

INTEGRABLE ALMOST TANGENT STRUCTURES

F. BRICKELL & R. S. CLARK

Suppose that Γ is a given pseudogroup of local diffeomorphisms $f: R^n \rightarrow R^n$. A maximal atlas of charts of a manifold M whose changes of coordinates belong to Γ determines a Γ structure on M . These particular charts of M are said to be *adapted* for the Γ structure.

The set Γ_G of all local diffeomorphisms f whose derivatives Df have values in some given Lie subgroup G of $GL(R^n)$ is an important example of a pseudogroup. A Γ_G structure on M is called an *integrable G structure*. Examples of these are integrable almost complex structures and integrable almost tangent structures.

Γ structures on manifolds M and M_1 are said to be *isomorphic* if there exists a bijection

$$\phi: M_1 \rightarrow M$$

such that x is an adapted chart of M iff $x \circ \phi$ is an adapted chart of M_1 .

Any complex manifold has a standard integrable almost complex structure. A well-known theorem states that any integrable almost complex structure on a manifold M is isomorphic to this standard structure on some complex manifold.

Any tangent manifold has a standard integrable almost tangent structure. But an integrable almost tangent structure on a manifold M is not necessarily isomorphic to this standard structure on some tangent manifold. In this paper we find necessary and sufficient conditions for the existence of such a tangent manifold.

1. Locally affine structures

Suppose given a Γ structure on a manifold M . An atlas of adapted charts of M whose changes of coordinates belong to a given subpseudogroup Γ' of Γ determines a *subordinate Γ' structure* on M . In general such a subordinate structure does not exist.

A *locally affine structure* on M is a pseudogroup structure with coordinate transformations of the type

$$z \rightarrow Az + b,$$

where $A \in GL(R^n)$, $b \in R^n$. A manifold with such a structure carries a standard

flat linear connection whose components with respect to the adapted charts are zero.

A subpseudogroup structure with coordinate transformations of the type

$$z \rightarrow Az$$

is a locally *centro-affine structure*.

A locally affine structure does not necessarily admit a subordinate locally centro-affine structure. In order to be able to state conditions for it to do so we recall [3] that a vector field V on a manifold M is *concurrent* with respect to a linear connection ∇ on M if

$$\nabla_X V = X$$

for any vector field X in M .

Theorem 1. *A locally affine structure on M has a subordinate locally centro-affine structure iff M admits a global vector field concurrent with respect to its flat connection.*

Proof. Suppose that M has a subordinate locally centro-affine structure. For each chart y adapted for this structure we define the local vector field

$$y^j \partial / \partial y^j .$$

These local fields agree on the intersection of their domains and so define a global vector field V on M . This is concurrent with respect to the flat connection ∇ since, if $X = a^i \partial / \partial y^i$,

$$\nabla_X V = a^i \nabla_{\partial / \partial y^i} (y^j \partial / \partial y^j) = a^i \partial / \partial y^i = X .$$

Conversely, suppose that M carries a concurrent vector field V . For each chart y adapted for the locally affine structure let $V = v^i \partial / \partial y^i$. Then for any vector field $X = a^i \partial / \partial y^i$

$$\nabla_X V = a^i \frac{\partial v^j}{\partial y^i} \frac{\partial}{\partial y^j} .$$

Since this must be X it follows that

$$v = y + c$$

for some $c \in R^n$. Such functions v are therefore charts adapted for the locally affine structure and so, on any intersection of domains,

$$\bar{v} = Av + b ,$$

where A, b have values in $GL(R^n)$ and R^n respectively. But because V is a

global vector field, it follows that $b = 0$. The charts v therefore define a locally centro-affine structure on M subordinate to the given locally affine structure.

q.e.d.

We shall say that a locally affine structure on a manifold M is *complete* if the flat connection on M is complete. Any complete locally affine structure on a connected manifold M determines a complete locally affine structure on its simply connected covering manifold M' . This structure on M' is isomorphic to the standard locally affine structure on R^n determined by its identity chart [2].

