistence. Since by Theorem 4.1 if $\lambda_1(g) < 0$ then one can always pointwise conformally deform g to a metric of constant negative scalar curvature, we can without loss of generality restrict our attention to the case where the given metric already has a constant negative scalar curvature $k \equiv -c$, where $c > 0$ is a constant. Thus (1.5) reads

$$(4.2) \quad \frac{4(n-1)}{n-2} \Delta u + cu = -Ku^{(n+2)/(n-2)}, \quad u > 0.$$

In order to understand (4.2), one must first free it from geometric considerations and examine

$$(4.3) \quad \Delta u + \gamma u = -Hu^a, \quad u > 0,$$

where $a \equiv \text{const} > 1$, $\gamma \equiv \text{const} > 0$ and $H \in C^\infty(M)$ are not tied to geometry. Under the change of variable $v = u^{1-a}$, as in (2.13) and (2.14), we find that

$$(4.4) \quad \Delta v - \frac{\gamma}{b}v = \frac{H}{b} + (1+b)\frac{|\nabla v|^2}{v}, \quad v > 0,$$

with $b = (a-1)^{-1} > 0$.

First we show that if H is “too positive too often”, then a positive solution of (4.3) will not exist.

Proposition 4.5. *A necessary condition for a solution of (4.3) to exist is that the unique solution of*

$$(4.6) \quad \Delta \varphi - \frac{\gamma}{b}\varphi = \frac{H}{b}$$

be positive, i.e., $\varphi > 0$. A weaker necessary condition is $\bar{H} < 0$.

Proof. We work with (4.4), which we assume has a solution $v > 0$. Then $z = v - \varphi$ is a solution of

$$\Delta z - \frac{\gamma}{b}z = (1+b)\frac{|\nabla v|^2}{v} \geq 0,$$

which, by the maximum principle, implies that $v \leq \varphi$. But $v > 0$. Thus $\varphi > 0$. Integrating (4.6) shows that $\bar{H} < 0$, which is weaker than $\varphi > 0$ as we shall prove in Proposition 4.12.

Proposition 4.7. *If (4.3) has a solution for given (γ, H) , and if $0 < \gamma_1 \leq \gamma$ and either $H_1 \leq H$ or $H_1 = \alpha H$ for some constant $\alpha > 0$, then (4.3) has a solution given (γ_1, H_1) .*

Proof. Assume $\gamma_1 \leq \gamma$ and $H_1 \leq H$. In view of Lemmas 2.6 and 2.8, it is sufficient to find an upper solution u_+ of (4.3). Let $u > 0$ be the solution for the given (γ, H) . Then

$$\Delta u + \gamma_1 u + H_1 u^a = (\Delta u + \gamma u + H u^a) + (\gamma_1 - \gamma)u + (H_1 - H)u^a \leq 0.$$

Thus a solution exists given (γ_1, H_1) . Next, say $H_1 = \alpha H$, and u is a solution of (4.3). Then $\alpha^{-b}u$ is a solution of (4.3) which H replaced by $H_1 = \alpha H$. Therefore a solution exists for (γ, H_1) and hence for (γ_1, H_1) .

Proposition 4.8. *If $\bar{H} < 0$, then there is a constant $0 < \gamma_0(H) \leq \infty$ such that one can solve (4.3) for $0 < \gamma < \gamma_0(H)$ but not for $\gamma > \gamma_0(H)$.*

Proof. In view of Proposition 4.7, it is sufficient to show that given H with $\bar{H} < 0$, there is a constant $\gamma > 0$ small enough so that (4.7) has a solution. As before, Lemmas 2.6 and 2.8 reduce this to the existence of an upper solution u_+ . Writing $v = u_+^{1-a}$ as in (4.4), it is sufficient to find $v > 0$ satisfying

$$(4.9) \quad \Delta v - \frac{\gamma}{b}v - \frac{H}{b} - (1 + b)\frac{|\nabla v|^2}{v} \geq 0 .$$

Let $\Delta\psi = (H - \bar{H})/b$, and let $v = \psi + \mu$, where the constant μ is chosen so large that both $v > 0$ and

$$\frac{\bar{H}}{2b} < -(1 + b)\frac{|\nabla\psi|^2}{\psi + \mu} .$$

Then pick $\gamma > 0$ so small that

$$\frac{1}{2}\bar{H} < -\gamma(\psi + \mu) .$$

It is clear that v satisfies (4.9). **q.e.d.**

Our next proposition discusses when $\gamma_0(H) = \infty$.

