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0-DEFORMABLE (1, 1)-TENSOR FIELDS

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1. Introduction

Given a smooth real vector bundle $\pi: V \to M$, a $(1,1)$-tensor is a smooth map $J: V \to V$ which satisfies $\pi J = \pi$ and is linear on each fibre.

J is said to be *0-deformable* if for any two fibres $\pi^{-1}(x)$ and $\pi^{-1}(y)$ an isomorphism $\tau_{xy}: \pi^{-1}(x) \to \pi^{-1}(y)$ exists such that $\tau_{xy}J_x = J_y\tau_{xy}$. It follows from a result of J. R. Vanstone [10] that J is 0-deformable if and only if there exists a connection ∇ in V satisfying $\nabla J = 0$.

This note contains four results from the author's Ph. D. thesis written at the University of Toronto. The guidance of Professor Vanstone in the development of the thesis is gratefully acknowledged. Suggestions by Professor S. Halperin figure prominently in the thesis as well.

The referee has brought to our attention an article by R. Crittenden [2]. The article established the equivalence of covariant constancy with respect to some connection and the existence of *local smooth frames* with respect to which the coefficients of a tensor are constants. However, 0-deformability implies a priori only the *pointwise existence* of such frames. Some results mentioned in Crittenden's article are corollaries of our results.

2. Semisimple (1, 1)-tensors

Theorem 1. *Let*

$$p(x) = x^s + \cdots + b_1 x + b_0 \tag{2.1}$$

be a real polynomial with s distinct roots. Then associated with p are $s(s-1)$ real numbers $\{a_{ij}\}$ with the following property: if ∇ is an arbitrary connection in V, and J is an arbitrary $(1,1)$*-tensor solution of*

$$p(J) = J^s + \cdots + b_1 J + b_0 I = 0\ , \tag{2.2}$$

then the new connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X v = \nabla_X v + \sum_{i=0}^{s-1}\sum_{j=1}^{s-1} a_{ij}J^i(\nabla_X J^j)v\ , \tag{2.3}$$

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(*where X is a tangent vector field, and v is a cross section of V*) *satisfies*

$$\tilde{\nabla} J = \tilde{\nabla} \circ J - J \circ \tilde{\nabla} = 0 . \tag{2.4}$$

Proof. It is easy to show that the above result would follow from a complex version of the result by simply taking real parts of the $\{a_{ij}\}$. Thus this proof actually treats the complex case only.

We will need the following

Lemma. *Suppose π_i, $i = 1, \cdots, s$, are $(1,1)$-tensors in V satisfying*

$$\pi_i^2 = \pi_i , \quad \pi_i \pi_j = 0 \quad \text{for} \quad i \neq j , \quad \sum \pi_i = I . \tag{2.5}$$

If ∇ is an arbitrary connection in V, then the new connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X v = \nabla_X v + \sum_{i=1}^{s} \pi_i (\nabla_X \pi_i) v \tag{2.6}$$

satisfies

$$\tilde{\nabla} \pi_\alpha = 0 , \qquad \alpha = 1, \cdots, s . \tag{2.7}$$

Proof. $\tilde{\nabla} \pi_\alpha = \nabla \pi_\alpha + \sum \pi_i (\nabla \pi_i) \pi_\alpha - \sum \pi_\alpha \pi_i (\nabla \pi_i)$
$= \nabla \pi_\alpha + \sum \pi_i \nabla (\pi_i \pi_\alpha) - \sum \pi_i \pi_i (\nabla \pi_\alpha) - \pi_\alpha (\nabla \pi_\alpha) = 0 .$ q.e.d.