Theorem 2. *A locally affine structure on a connected manifold M which*

(i) *is complete,*

(ii) *admits a subordinate locally centro-affine structure,*

is isomorphic to the standard structure on R^n .

Proof. Theorem 1 shows that M carries a concurrent vector field V . This lifts to a concurrent vector field W on R^n . If y is the identity chart on R^n , an argument used in the previous proof shows that

$$W = (y^i + c^i) \frac{\partial}{\partial y^i}$$

for some $c^i \in R$, and so W has just one zero. This arises from a zero of the vector field V . But since W has only one zero, R^n must cover M just once.

2. Integrable almost tangent structures

A manifold M modelled on $R^n \oplus R^n$ carries a *foliation* (of dimension n) if it has a pseudogroup structure whose coordinate transformations are local diffeomorphisms of the type

$$(z, w) \rightarrow (fz, g(z, w)) ,$$

where f is a local diffeomorphism in R^n . If (x, y) is an adapted chart at a point $m \in M$, the leaf F_m containing m is determined locally by $x = xm$ and it admits $y|_{F_m}$ as a chart.

An *integrable almost tangent structure* [1] on M is a pseudogroup structure with coordinate transformations of the type

$$(z, w) \rightarrow (fz, (Df)_z w + bz) ,$$

where f is a local diffeomorphism in R^n , and the local function $b: R^n \rightarrow R^n$ is differentiable. An integrable almost tangent manifold carries an underlying foliation \mathcal{F} . A chart (x, y) which is adapted for the almost tangent structure is necessarily adapted for \mathcal{F} , and the charts $y|_{F_m}$ determine a locally affine structure on the leaf F_m .

A subpseudogroup structure with coordinate transformations of the type

$$(z, w) \rightarrow (fz, (Df)_2w)$$

is a nearly tangent structure.

Theorem 3. *An integrable almost tangent structure on M has a subordinate nearly tangent structure iff M admits a global vector field which is tangent to the underlying foliation \mathcal{F} and concurrent with respect to the locally affine structure on each leaf of \mathcal{F} .*

Proof. Suppose that M has a subordinate nearly tangent structure with charts (x, y) . The local vector fields $y^i\partial/\partial y^i$ agree on the intersection of their domains since

$$y^i \frac{\partial}{\partial y^i} = y^i \left(\frac{\partial \bar{x}^j}{\partial y^i} \frac{\partial}{\partial \bar{x}^j} + \frac{\partial \bar{y}^j}{\partial y^i} \frac{\partial}{\partial \bar{y}^j} \right) = y^i \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial}{\partial \bar{y}^j} = \bar{y}^i \frac{\partial}{\partial \bar{y}^i},$$

and together they define a global vector field A on M . This vector field is tangent to the foliation \mathcal{F} , and is concurrent with respect to the locally affine structures on the leaves.

Conversely suppose that M carries such a vector field A . For each chart (x, y) adapted for the almost tangent structure let $A = v^i\partial/\partial y^i$, where v^i depend on x, y . Then for any vector field $X = a^i\partial/\partial y^i$ on a leaf

$$\nabla_X A = a^i \frac{\partial v^j}{\partial y^i} \frac{\partial}{\partial y^j} .$$

Since this must be X , it follows that

$$v = y + c(x)$$

for some local differentiable function $c: R^n \rightarrow R^n$. Consequently (x, v) are also charts of M adapted for the almost tangent structure and so on any intersection of their domains

$$\bar{x}^i = f^i(x) , \quad \bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} v^j + b^i(x) .$$

In terms of such a chart, $A = v^i\partial/\partial v^i$, and because A is a global vector field it follows that $b^i = 0$. The charts (x, v) therefore define a subordinate nearly tangent structure on M . q.e.d.

Suppose that M is a manifold with a nearly tangent structure, and that A is the associated vector field defined in Theorem 3.