Proposition 4.10. *If $H < 0$, then $\gamma_0(H) = \infty$. However, if $H(x_0) > 0$ for some $x_0 \in M$, then $\gamma_0(H) < \infty$.*

It would be pleasant if one could improve this to read “ $\gamma_0(H) = \infty$ if and only if $H \leq 0$ ($\neq 0$)”, much as in Theorem 10.5 (a) of [10].

Proof. If $H < 0$, then $\gamma_0(H) = \infty$ by Theorem 2.11 (in this case, one can even use constants for u_+ and u_-).

To prove the second half, recall that by Proposition 4.5 if a solution exists, then the unique solution φ of (4.6) must be positive. But the Asymptotic Theorem 4.4 in [10] asserts that

$$\lim_{\gamma \rightarrow \infty} \gamma\varphi(x; \gamma) = -H(x)$$

uniformly in x . Since $H(x_0) > 0$, $\varphi(x_0; \gamma) < 0$ for $\gamma > 0$ sufficiently large. Hence $\gamma_0(H) < \infty$. **q.e.d.**

The next proposition shows that if one allows H to be positive occasionally, then the critical constant $\gamma_0(H)$ may be arbitrarily small. This shows why one needs the technical hypothesis in Lemma 2.15 to obtain existence in this case.

Proposition 4.11. *Given any fixed $\gamma > 0$, there exists $H \in C^\infty(M)$ with $\bar{H} < 0$ such that $\gamma > \gamma_0(H)$; i.e., for these γ and H , (4.3) has no solution.*

Proof. By Proposition 4.5, we need only to find H such that the solution of (4.6) is not everywhere positive. Pick a smooth function $\psi \not\equiv 0$ with $\bar{\psi} = 0$. Let $\alpha > 0$ be so small that $\psi + \alpha$ still changes sign, and let $H = b\Delta\psi -$

$\gamma(\psi + \alpha)$. Then $\bar{H} = -\gamma\alpha < 0$, and the unique solution of (4.6) is $\varphi = \psi + \alpha$ which is not everywhere positive. q.e.d.

Finally, we apply these results for (4.3) to (4.2).

Theorem 4.12. *Let M be a compact connected manifold of dimension ≥ 3 with a metric g of constant negative scalar curvature $k \equiv -c$, and let $K \in C^\infty(M)$.*

(a) *If $K \in \text{PC}(g)$, then the unique solution of*

$$(n - 1)\Delta\varphi - c\varphi = K$$

must be positive, i.e., $\varphi > 0$.

(b) *If $K \in \text{PC}(g)$ and either $K_1 \leq K$ or else $K_1 = \alpha K$ for some constant $\alpha > 0$, then $K_1 \in \text{PC}(g)$.*

(c) *If $\bar{K} < 0$, there exists a constant $\gamma_0(K) > 0$ such that $K \in \text{PC}(g)$ for $c < \gamma_0(K)$ but not for $c > \gamma_0(K)$.*

(d) *There exists $K \in C^\infty(M)$ with $\bar{K} < 0$ such that $K \notin \text{PC}(g)$.*

Proof. These follow from Propositions 4.5, 4.7, 4.8, and 4.11 respectively.