Now let $\{\lambda_1, \cdots, \lambda_s\}$ be the distinct roots of p. Then

$$(J - \lambda_1 I) \cdots (J - \lambda_s I) = 0 .$$

Define new polynomials p_i, $i = 1, \cdots, s$, by

$$p_i(x) = \prod_{j \neq i} (x - \lambda_j) .$$

Then $p_i(\lambda_i) \neq 0$ for each $i = 1, \cdots, s$, and therefore $\pi_i = p_i(\lambda_i)^{-1} p_i(J)$, $i = 1, \cdots, s$, are s new $(1,1)$-tensors in V. Using elementary linear algebra it follows that (2.5) holds, and $J = \sum \lambda_i \pi_i$. Let ∇ be an arbitrary connection in V. Then from the lemma it follows that the connection $\tilde{\nabla}$ defined by (2.6) satisfies (2.7). Thus $\tilde{\nabla} J = \sum \lambda_i \tilde{\nabla} \pi_i = 0$. Define $c_{ij} \in C$, $i = 1, \cdots, s$, $j = 0, 1, \cdots, s-1$, by $\pi_i = \sum c_{ij} J^j$. It follows that (2.3) holds with $a_{ij} = \sum_{k=1}^{s} c_{ki} c_{kj}$. q.e.d.

Note that the proof is completely algebraic and uses only the derivational properties of connections.

In the case $s = 2$, the constants $\{a_{ij}\}$ are

$$a_{01} = D^{-1}\{-b_1\} , \qquad a_{11} = D^{-1}\{2\} ,$$

where $D = b_1^2 - 4b_0$ is the discriminant of p. In the case $s = 3$, the constants $\{a_{ij}\}$ are

$$\begin{aligned}
a_{01} &= D^{-1}\{-4b_1^2 b_2 + 6b_0 b_1 + b_0 b_2^2 + b_1 b_2^3\} , \\
a_{02} &= D^{-1}\{-4b_1^2 + b_1 b_2^2 + 3b_0 b_2\} , \\
a_{11} &= D^{-1}\{2b_1^2 + 2b_2^4 + 12b_0 b_2 - 8b_1 b_2^2\} , \\
a_{12} &= a_{21} = D^{-1}\{2b_2^3 + 9b_0 - 7b_1 b_2\} , \\
a_{22} &= D^{-1}\{2b_2^2 - 6b_1\} ,
\end{aligned}$$

where $D = -4b_1^3 - 27b_0^2 + b_1^2 b_2^2 + 18b_0 b_1 b_2 - 4b_0 b_2^3$ is the discriminant of p. We note that S. Tachibana [9] and C. J. Hsu [4] derived the above result for the cases $p(x) = x^s \pm 1$ and $p(x) = x^s \pm \lambda^s$, $\lambda \neq 0$. K. Yano [12] has studied (1, 1)-tensors which satisfy $J^3 + J = 0$. In this case

$$\tilde{\nabla}_X v = \nabla_X v - \tfrac{1}{2}J(\nabla_X J)v + (\nabla_X J^2)v + \tfrac{3}{2}J^2(\nabla_X J^2)v$$

defines a connection $\tilde{\nabla}$ in terms of an arbitrary connection ∇ which satisfies $\tilde{\nabla}J = 0$. Finally, we note that the requirement that p have distinct roots is a necessary condition. Let (x_1, x_2) be the usual coordinate system for $\boldsymbol{R}^2$. Define J as $J\partial x_1 = x_1 \partial x_2$, $J\partial x_2 = 0$. Clearly J satisfies $J^2 = 0$, but J is not 0-deformable. It is easy to construct such matrix examples for any polynomial with a multiple real or complex root.

3. 0-deformable (1, 1)-tensors and Riemann structures

We will call a symmetric positive-definite (0, 2)-tensor field $\mathscr{G}$ in V a Riemann structure.

Theorem 2. *Suppose a* (1, 1)*-tensor J is constant with respect to a connection ∇. Then V admits a Riemann structure $\mathscr{G}$ and a connection $\tilde{\nabla}$ which satisfy $\tilde{\nabla}J = 0$ and $\tilde{\nabla}\mathscr{G} = 0$.*

Proof. Our proof is an explicit construction of the promised Riemann structure and connection.

Let $J = J_S + J_N$ be the decomposition of J into its semisimple and nilpotent parts. Since J is 0-deformable, we have that both J_S and J_N are polynomials in J with constant coefficients. Thus $\nabla J_S = \nabla J_N = \nabla J = 0$. The local eigenspaces of J_S are naturally preserved by ∇-parallel translation over M and are global subbundles of V. Thus we may assume that V is an eigenbundle of J_S, the general case being given by the obvious direct sums.

