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SUBSCALAR PAIRS OF METRICS AND HYPERSURFACES WITH A NONDEGENERATE SECOND FUNDAMENTAL FORM

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0. Introduction

In this paper we establish an integral formula which holds on any compact oriented manifold without boundary equipped with a pair of Riemannian metrics. The natural assumptions needed to exploit this formula depend on positivity properties of a quadratic form constructed from the difference tensor of the Levi-Civita connections. We call two metrics satisfying this positivity property a subscalar pair.

The results are first applied to prove that subscalar pairs of Einstein metrics inducing the same element of volume are isometric. This generalizes a result of Munzner [10] on volume-preserving maps of the two-sphere in euclidean space.

Next we study the pseudo-Riemannian geometry of a hypersurface with a nondegenerate second fundamental form. In particular, we give a geometric interpretation of the rank of the difference tensor of the first and second fundamental forms, and establish local rigidity theorems on hypersurfaces with a given second fundamental form. In order to establish global results we assume that the hypersurfaces are convex, which for us means the second fundamental form of each convex hypersurface is negative definite. Under this assumption we give characterizations of the euclidean sphere in terms of various integral inequalities and prove a uniqueness theorem characterizing spheres as the only compact convex solutions of a differential inequality of 4th order in the derivatives of the position vector.

Finally we study the third fundamental form geometry of a convex hypersurface and prove that two compact convex hypersurfaces having the same second fundamental form and Gauss-Kronecker curvature differ by a rigid motion. This generalizes a result of Grove [6] on convex surfaces.

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1. Preliminary notations and conventions

Let I be a Riemannian metric on a manifold M, and $\{w^\alpha\}$ be a local coframe in terms of which ϕ_α^γ is the matrix of one-forms defining the Levi-Civita connection. We recall that if $I = \Sigma g_{\alpha\beta} w^\alpha w^\beta$ then ϕ_α^γ is characterized by the equations (see [11])

$$dw^\alpha = \Sigma w^\gamma \wedge \phi_\gamma^\alpha , \qquad dg_{\alpha\beta} = \Sigma \phi_\alpha^\gamma g_{\gamma\beta} + g_{\alpha\gamma}\phi_\beta^\gamma ,$$

and the associated curvature tensor is defined by

$$\Theta_\alpha^\beta = \tfrac{1}{2}\Sigma R_{\alpha\gamma\lambda}^\beta w^\gamma \wedge w^\lambda ,$$

where

$$\Theta_\alpha^\beta = d\phi_\alpha^\beta - \Sigma \phi_\alpha^\gamma \wedge \phi_\gamma^\beta .$$

We will adopt the notation that the element of volume defined by I is denoted by $*1$; the thus

$$*1 = (\det g_{\alpha\beta})^{1/2} w^1 \wedge \cdots \wedge w^m .$$

Now given any real valued functions $f\colon M \to R$ the equations

$$df = \Sigma f_\alpha w^\alpha , \qquad df_\alpha - \Sigma f_\gamma \phi_\alpha^\gamma = \Sigma f_{\alpha;\beta} w^\beta$$

define the components $f_{\alpha;\beta}$ of $H_I(f)$, the Hessian of f with respect to the metric I relative to the local coframe $\{w^\alpha\}$. Taking the exterior derivative of the equation defining f_α gives

$$0 = \Sigma df_\alpha \wedge w^\alpha + f_\alpha w^\gamma \wedge \phi_\gamma^\alpha = \Sigma f_{\alpha;\beta} w^\alpha \wedge w^\beta ,$$

and implies that $f_{\alpha;\beta} = f_{\beta;\alpha}$ or that $H_I(f)$ is a symmetric matrix.

Matters being so, if $I = \Sigma g_{\alpha\beta} w^\alpha w^\beta$ and $g^{\alpha\beta}$ is the matrix inverse to $g_{\alpha\beta}$, then we may introduce the Laplacian of f with respect to I as the I-trace of $H_I(f)$, that is,

$$\mathrm{Lap}_I\,(f) = \mathrm{tr}_I\, H_I(f) = \Sigma g^{\alpha\beta} f_{\alpha;\beta} .$$

2. Difference tensor

Now let I, I' be a pair of Riemannian metrics defined on a manifold M, and $\{w^\alpha\}$ be a local coframe in terms of which ϕ_α^γ and $\phi_\alpha^{\gamma\prime}$ are the matrices of one-forms which define the Levi-Civita connections of I and I' respectively. Then since

$$df_\alpha - f_\gamma \phi_\alpha^{\gamma\prime} - (df_\alpha - f_\gamma \phi_\alpha^\gamma) = f_\gamma(\phi_\alpha^\gamma - \phi_\alpha^{\gamma\prime}) ,$$

we may use the difference of the Hessians of real valued functions to define a tensor $\Delta(I, I')$ via the equation

$$H_{I'}(f) - H_I(f) = \langle \Delta(I, I'), df \rangle , \tag{1}$$

where $\langle , \rangle$ is the canonical bilinear pairing between the tangent space T and the cotangent space T^*.