Remark 4.13. In calculus of variations approaches to solving (2.2), such as used by Yamabe, Trudinger, and Eliasson, one attempts to realize a solution as an extremum of a suitable functional defined on $H_{1,2}(M)$. In this approach one needs to have $H_{1,2}$ compactly imbedded in L_{a+1} where a is the exponent in (2.2). There is, therefore, difficulty with exponents $a \geq (n + 2)/(n - 2)$ since the imbedding $H_{1,2} \rightarrow L_p$ is not compact for $p = 1 + (n + 2)/(n - 2) = 2n/(n - 2)$ and since there is no imbedding of $H_{1,2}$ into L_p for $p > 2n/(n - 2)$. Using the non-variational technique of upper and lower solutions one can circumvent these difficulties. We have in (4.1) generalized Corollary 1 of [19] to show that with $K < 0$ and $\bar{k} \leq 0$ ($k \neq 0$), (1.5) has solutions with the exponent $(n + 2)/(n - 2)$ replaced by any constant $a > 1$.

5. Pointwise conformal deformation of scalar curvature: The case $\lambda_1(g) \geq 0$

Little is known about the existence of a positive solution to (1.5) if $\lambda_1(g) \geq 0$. We shall give a necessary condition for a solution to exist in the case $\lambda_1(g) = 0$ and also in the case where $M = S^n$ with the standard metric. To begin, we note the following which is an immediate consequence of Theorem 2.11 (a) applied to (1.5).

Proposition 5.1. *The function $K \equiv 0$ belongs to $\text{PC}(g)$ if and only if $\lambda_1(g) = 0$.*

Consequently, for Question (C) in the case $\lambda_1(g) = 0$ one needs only to find which functions K are scalar curvatures of a metric pointwise conformal to a metric with scalar curvature zero, that is, for which $K \in C^\infty(M)$ one can find a positive solution of (see (1.5))

$$(5.2) \quad \frac{4(n-1)}{n-2} \Delta u = -Ku^{(n+2)/(n-2)} .$$

As we saw in Lemma 2.5, an obvious necessary condition for a positive solution to exist is that either $K \equiv 0$ or else K changes sign. There is another necessary condition too.

Proposition 5.3. *If a positive solution u of (5.2) exists and $K \not\equiv 0$, then K must change sign and $\int_M K dV < 0$.*

Proof. To simplify notation, rewrite (5.2) as $\Delta u = -Hu^a$. Then multiply by u^{-a} and integrate by parts to find

$$-\int_M H dV = \int_M \frac{\Delta u}{u^a} dV = a \int_M \frac{|\nabla u|^2}{u^{a+1}} dV > 0 . \quad \text{q.e.d.}$$

It is not known if these two necessary conditions on K for solvability of (5.2) are sufficient, although by analogy with Theorem 5.3 of [10], one might suspect that they are.

Given a metric g and a function $K \in C^\infty$, if $\lambda_1(g)$ and K have opposite signs, then $K \notin \text{PC}(g)$ by the sign condition of Lemma 2.5. On the other hand, if $\lambda_1(g) < 0$ and $K < 0$, then $K \in \text{PC}(g)$ by Theorem 4.1. This leads one to guess that if $\lambda_1(g) > 0$ and $K > 0$, then $K \in \text{PC}(g)$. The main result of this section is to prove that in certain cases this is false. We will consider (1.5) on the sphere $S^n \subset \mathbf{R}^{n+1}$ with its standard metric g . Then the scalar curvature is a positive constant, so $\lambda_1(g) > 0$. We will exhibit positive functions K which do not belong to $\text{PC}(g)$. Our proof is similar to Theorem 8.8 of [10] concerning $\Delta u = 1 - Ke^{2u}$ on S^2 . As we shall see, the sphere S^n appears to be the only compact manifold on which this type of obstruction to the solvability of (1.5) can occur.

The Basic Identity (8.1) of [10] asserts that for any functions u and F

$$(5.4) \quad 2\Delta u \nabla u \cdot \nabla F \equiv -(2H_F - (\Delta F)g)(\nabla u, \nabla u) ,$$

where H_F denotes the Hessian (2nd covariant derivative) of F , g is the metric tensor, and the symbol " \equiv " is used to denote equality modulo terms which are divergences. (On \mathbf{R}^n with its standard basis, g is of course the identity matrix, and H_F the matrix of second derivatives.)