Case 1. Suppose $J_S - \lambda I = 0$ for some real λ. If $J_N = 0$, then we may choose any Riemann structure $\mathscr{G}$ and connection $\tilde{\nabla}$ satisfying $\tilde{\nabla}\mathscr{G} = 0$. Of course, $\tilde{\nabla}J = 0$ as well. Thus we assume $J_N^P \neq 0$, $J_N^{P+1} = 0$, $P \geq 1$. Let $\backslash$ denote orthogonal complement in V with respect to a fixed but arbitary Riemann structure. Although what follows seems complicated, it is nothing more that a smooth global decomposition of V which, on each fibre, amounts to the usual decomposition of a vector space in terms of a nilpotent endomorphism. Namely, we decompose I as the following hierarchy of orthogonal projection tensors ${}_i\pi_j$ and V itself as the following hierarchy of subbundles:

$$
\begin{aligned}
{}_0\pi_0&: V \to \operatorname{Ker} J_N^{P+1} \backslash \operatorname{Ker} J_N^P ,\\
{}_i\pi_0&: V \to J_N^i\, {}_0\pi_0 V, \quad i = 1, \cdots, P,\\
{}_0\pi_1&: V \to \operatorname{Ker} J_N^P \backslash \{{}_1\pi_0 V \oplus \operatorname{Ker} J_N^{P-1}\} ,\\
{}_i\pi_1&: V \to J_N^i\, {}_0\pi_1 V , \quad i = 1, \cdots, P-1 ,\\
{}_0\pi_2&: V \to \operatorname{Ker} J_N^{P-1} \backslash \{{}_2\pi_0 V \oplus {}_1\pi_1 V \oplus \operatorname{Ker} J_N^{P-2}\} ,\\
&\ \vdots\\
{}_0\pi_P&: V \to \operatorname{Ker} J_N \backslash \{{}_P\pi_0 V \oplus {}_{P-1}\pi_1 V \oplus \cdots \oplus {}_1\pi_{P-1} V\} .
\end{aligned}
$$

Thus

$$V = \underbrace{{}_0\pi_0 V \oplus {}_1\pi_0 V \oplus \cdots \oplus {}_P\pi_0 V}_{P+1} \oplus \underbrace{{}_0\pi_1 V \oplus \cdots \oplus {}_{P-1}\pi_1 V}_{P} \oplus \cdots \oplus \underbrace{{}_0\pi_P V}_{1} .$$

In each *list* ${}_0\pi_i V \oplus {}_1\pi_i V \oplus \cdots \oplus {}_{P-i}\pi_i V$, J_N acts precisely as follows: J_N maps ${}_0\pi_i V$ isomorphically onto ${}_1\pi_i V$; J_N maps ${}_1\pi_i V$ isomorphically onto ${}_2\pi_i V$; so on; and J_N maps ${}_{P-i}\pi_i V$ to 0.

First we use a much coarser decomposition of V, namely, just the decomposition of V into the above $P+1$ lists. That is, let $\pi_\alpha = \sum {}_j\pi_\alpha$, $\alpha = 0, \cdots, P$. Thus $V = \pi_0 V \oplus \pi_1 V \oplus \cdots \oplus \pi_P V$. It follows easily that $J_N \pi_\alpha = \pi_\alpha J_N$. Now define a new connection $\overset{*}{\nabla}$ in terms of ∇ by

$$\overset{*}{\nabla}_X v = \nabla_X v + \sum_{j=0}^{P} \pi_j (\nabla_X \pi_j) v .$$

In view of the lemma in § 2, we have $\overset{*}{\nabla} \pi_\alpha = 0$, and also

$$\overset{*}{\nabla} J_N = \sum \pi_j (\nabla \pi_j) J_N - \sum J_N \pi_j (\nabla \pi_j) = 0 .$$

Therefore we may as well assume $V = \pi_0 V$, the generalization to $\pi_0 V \oplus \cdots \pi_P V$ again being the obvious direct sum.