Theorem 4. *The set of zeros of A can be given the structure of a regular submanifold M' of M of dimension n .*

Proof. Let M' be the set of zeros of A , and p a given point of M' . Choose a chart $\xi = (x, y)$ at p adapted for the nearly tangent structure and having range $U \times V \subset R^n \oplus R^n$, where U and V are open cubes in R^n . The inter-

section $M' \cap (\text{domain } \xi)$ is just the set of points for which $y = 0$.

If $y: M' \rightarrow M$ is the natural injection, then $x' = x \circ j$ is an injection $M' \rightarrow R^n$ with range U . It is therefore a chart for M' . If \bar{x}' is another chart for M' at p obtained from $\bar{\xi} = (\bar{x}, \bar{y})$, then $\bar{x}' \circ x'^{-1} = \bar{x} \circ x^{-1}$, and this is a local diffeomorphism in R^n . As p varies over M' , the charts x' obtained in this way define a manifold structure of dimension n on M' .

The representative for j in terms of the charts x', ξ is the function $z \rightarrow (z, 0)$. Consequently M' is a submanifold of M . The domain of x' is the intersection of the domain of ξ with M' . This implies that M' is a regular submanifold of M .

3. The main theorem

We shall say that an integrable almost tangent structure on a manifold M is complete if the locally affine structure on each leaf of the foliation \mathcal{F} is complete.

Suppose that such a structure is given on M , and that it admits a subordinate nearly tangent structure. Each leaf then admits a subordinate locally centro-affine structure. According to Theorem 2, the locally affine structure on each leaf is isomorphic to the standard structure on R^n , and therefore admits a global adapted chart s .

The vector field A , introduced in Theorem 3, has just one zero in each leaf. We can therefore define the function $\pi: m \rightarrow p$, where p is the zero in the leaf F_m . This maps M onto M' .

Choose a point $p \in M'$, and let $j_p: F_p \rightarrow M$ be the natural injection. Let $\xi = (x, y)$ be a chart of M at p chosen as in the proof of Theorem 4. Then the chart $y \circ j_p$ is adapted for the locally centro-affine structure on F_p and on its domain

$$y \circ j_p = As + b$$

for some $A \in GL(R^n)$, $b \in R^n$. The function $As + b$ is a global chart for the manifold F_p . It can be shown to be adapted to the locally centro-affine structure on F_p . This extension of the chart $y \circ j_p$ can be carried out for each point p in the open set

$$W' = (x')^{-1}U$$

of M' . Consequently it defines a function Y on $\pi^{-1}(W')$ with values in R^n .

The function x is constant on the slices of ξ , and so it also can be extended to a function X on $\pi^{-1}(W')$ with values in R^n . This is constant on the leaves of \mathcal{F} .

Lemma. *The function $(X, Y): M \rightarrow R^n \oplus R^n$ is a chart adapted to the nearly tangent structure on M .*

Proof. In the first place we observe that (X, Y) is a local injection onto the open set $U \times R^n$.

We shall say that (X, Y) is *nearly tangent at a point m* , if it is defined on some neighborhood of m , and if its restriction to this neighborhood is a chart adapted to the nearly tangent structure on M . To prove the lemma it is sufficient to show that (X, Y) is nearly tangent at each point of its domain $\pi^{-1}(W')$.

Let p be a point of W' , and consider the set S of points in F_p at which (X, Y) is nearly tangent. S is not empty because (X, Y) is nearly tangent at all points of the domain of the chart ξ . S is, of course, an open subset of F_p . We complete the proof of this lemma by showing that S is also a closed subset of F_p and will therefore coincide with F_p .

Choose a point m in the closure of S , and then choose a chart $\bar{\xi} = (\bar{x}, \bar{y})$ at m , with range $\bar{U} \times \bar{V}$ (where \bar{U}, \bar{V} are open in R^n), adapted for the nearly tangent structure. The domain of $\bar{\xi}$ will meet S , and we choose a point q of the intersection. Since (X, Y) is nearly tangent at q , there is a neighborhood

$$W_1 = \bar{\xi}^{-1}(U_1 \times V_1)$$

of q , where U_1 and V_1 are cubes contained in \bar{U} and \bar{V} , such that (X, Y) is defined on W_1 and its restriction to W_1 is a chart adapted to the nearly tangent structure. Consequently on W_1

$$X^i = f^i(\bar{x}), \quad Y^i = \frac{\partial X^i}{\partial \bar{x}^j} \bar{y}^j,$$

where f is a local diffeomorphism in R^n with domain U_1 .