We now assume that our underlying manifold is the unit sphere S^n with g its standard metric. If F is a first order spherical harmonic on S^n , that is, a non-trivial solution of

$$(5.5) \quad \Delta F = -nF$$

on S^n , then $H_F = -Fg$ so that

$$(5.6) \quad 2H_F - (\Delta F)g = (n - 2)Fg .$$

Substituting (5.6) in (5.4) we obtain

$$(5.7) \quad 2\Delta u \nabla u \cdot \nabla F \equiv -(n - 2)F |\nabla u|^2 .$$

But

$$F |\nabla u|^2 = \frac{1}{2}F\Delta(u^2) - Fu\Delta u \equiv -\frac{1}{2}nu^2F - Fu\Delta u \quad (\text{by (5.5)}) .$$

Thus (5.7) becomes

$$(5.8) \quad 2\Delta u \nabla u \cdot \nabla F \equiv (n - 2)(\frac{1}{2}nu^2F + Fu\Delta u)$$

for any function u on S^n .

Consider now u to be a solution of the equation

$$(5.9) \quad \Delta u = -q(x, u)$$

on S^n , and let

$$(5.10) \quad Q(x, u) = \int_{\gamma}^u q(x, s)ds ,$$

where γ is a constant to be chosen at one's convenience depending on the specific form of q . Considering u as a function of $x \in S^n$ we have

$$(5.11) \quad \nabla Q = Q_x + q(x, u)\nabla u ,$$

where Q_x denotes the gradient of $Q(x, u)$ on S^n with the variable u held constant. Now use (5.9) on both sides of (5.8) to obtain

$$(5.12) \quad -2q\nabla u \cdot \nabla F \equiv (n - 2)(\frac{1}{2}nu^2F - Fuq) .$$

Applying (5.11) and (5.5) to the left hand side of (5.12) gives

$$(5.13) \quad \begin{aligned} -2q\nabla u \cdot \nabla F &= 2Q_x \cdot \nabla F - 2\nabla Q \cdot \nabla F \equiv 2Q_x \cdot \nabla F + 2Q\Delta F \\ &= 2Q_x \cdot \nabla F - 2nQF . \end{aligned}$$

From (5.13) and (5.12) it follows that

$$(5.14) \quad 2Q_x \cdot \nabla F \equiv \frac{1}{2}n(n - 2)u^2F + [2nQ - (n - 2)uq]F .$$

This identity will yield the desired obstructions since for certain functions q the integral of the left side is positive while the integral of the right is negative or zero. We consider a special case.

Assume that $n \geq 3$ and that $q(x, u)$ has the form

$$(5.15) \quad q(x, u) = cu + H(x)u^a ,$$

where both c and a are constants with $a > 1$ and $H \in C^\infty(M)$. Then with $\gamma = 0$ in (5.10), one finds that (5.14) becomes

$$(5.16) \quad \frac{2u^{a+1}}{a+1} \nabla H \cdot \nabla F \equiv \left[\frac{1}{2}n(n-2) + 2c \right] u^2 F + \left[\frac{2n}{a+1} - (n-2) \right] H u^{a+1} F .$$

In order to get an obstruction upon integrating (5.16), one needs control over the signs of the individual terms. Since each spherical harmonic F of degree 1 changes sign, the integral of the first term on the right side of (5.16) will be of indeterminate sign unless its coefficient is zero. Therefore we assume that c has the particular value

$$c = -\frac{1}{4}n(n-2) .$$

Note that the scalar curvature k of S^n is $n(n-1)$ so that the coefficient of u in (1.5) is $-\frac{1}{4}n(n-2)$ once the equation has been multiplied through by $\frac{1}{4}(n-2)/(n-1)$ to make the coefficient of Δu equal to 1. Thus the above value for c is precisely the value of geometric interest. We now integrate (5.16).