From the definitions of ${}_i\pi_0$ it follows that

$${}_0\pi_0 J_N = 0 , \quad {}_{i+1}\pi_0 J_N = J_N\, {}_i\pi_0 \quad \text{for} \quad i < P , \quad J_N\, {}_P\pi_0 = 0 .$$

Now define a new connection $\bar{\nabla}$ in terms of $\overset{*}{\nabla}$ by

$$\bar{\nabla}_X v = \overset{*}{\nabla}_X v + \sum_{i=0}^{P} {}_i\pi_0 (\overset{*}{\nabla}_X\, {}_i\pi_0) v .$$

Again $\bar{\nabla}\, {}_i\pi_0 = 0$ follows from the lemma. Also,

$$\begin{aligned}\bar{\nabla} J_N &= \sum_{i=0}^{P} {}_i\pi_0(\overset{*}{\bar{\nabla}}{}_i\pi_0)J_N - \sum_{i=0}^{P} J_N\, {}_i\pi_0(\overset{*}{\bar{\nabla}}{}_i\pi_0) \\ &= \sum_{i=1}^{P} {}_i\pi_0 J_N(\overset{*}{\bar{\nabla}}{}_{i-1}\pi_0) - \sum_{i=0}^{P-1} J_N\, {}_i\pi_0(\overset{*}{\bar{\nabla}}{}_i\pi_0) \\ &= \sum_{i=1}^{P} J_N\, {}_{i-1}\pi_0(\overset{*}{\bar{\nabla}}{}_{i-1}\pi_0) - \sum_{i=0}^{P-1} J_N\, {}_i\pi_0(\overset{*}{\bar{\nabla}}{}_i\pi_0) = 0\ .\end{aligned}$$

Let $\mathscr{G}_1$ denote an arbitrary Riemann structure in ${}_0\pi_0 V$. Using J_N, $\mathscr{G}_1$ may be extended to a Riemann structure $\mathscr{G}$ in V in the obvious way, that is, so that each $J_N^j\colon {}_0\pi_0 V \to {}_j\pi_0 V$, $j = 1, \cdots, P$, is an isometry. Define a new $(1, 1)$-tensor $\tilde{J}_N$ by

$$\tilde{J}_N = \begin{cases} J_N^{-1} & \text{on } {}_i\pi_0 V, \quad i > 0\ , \\ 0 & \text{on } {}_0\pi_0 V\ . \end{cases}$$

Of course, $\tilde{J}_N$ is simply the transpose of J_N with respect to $\mathscr{G}$. Furthermore, $\tilde{J}_N$ satisfies

$$\tilde{J}_N\, {}_0\pi_0 = 0\ , \quad \tilde{J}_N\, {}_{i+1}\pi_0 = {}_i\pi_0 \tilde{J}_N \qquad \text{for } i < P\ , \quad {}_P\pi_0 \tilde{J}_N = 0\ .$$

Thus $J_N\tilde{J}_N = I - {}_0\pi_0$, so that $J_N(\bar{\nabla}\tilde{J}_N) = 0$. Therefore $\bar{\nabla}\tilde{J}_N = {}_P\pi_0(\bar{\nabla}\tilde{J}_N) = \bar{\nabla}({}_P\pi_0 \tilde{J}_N) = 0$. Hence $\bar{\nabla} J_N = \bar{\nabla}\tilde{J}_N = 0$.

Finally, define a new connection $\tilde{\nabla}$ in terms of $\bar{\nabla}$ by

$$\mathscr{G}(\tilde{\nabla}_X v_1, v_2) = \mathscr{G}(\bar{\nabla}_X v_1, v_2) + \tfrac{1}{2}(\bar{\nabla}_X \mathscr{G})(v_1, v_2)\ .$$

It follows that

$$\begin{aligned}(\tilde{\nabla}_X \mathscr{G})(v_1, v_2) &= X\mathscr{G}(v_1, v_2) - \mathscr{G}(\tilde{\nabla}_X v_1, v_2) - \mathscr{G}(v_1, \tilde{\nabla}_X v_2) \\ &= X\mathscr{G}(v_1, v_2) - (\bar{\nabla}_X \mathscr{G})(v_1, v_2) - \mathscr{G}(\bar{\nabla}_X v_1, v_2) - \mathscr{G}(v_1, \bar{\nabla}_X v_2) \\ &= 0\ .\end{aligned}$$