If we let

$$\phi_\alpha^\gamma - \phi_\alpha^{\gamma\prime} = K_{\alpha\beta}^\gamma w^\beta ,$$

then the symmetry of the Hessians in (1) implies

$$K_{\alpha\beta}^\gamma = K_{\beta\alpha}^\gamma . \tag{2}$$

As a result if we let $\{e_\gamma\}$ denote the dual basis of $\{w^\alpha\}$, we have by (1) and (2) that

$$\Delta(I, I') = \Sigma K_{\alpha\beta}^\gamma w^\alpha \circledcirc w^\beta \otimes e_\gamma \in (T^* \circledcirc T^*) \otimes T .$$

In the case that one of the metrics is the induced metric of an immersion in an arbitrary codimension euclidean space there is a direct definition in terms of the coordinate functions, which is of interest. Thus let

$$X : M_m \to R^{m+p}$$

be an arbitrary codimension immersion of an m-manifold in euclidean $(m + p)$-dimensional space, and choose frames with $\{e_\alpha\}$, $1 \le \alpha \le m$, tangent to $X(M_m)$ and $\{e_a\}, m + 1 \le a \le m + p$, normal to $X(M_m)$ in such a way that the restrictions of the dual coframes satisfy

$$I = dX \cdot dX = \Sigma(\tau^\alpha)^2 , \qquad I' = \Sigma g_{\alpha\beta}{}' \tau^\alpha \tau^\beta .$$

As such given a fixed vector $a \in R^{m+p}$

$$d(X \cdot a) = \Sigma(e_\alpha \cdot a)\tau^\alpha ,$$

and using the structure equations of euclidean space

$$d(e_\alpha \cdot a) - (e_\gamma \cdot a)\phi_\alpha^{\gamma\prime} = (\phi_\alpha^\beta - \phi_\alpha^{\beta\prime})e_\beta \cdot a + \phi_\alpha^b e_b \cdot a .$$

Thus the Hessian

$$H_{I'}(X \cdot a) = \Delta(I, I') \cdot a + II \cdot a ,$$

where II is the vector-valued second fundamental form. In particular, the tension field of X with respect to I' (see [3]) defined by

$$\mathrm{Lap}_{I'}\,(X\cdot a) = h^*\cdot a$$

is given by

$$h^* = \mathrm{tr}_{I'}\,(\Delta(I,I') + II) = \Sigma g^{\alpha\beta'}K^{\gamma}_{\alpha\beta}e_{\gamma} + \Sigma g^{\alpha\beta'}h^{b}_{\alpha\beta}e_{b}\ ,$$

where $h^b_{\alpha\beta}$ are the components of II. We remark that this calculation simplifies a calculation in [5].

In particular, if we take $I = I'$ we get

$$H_I(X\cdot a) = II\cdot a\ ,$$

and have the direct definition of $\Delta(I,I')$ as a tangent pencil of symmetric matrices via the characterization

$$H_{I'}(X\cdot a) - H_I(X\cdot a) = \Delta(I,I')\cdot a\ , \qquad a \in R^{m+p}\ .$$

We now return to the general situation and establish a basic interchange formula. As we have already noted the difference tensor

$$\Delta = K^{\gamma}_{\alpha\beta}w^{\alpha} \circledcirc w^{\beta} \otimes e_{\gamma} \in (T^* \circledcirc T^*) \otimes T$$

can be written as

$$\Delta = (\phi^{\gamma}_{\alpha} - \phi^{\gamma}_{\alpha}{}') \otimes w^{\alpha} \otimes e_{\gamma} \in T^* \otimes (T^* \otimes T)\ ,$$

and as such we naturally define $\Delta \wedge \Delta \in \Lambda^2T^* \otimes (T^* \otimes T)$ by

$$\begin{aligned}\Delta \wedge \Delta &= \Sigma(\phi^{\gamma}_{\alpha} - \phi^{\gamma}_{\alpha}{}') \wedge (\phi^{\beta}_{\gamma} - \phi^{\beta}_{\gamma}{}') \otimes w^{\alpha} \otimes e_{\beta} \\ &= \Sigma(K^{\gamma}_{\alpha\lambda}K^{\beta}_{\gamma\mu} - K^{\gamma}_{\alpha\mu}K^{\beta}_{\gamma\lambda})w^{\lambda} \wedge w^{\mu} \otimes w^{\alpha} \otimes e_{\beta}\ .\end{aligned}$$