Theorem 5.17. *If u is a positive solution of the equation*

$$(5.18) \quad \Delta u - \frac{1}{4}n(n-2)u + Hu^a = 0$$

on the standard n -sphere ($n \geq 3$), then

$$(5.19) \quad \int_{S^n} u^{a+1} \nabla H \cdot \nabla F dV = \frac{1}{2}(n-2) \left(\frac{n+2}{n-2} - a \right) \int_{S^n} u^{a+1} H F dV$$

for all spherical harmonics F of degree 1.

For $a = (n+2)/(n-2)$, the right side of (5.19) is zero, and we see that (5.18) has no positive solutions if H is any function such that $\nabla H \cdot \nabla F_0$ has a fixed sign for some spherical harmonic F_0 of degree 1. In particular, there are no positive solutions for H of the form $H = \text{const} + F_0$. If $a > (n+2)/(n-2)$ and if H is a spherical harmonic F_0 of degree 1, the two sides of (5.19) have opposite signs for $F = F_0$, so again there can be no positive solution of (5.18).

Corollary 5.20. *If $K \in C^\infty(S^n)$ is a spherical harmonic of degree 1, or more generally if $\nabla K \cdot \nabla F_0$ has a fixed sign for some spherical harmonic F_0 of degree 1, then K cannot be realized as the scalar curvature of a metric point-wise conformal to the standard metric on S^n .*

Remark 5.21. It is natural to inquire if one can obtain obstructions to the

existence of solutions as in Theorem 5.17 on manifolds other than S^n . The key role of the sphere in the above derivation appears in the existence of a nontrivial function F satisfying (5.5), (5.6) and therefore $H_F + Fg = 0$. A theorem of Obata [15] asserts that if a complete connected Riemannian manifold M of dimension $n \geq 2$ admits a nontrivial solution φ of $H_\varphi + c\varphi g = 0$, $c > 0$, where g is the metric tensor, then M is isometric to a standard n -sphere of radius $1/\sqrt{c}$ in Euclidean $(n + 1)$ -space. Hence it appears that obstructions of the form (5.19) to the existence of solutions of (5.18) are peculiar to the sphere.

In this regard, it is interesting to note another, perhaps more conceptual, method for deriving (5.14) and hence Theorem 5.17 and Corollary 5.20. It is based on an idea of G. Rosen [18]. A solution u of (5.9) is a critical point of the functional

$$(5.22) \quad J(u) = \int_M [|\nabla u|^2 - 2Q(x, u)]dV .$$

Let $\varphi_\lambda: S^n \rightarrow S^n$ be the (conformal) diffeomorphism induced on S^n under stereographic projection from \mathbf{R}^n of the map $\bar{\Phi}_\lambda: \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by $\bar{\Phi}_\lambda(z) = \lambda z$. Then $\varphi_1 = \text{id}$, so

$$(5.23) \quad \left. \frac{dJ(u \circ \varphi_\lambda)}{d\lambda} \right|_{\lambda=1} = 0 .$$

If one carries out the computation of (5.23), using $F|\nabla u|^2 \equiv -\frac{1}{2}nu^2F + Fuq(x, u)$ (see before (5.8)), one obtains (5.14).

Remark 5.24. If H is positive somewhere, one can solve (5.18) by the calculus of variations for exponents $a < (n + 2)/(n - 2)$. Hence if $H = F_0$, a spherical harmonic of degree 1, then (5.18) has solutions for all $1 < a < (n + 2)/(n - 2)$ but, according to Theorem 5.17, not for $a = (n + 2)/(n - 2)$. This has some bearing on Yamabe's attempt to prove that on any compact Riemannian manifold of dimension $n \geq 3$, (1.5) has a solution with K a suitable constant [20]. In order to circumvent difficulties with the exponent $(n + 2)/(n - 2)$ (see Remark (4.13)), Yamabe looked for the solution of (1.5) as a limit of solutions u_a of this equation with exponent a approaching $(n + 2)/(n - 2)$ from below. To find the solutions u_a he used the calculus of variations. The case of (5.18) in which $H = F_0$ is a special case of (1.5) in which Yamabe's method necessarily fails. Yamabe's situation differs in the respect that he was looking for a solution with K constant. If his method is to work, it must make critical use of this fact. The above suggests that his method may well fail even with K constant.