Also,

$$\begin{aligned}\mathscr{G}((\tilde{\nabla}_X J_N)v_1, v_2) &= \mathscr{G}(\tilde{\nabla}_X J_N v_1, v_2) - \mathscr{G}(\tilde{\nabla}_X v_1, \tilde{J}_N v_2) \\ &= \mathscr{G}(\bar{\nabla}_X J_N v_1, v_2) + \tfrac{1}{2}(\bar{\nabla}_X \mathscr{G})(J_N v_1, v_2) \\ &\quad - \mathscr{G}(\bar{\nabla} v_1, \tilde{J}_N v_2) - \tfrac{1}{2}(\bar{\nabla}_X \mathscr{G})(v_1, \tilde{J}_N v_2) = 0\ .\end{aligned}$$

Since $\tilde{\nabla} J_S = 0$, $\tilde{\nabla} J = \tilde{\nabla} J_S + \tilde{\nabla} J_N = 0$.

Case 2. We now suppose $(J_S - \alpha I)^2 + \beta^2 I = 0$, $\beta \neq 0$. Since $\nabla J = 0$, $\nabla(\beta^{-1})(J - \alpha I)) = 0$. Thus we may assume $\alpha = 0$ and $\beta = 1$, so $J_S^2 + I = 0$. Suppose $J_N = 0$. If $\mathscr{G}_1$ is an arbitrary Riemann structure in V, then define a new Riemann structure $\mathscr{G}$ by

$$\mathscr{G}(v_1, v_2) = \mathscr{G}_1(v_1, v_2) + \mathscr{G}(J_S v_1, J_S v_2) \ .$$

It follows that J_S is skew with respect to $\mathscr{G}$. Define a new connection $\tilde{\nabla}$ in terms of ∇ by

$$\mathscr{G}(\tilde{\nabla}_X v_1, v_2) = \mathscr{G}(\nabla_X v_1, v_2) + \tfrac{1}{2}(\nabla_X \mathscr{G})(v_1, v_2) \ .$$

Then

$$\begin{aligned}(\tilde{\nabla}_X \mathscr{G})(v_1, v_2) &= X\mathscr{G}(v_1, v_2) - (\nabla_X \mathscr{G})(v_1, v_2) - \mathscr{G}(\nabla_X v_1, v_2) - \mathscr{G}(v_1, \nabla_X v_2) \\ &= 0 \ ,\end{aligned}$$

and also

$$\begin{aligned}\mathscr{G}((\tilde{\nabla}_X J_S)v_1, v_2) = \mathscr{G}(\nabla_X J_S v_1, v_2) &+ \tfrac{1}{2}(\nabla_X \mathscr{G})(J_S v_1, v_2) \\ &+ \mathscr{G}(\nabla_X v_1, J_S v_2) + \tfrac{1}{2}(\nabla_X \mathscr{G})(v_1, J_S v_2) = 0 \ .\end{aligned}$$

Therefore it remains to consider the posibility that

$$J_N^P \neq 0 \ , \quad J_N^{P+1} = 0 \ , \quad P \geq 1 \ .$$

Just as in Case 1, we construct projection tensors ${}_i\pi_j$ and $\pi_\alpha = \sum {}_i\pi_\alpha$ using J_N and some Riemann structure. It follows that each π_α has even rank and commutes with J_S. Thus, if we define a new connection $\overset{*}{\nabla}$ in terms of ∇ by

$$\overset{*}{\nabla}_X v = \nabla_X v + \sum_{i=0}^{P} \pi_i(\nabla_X \pi_i)v \ ,$$

then we have as before that $\overset{*}{\nabla}\pi_\alpha = \overset{*}{\nabla} J_N = 0$. Also,

$$\overset{*}{\nabla} J_S = \sum \pi_i(\nabla \pi_i)J_S - \sum J_S \pi_i(\nabla \pi_i) = 0 \ .$$

Thus, as in Case 1, we may restrict our attention to the case $V = \pi_0 V$. Each ${}_i\pi_0$ has even rank and satisfies ${}_i\pi_0 J_S = J_S\, {}_i\pi_0$. Define a new connection $\bar{\nabla}$ in terms of $\overset{*}{\nabla}$ by

$$\bar{\nabla}_X v = \overset{*}{\nabla}_X v + \sum_{i=0}^{P} {}_i\pi_0(\overset{*}{\nabla}_X {}_i\pi_0)v \ .$$