The first relation holds on $\bar{W} = \bar{\xi}^{-1}(U_1 \times \bar{V})$. The second relation also holds on \bar{W} because the functions Y and \bar{y} induce the same locally centro-affine structure on each leaf of \mathcal{F} which meets W_1 . Consequently the function (X, Y) is defined on \bar{W} and its restriction to \bar{W} is a chart adapted to the nearly tangent structure on M . But $m \in \bar{W}$ and therefore $m \in S$. It follows that S is a closed subset of F_p . q.e.d.

Suppose that M' is any manifold modelled on R^n , and that TM' is its *tangent manifold*. Let $\pi': TM' \rightarrow M'$ be the natural projection. Associated with any chart x' of M' with domain W' we have a standard chart (X', Y') of TM' with domain $(\pi')^{-1}W'$ defined by

$$X': v \rightarrow x'm, \quad Y': v \rightarrow a,$$

where $v = a^i(\partial/\partial x'^i)_m$. These charts define a nearly tangent structure on TM' whose underlying integrable almost tangent structure is complete.

Conversely, we have

Theorem 5. *An integrable almost tangent structure on a manifold M which*
 (i) *is complete,*

(ii) admits a subordinate nearly tangent structure, is isomorphic to the standard structure on a tangent manifold.

Proof. Let M' be the submanifold of M defined in Theorem 4. Choose any point $m \in M$ and a chart $\xi = (x, y)$ of M at $p = \pi m$ as in the proof of Theorem 4. The previous lemma shows that this can be extended to a chart (X, Y) at m .

Let x' be the chart of M' at p associated with ξ . The local function

$$m \rightarrow Y^i(m)(\partial/\partial x'^i)_p$$

is independent of the choice of ξ . Such functions determine a bijection ϕ between M and TM' . In terms of the chart (X, Y) of M and the standard chart (X', Y') of TM' associated with x' , the representative of ϕ is the identity function on $U \times R^n$. Consequently ϕ is a diffeomorphism of M onto TM' .

Since it maps the atlas of adapted charts (X, Y) of M to the atlas of adapted charts (X', Y') of TM' , ϕ is an isomorphism of the integrable almost tangent structures on these manifolds. q.e.d.

A manifold with an integrable almost tangent structure does not necessarily admit any subordinate nearly tangent structure.

To illustrate this, consider the circle S and its atlas \mathcal{A} whose charts are restrictions of the global function $(\cos \alpha, \sin \alpha) \rightarrow \alpha$. Two such charts differ by a coordinate transformation given locally by

$$z \rightarrow z + c$$

for some $c \in R$, and so \mathcal{A} determines a locally affine structure $\Gamma_{\mathcal{A}}$ on S . This structure is complete. Now consider the torus $T = S \times S$. The atlas of charts $x \times y$, where $x, y \in \mathcal{A}$, defines an integrable almost tangent structure Σ on T . The leaves of the foliation \mathcal{F} are the circles $F_p = p \times S$. The locally affine structure on any leaf is isomorphic to $\Gamma_{\mathcal{A}}$ and so it is complete. Since T is compact, it cannot be a tangent manifold. It follows from Theorem 5 that the integrable almost tangent structure Σ cannot admit any subordinate nearly tangent structure.

References

- [1] R. S. Clark & M. R. Bruckheimer, *Sur les structures presque tangents*, C. R. Acad. Sci. Paris **251** (1960) 627–629.
- [2] J. A. Wolf, *Spaces of constant curvature*, McGraw-Hill, New York, 1967, Chapter I, §1.9.
- [3] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1955, 168.

UNIVERSITY OF SOUTHAMPTON