The gap in Yamabe's proof was indicated by Trudinger [19] who also gave an objection to Yamabe's method similar to ours above but for a different equation. Trudinger pointed out that Yamabe's method fails in the analogous case of the Dirichlet problem for the equation $\Delta u = -\lambda u^a$, $\lambda > 0$, on a star-

like domain Ω in \mathbf{R}^n . For this equation Pohožaev [17] has shown that there are positive solutions (with $u = 0$ on $\partial\Omega$) for $a < (n + 2)/(n - 2)$ but no positive solutions for the critical exponent $(n + 2)/(n - 2)$. On the other hand, as is pointed out in [12], for any $a > 1$ there is a positive solution if $\Omega \subset \mathbf{R}^n$ is a shell $\{r_1 < |x| < r_2\}$. Thus the existence of a positive solution in this case depends on geometric aspects of Ω (for further discussion, see [12]).

6. Scalar curvature on open manifolds

Proof of Theorem 1.4. M is now a noncompact manifold of dimension ≥ 3 diffeomorphic to an open submanifold of a compact manifold M_1 , and $K \in C^\infty(M)$. With no loss of generality, we can assume that $M_1 - M$ contains an open set, that M and M_1 are connected, and that M_1 has a metric g of constant scalar curvature $k \equiv -1$ (such a metric exists by Theorem 1.1). All L_p statements on M and M_1 will be with respect to this metric.

By Proposition 2.5 of [11] there is a diffeomorphism φ of M such that $K_1 \equiv K \circ \varphi \in L_p(M)$ for some $p > \dim M$. Extend K_1 to M_1 by defining it to be identically equal to $-1/s$, where $s = \frac{1}{4}(n - 2)/(n - 1)$, on $M_1 - M$. Then the extended $K_1 \in L_p(M_1)$ and equals $-1/s$ on some open set in M_1 . Therefore, given any $\varepsilon > 0$, by Proposition 2.6 of [11] there is a diffeomorphism ψ of M_1 such that $\|sK_1 \circ \psi + 1\|_p < \varepsilon$. This fact together with Lemma 2.15 guarantees that there is a diffeomorphism ψ of M_1 such that there exists a positive solution $u \in H_{2,p}(M)$ of

$$\frac{4(n - 1)}{n - 2} \Delta u + u = -(K_1 \circ \psi)u^{(n+2)/(n-2)},$$

with u smooth on any open set where $K_1 \circ \psi$ is smooth.

In particular, u is C^∞ on $\psi^{-1}(M) \subset M_1$. Consequently $g_1 = u^{4/(n-2)}g$ is a metric on $\psi^{-1}(M)$ with scalar curvature $K_1 \circ \psi$, so that K_1 is the curvature of the pulled-back metric $\psi^{-1*}(g_1)$ on M , and finally K is the curvature of the metric $\varphi^{-1*}[\psi^{-1*}(g_1)]$ on M . q.e.d.

Let us consider Questions (B) and (C). An immediate consequence of our proof of Theorem 1.4 is the following.

Proposition 6.1. *If M is as in Theorem 1.4, then any $K \in C^\infty(M)$ is the scalar curvature of a metric which is conformally equivalent to some metric of constant scalar curvature -1 .*

Because the above construction involves a diffeomorphism ψ of M_1 which depends on K , one can not prescribe the metric of scalar curvature -1 on M in advance. Further investigation is needed to fully resolve Question (B).

Beyond the meager pointwise conformal statement implicit in Theorem 6.1, we can only mention some situations where it is not possible to solve (1.5).

Proposition 6.2. *Let M be a complete Riemannian manifold of dimension*

≥ 3 whose metric g has nonnegative Ricci curvature. If $K \in C^\infty(M)$ has the property $K \leq \text{const} < 0$, then $K \notin \text{PC}(g)$.

A special case is $M = \mathbf{R}^n$, $n \geq 3$ with the standard metric.