Again, $\bar{\nabla}\, {}_i\pi_0 = 0$, $i = 0, 1, \cdots, P$, and $\bar{\nabla} J_N = 0$. Also,

$$\bar{\nabla} J_S = \sum {}_i\pi_0(\overset{*}{\nabla} {}_i\pi_0)J_S - \sum J_S\, {}_i\pi_0(\overset{*}{\nabla} {}_i\pi_0) = 0 \ .$$

Let $\mathscr{G}_1$ be a Riemann structure in ${}_0\pi_0 V$. Extend $\mathscr{G}_1$ to a Riemann structure $\mathscr{G}_2$ in V using J_N in the obvious way. Let $\tilde{J}_N$ be a $(1, 1)$-tensor in V defined by

$$\tilde{J}_N = \begin{cases} J_N^{-1} & \text{on } {}_i\pi_0 V, \quad i > 0 \ , \\ 0 & \text{on } {}_0\pi_0 V \ . \end{cases}$$

As in Case 1, $\tilde{J}_N$ is just the transpose of J_N with respect to $\mathscr{G}_2$, and $\bar{\nabla}\tilde{J}_N = 0$. Also $\tilde{J}_N J_S = J_S \tilde{J}_N$. Define a new Riemann structure $\mathscr{G}$ in terms of $\mathscr{G}_2$ by

$$\mathscr{G}(v_1, v_2) = \mathscr{G}_2(v_1, v_2) + \mathscr{G}_2(J_S v_1, J_S v_2) .$$

It follows that $\tilde{J}_N$ is the transpose of J_N with respect to $\mathscr{G}$, and J_S is skew with respect to $\mathscr{G}$.

Finally, define a connection $\tilde{\nabla}$ in terms of $\bar{\nabla}$ by

$$\mathscr{G}(\tilde{\nabla}_X v_1, v_2) = \mathscr{G}(\bar{\nabla}_X v_1, v_2) + \tfrac{1}{2}(\bar{\nabla}_X \mathscr{G})(v_1, v_2) .$$

It follows that $\tilde{\nabla}\mathscr{G} = 0$ and $\tilde{\nabla}J = \tilde{\nabla}J_S + \tilde{\nabla}J_N = 0$. q.e.d.

Theorem 2 was announced in [6]. A much more general result follows.

Theorem 3. *Suppose some set of tensors $\{T_\alpha\}$ in V, each of arbitrary type, satisfies $\nabla T_\alpha = 0$ for some connection ∇. Then V admits a Riemann structure $\mathscr{G}$ and a connection $\tilde{\nabla}$ such that $\tilde{\nabla}T_\alpha = 0$ and $\tilde{\nabla}\mathscr{G} = 0$.*

Proof. We restrict our attention to a fixed fibre $\pi^{-1}(x)$ of V identified with $\boldsymbol{R}^n$. Let $G \subset GL(n)$ be the Lie group of invertible linear transformations of $\boldsymbol{R}^n$ which leave each T_α invariant. Since $\nabla T_\alpha = 0$, V is endowed with a G-structure. Clearly G is an algebraic group, and so, according to H. Whitney [11], has a finite number of topological components. A generalization by G. Hochschild [3, p. 180] of a theorem of K. Iwasawa [5] states that any Lie group with a finite number of components is diffeomorphic to the manifold product of a maximal compact subgroup with a Euclidean space. Thus

$$G \overset{\text{diffeo}}{=} K \times \boldsymbol{R}^p ,$$

K being a maximal compact subgroup of G, and the structure group of V may be reduced from G to K. Since K is compact, a standard result in Lie theory provides that an inner product exists for $\boldsymbol{R}^n$ with respect to which K is a subgroup of $\mathscr{O}(n)$. q.e.d.

In view of the sophisticated nature of the theorems of Whitney, Iwasawa, and Hochschild used in our proof, Theorem 3 can only be regarded as an easy application. Our simple proof of Theorem 2 contrasts with the proof of the more general theorem.