Proposition 1. *Let $\Delta = \Delta(I,I')$. Then*

$$\Theta' - \Theta = D_I\Delta - \Delta \wedge \Delta\ , \tag{3}$$

where $D_I\Delta$ is the covariant derivative of Δ in the I-metric.

Proof. We differentiate the defining equations

$$\phi^{\gamma}_{\alpha} - \phi^{\gamma}_{\alpha}{}' = \Sigma K^{\gamma}_{\alpha\beta}\tau^{\beta}\ ,$$

and see

$$\begin{aligned}&d\phi^{\gamma}_{\alpha} - \phi^{\beta}_{\alpha} \wedge \phi^{\gamma}_{\beta} - d\phi^{\gamma}_{\alpha}{}' + \phi^{\beta}_{\alpha}{}' \wedge \phi^{\gamma}_{\beta}{}' \\ &\quad = (dK^{\gamma}_{\alpha\beta} - K^{\gamma}_{\alpha\sigma}\phi^{\sigma}_{\beta} - K^{\gamma}_{\sigma\beta}\phi^{\sigma}_{\alpha} + K^{\sigma}_{\alpha\beta}\phi^{\gamma}_{\sigma}) \wedge \tau^{\beta} \\ &\qquad - \phi^{\beta}_{\alpha} \wedge \phi^{\gamma}_{\beta} + \phi^{\beta}_{\alpha}{}' \wedge \phi^{\gamma}_{\beta}{}' + K^{\gamma}_{\sigma\beta}\phi^{\sigma}_{\alpha} \wedge \tau^{\beta} + K^{\sigma}_{\alpha\beta}\phi^{\gamma}_{\sigma} \wedge \tau^{\beta}\ ,\end{aligned}$$

or

$$\begin{aligned}\Theta^\gamma_\alpha - \Theta^{\gamma\prime}_\alpha &= \tfrac{1}{2}\Sigma(K^\gamma_{\alpha\beta;\lambda} - K^\gamma_{\alpha\lambda;\beta})\tau^\lambda \wedge \tau^\beta + \Sigma(\phi^\sigma_\alpha - \phi^{\sigma\prime}_\alpha) \wedge (\phi^\gamma_\sigma - \phi^{\gamma\prime}_\sigma) \\ &= -D_I\Delta + \Delta \wedge \Delta\end{aligned}$$

as claimed.

We will often need (3) explicitly in local coframes and hence record that equation with its contraction. Thus in components (3) reads

$$(4) \qquad K^\gamma_{\alpha\beta;\lambda} - K^\gamma_{\alpha\lambda;\beta} = -R^\gamma_{\alpha\beta\lambda} + R^\gamma_{\alpha\beta\lambda}{}' + K^\sigma_{\alpha\beta}K^\gamma_{\sigma\lambda} - K^\sigma_{\alpha\lambda}K^\gamma_{\sigma\beta} ,$$

and its contraction on γ, β gives

$$(5) \qquad K^\gamma_{\gamma\alpha;\lambda} - K^\gamma_{\alpha\lambda;\gamma} = -R_{\alpha\lambda} + R_{\alpha\lambda}{}' + K^\sigma_{\gamma\alpha}K^\gamma_{\sigma\lambda} - K^\sigma_{\alpha\lambda}K^\gamma_{\gamma\sigma} ,$$

where $R_{\alpha\lambda}$ is the Ricci tensor of I which will alternatively be denoted by Ric when we want to surpress indices.

The tensor

$$\Delta \wedge \Delta \in \Lambda^2T^* \otimes (T^* \otimes T)$$

may be contracted by the action of T on Λ^2T^* to give

$$C(\Delta \wedge \Delta) = \Sigma(K^\gamma_{\beta\alpha}K^\beta_{\gamma\lambda} - K^\gamma_{\alpha\lambda}K^\beta_{\beta\gamma})w^\lambda \circledcirc w^\alpha \in T^* \circledcirc T^* ,$$

a symmetric quadratic differential form.