Proof. If k is the scalar curvature of g , then $k \geq 0$. We claim there is no positive solution defined on all of M of

$$(6.3) \quad \frac{4(n-1)}{n-2} \Delta u - ku = -Ku^{(n+2)/(n-2)}.$$

In fact, if $u > 0$ were such a solution, then

$$\frac{n-2}{4(n-1)} [ku - Ku^{(n+2)/(n-2)}] \geq \varepsilon u^{(n+2)/(n-2)}$$

for some constant $\varepsilon > 0$. Therefore there would be a solution of

$$\Delta u \geq \varepsilon u^{(n+2)/(n-2)},$$

which is impossible by a theorem of Osserman [16, Remark 3, p. 1645] and its generalization by Calabi [5, Theorem 4].

Added in proof. In [21] we show that there are topological obstructions to zero scalar curvature and hence to $\lambda_1(g) = 0$. Indeed, if M is a compact spin manifold with \hat{A} genus not zero and with first Betti number not zero, then M does not admit a metric of zero scalar curvature. In addition, [21] contains a computation of the 1st and 2nd variations of λ_1 . In [22] we have obtained a more direct proof of Theorems 1.1 to 1.4. This proof does not yield information on pointwise conformal change of metrics, but it does yield additional facts on conformal change. For example, Question (B) in the compact case is answered for $\lambda_1 = 0$ and partially for $\lambda_1 > 0$. It is shown there that every C^∞ function on S^n is the scalar curvature of some metric. Thus the functions K in Corollary 5.20 which cannot be realized as scalar curvatures of metrics pointwise conformal to the standard metric on S^n are, nevertheless scalar curvatures of some metrics.

Bibliography

- [1] M. F. Atiyah & I. M. Singer, *The index of elliptic operators: III*, Ann. of Math. **87** (1968) 546–604.
- [2] T. Aubin, *Métriques Riemanniennes et courbure*, J. Differential Geometry **4** (1970) 383–424.
- [3] A. Avez, *Valeur moyenne du scalaire de courbure sur une variété compacte. Applications relativistes*, C. R. Acad. Sci. Paris **256** (1963) 5271–5273.
- [4] L. Bers, F. John & M. Schechter, *Partial differential equations*, Interscience-Wiley, New York, 1964.
- [5] E. Calabi, *An extension of E. Hopf's maximum principle with an application to Riemannian geometry*, Duke Math. J. **25** (1958) 45–46.

- [6] R. Courant & D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience-Wiley, New York, 1962.
- [7] H. Eliasson, *On variations of metrics*, Math. Scand. **29** (1971) 317–327.
- [8] N. Hitchin, *The space of harmonic spinors*, to appear.
- [9] T. Kato, *Perturbation theory for linear operators*, Grundlehren Math. Wiss., Vol. 132, Springer, New York, 1966.
- [10] J. L. Kazdan & F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. **99** (1974) 14–47.
- [11] —, *Curvature functions for open 2-manifolds*, Ann. of Math. **99** (1974) 203–219.
- [12] —, *Remarks on some nonlinear elliptic equations*, to appear.
- [13] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963) 7–9.
- [14] J. W. Milnor, *Remarks concerning spin manifolds*, Differential and combinatorial topology, Princeton University Press, Princeton, 1965, 55–62.
- [15] M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, Nippon Hyoronsha, Tokyo, 1966, 101–114.
- [16] R. Osserman, *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math. **7** (1957) 1641–1647.
- [17] S. I. Pohožaev, *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Soviet Math. Dokl. **6** (1965) 1408–1411.
- [18] G. Rosen, *Existence of particlelike solutions to nonlinear field theories*, J. Mathematical Phys. **7** (1966) 2066–2070.
- [19] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa, **22** (1968) 265–274.
- [20] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960) 21–37.

Added in proof:

- [21] J. L. Kazdan & F. W. Warner, *Prescribing curvatures*, to appear in Proc. Sympos. Pure Math., Amer. Math. Soc.
- [22] —, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, to appear in Ann. of Math.

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