4. (1, 1)-tensors in tangent bundles which are covariant constant with respect to Riemann connections

Suppose $V = T(M)$, and suppose ∇ is the unique torsion-free connection associated with a Riemann structure $\mathscr{G}$. By the de Rham Decomposition Theorem [8, pp. 187–193], each tangent space $T_x(M)$ decomposes as

$$T_x(M) = T_0 \oplus T_1 \oplus \cdots \oplus T_t ,$$

and the restricted holonomy group H_x decomposes as

$$H_x = H_0 \times H_1 \times \cdots \times H_t ,$$

where H_0 acts as the identity on T_0, and each H_j acts as an irreducible subgroup of SO(dim T_j) on T_j, $j > 0$.

Theorem 4. *Suppose J is a $(1, 1)$-tensor in $T(M)$, and suppose $\nabla J = 0$. Then just as each tangent space $T_x(M)$ decomposes and just as the restricted holonomy group H_x decomposes, J_x decomposes as*

$$J_x = J_0 \oplus (\alpha_1 I_1 + \beta_1 \mathscr{I}_1) \oplus \cdots \oplus (\alpha_t I_t + \beta_t \mathscr{I}_t) ,$$

where J_0 is, as far as H_x is concerned, arbitrary, $\alpha_i, \beta_i \in \boldsymbol{R}$, I_j is the identity transformation on T_j, and $(\beta_j \mathscr{I}_j)^2 = -\beta_j^2 I_j$. Since x is arbitrary and J is 0-deformable, the decomposition holds globally.

Proof. The canonical form for J follows from commutativity with H and elementary linear algebra. We note that the eigenvalues of J on T_j, $i > 0$, are precisely $\alpha_j \pm i\beta_j$. Of course, J is also required to commute with the full holonomy group of ∇. This additional requirement may further restrict the form of J. q.e.d.

If in addition M is simply connected, and $\mathscr{G}$ is complete, then M decomposes as

$$M = \boldsymbol{R}^q \times M_1 \times \cdots \times M_t ,$$

where each M_j on which $\beta_j \mathscr{I}_j \neq 0$ is a Kähler manifold. It follows directly from Theorem 4 that if J is required to have $m = \dim M$ distinct real eigenvalues, then $H = H_0$, a result of D. Blair and A. Stone [1]. From a result of S. Kobayashi [7] it follows that if M is a compact hypersurface of $\boldsymbol{R}^{m+1}$ and the Riemann structure $\mathscr{G}$ and connection ∇ in $T(M)$ are naturally induced from the Euclidean imbedding, then the corresponding holonomy group is SO(m). Thus when $m = 2$, $T(M)$ admits an almost complex structure (in fact, Kähler structure, of course); otherwise, the only ∇-parallel (1, 1)-tensors are constant multiples of the identity tensor.

References

[1] D. Blair & A. Stone, *A note on the holonomy group of manifolds with certain structures,* Proc. Amer. Math. Soc. **21** (1969) 73–76.

[2] R. Crittenden, *Covariant differentiation,* Quart. J. Math. Oxford Ser. **13** (1962) 285–298.

[3] G. Hochschild, *The structure of Lie groups, Holden-Day, San Francisco,* 1965.

[4] C. J. Hsu, *On some properties of π-structures on differentiable manifold,* Tôhoku Math. J. **12** (1960) 429–454.

[5] K. Iwasawa, *On some types of topological groups,* Ann. of Math. **50** (1949) 507–558.

[6] C. D. Jeffries, *A theorem on covariant constant* (1,1)*-tensor fields,* Bull. Amer. Math. Soc. **76** (1970) 1030–1031.
[7] S. Kobayashi, *Holonomy groups of hypersurfaces,* Nagoya Math. J. **10** (1956) 9–14.
[8] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. I, Interscience, New York, 1963.
[9] S. Tachibana, *A remark on linear connections with some properties,* Natur. Sci. Rep. Ochanomizu Univ. **1** (1959) 1–6.
[10] J. R. Vanstone, *Connections satisfying a generalized Ricci lemma,* Canad. J. Math. **16** (1964) 549–560.
[11] H. Whitney, *Elementary structure of real algebraic varieties,* Ann. of Math. (2) **66** (1957) 545–556.
[12] K. Yano, *On a structure* f *satisfying* $f^3+f=O$, Tech. Rep., No. 2, 1961, Univ. of Washington.

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