Matters being so we say that I and I' form a

$$\left.\begin{array}{l}\text{subscalar}\\ \text{scalar}\\ \text{superscalar}\end{array}\right\} \text{ pair if } C(\Delta \wedge \Delta) \text{ is } \left\{\begin{array}{l}\text{positive semidefinite}\\ \text{zero}\\ \text{negative semidefinite} .\end{array}\right.$$

The motivation behind the nomenclature is the following integral formula.

Theorem 2. *Let M be a compact oriented manifold without boundary carrying a pair of metrics I and I'. Then*

$$0 = \int [\mathrm{tr}_I \,\mathrm{Ric}' - \mathrm{tr}_I \,\mathrm{Ric} + \mathrm{tr}_I \, C(\Delta \wedge \Delta)]*1 .$$

Proof. Let $v = \mathrm{tr}_I \,\Delta(I, I') \in T$, and $w = \mathrm{tr}\,\Delta(I, I') \in T^*$. Then the interchange formula of Proposition 1 gives with Stokes' theorem that

$$0 = \int(\mathrm{div}_I \, w - \mathrm{div}_I \, v)*1 = \int[\mathrm{tr}_I \,\mathrm{Ric}' - \mathrm{tr}_I \,\mathrm{Ric} + \mathrm{tr}_I \, C(\Delta \wedge \Delta)]*1 ,$$

where div_I is of course the divergence with respect to the metric I.

Since the scalar curvature is defined by $R = \mathrm{tr}_I \,\mathrm{Ric}$, by definition and Theorem 2 we have that if I and I' are subscalar, then

$$\int \mathrm{tr}_I \mathrm{Ric}' *1 \leq \int R*1 , \qquad \int \mathrm{tr}_{I'} \mathrm{Ric} *'1 \leq \int R'*'1 .$$

At this point we note that each result stated for a subscalar pair of metrics has an analog for superscalar pairs of metrics which we leave to the reader to formulate.

Now we define a diffeomorphism of a Riemannian manifold with metric I to be subscalar if I and f^*I form a subscalar pair, to be scalar if I and f^*I form a scalar pair, and to be superscalar if I and f^*I form a superscalar pair.

Proposition 3. *Let M be a compact oriented Riemannian manifold. Then a subscalar diffeomorphism f which preserves the Ricci tensor is a scalar diffeomorphism.*

Proof. Let I be the given metric and let $I' = f^*I$. Then Theorem 2 applies to give

$$\int \mathrm{tr}_I C(\Delta \wedge \Delta)*1 = 0 ,$$

which by the positivity of $C(\Delta \wedge \Delta)$ forces $C(\Delta \wedge \Delta) = 0$ as required.

3. Metrics inducing the same volume element

Let I, I' be a pair of Riemannian metrics on a manifold M, and let us choose local coframes $\{\tau^\alpha\}$ so that

$$I = \Sigma(\tau^\alpha)^2 , \qquad I' = \Sigma g_{\alpha\beta}'\tau^\alpha\tau^\beta .$$

As such the Levi-Civita connection of I' satisfies

$$dg_{\alpha\beta}' = g_{\alpha\gamma}'\phi_\beta^{\gamma}{}' + \phi_\alpha^{\gamma}{}'g_{\gamma\beta}' ,$$

and hence

$$d(\det_I g_{\alpha\beta}') = \det_I g_{\alpha\beta}'\Sigma \ \mathrm{tr}_I (dg_{\alpha\beta}'g^{\beta\gamma}{}') = 2 \det_I g_{\alpha\beta}'\Sigma\phi_\beta^{\beta}{}' ,$$

or

$$d(\tfrac{1}{2} \log \det_I g_{\alpha\beta}') = \Sigma\phi_\beta^{\beta}{}' = -\Sigma(\phi_\beta^\beta - \phi_\beta^{\beta}{}') = -\Sigma K_{\beta\gamma}^\beta\tau^\gamma .$$

Therefore, if I and I' induce the same volume element so that $\det_I I' = 1$, then

$$(6) \qquad \mathrm{tr}\, \Delta(I, I') = \Sigma K_{\beta\gamma}^\beta = 0 .$$

As a consequence of equations (6) we see that the quadratic differential form $C(\Delta \wedge \Delta)$ simplifies to

$$C(\Delta \wedge \Delta) = \Sigma K_{\beta\alpha}^\gamma K_{\gamma\lambda}^\beta .$$

As a first consequence of this observation we have

Theorem 4. *Let I, I' be two Riemannian metrics which induce the same volume element of a compact, oriented manifold without boundary. Then*

$$0 = \int [(\mathrm{tr}_{I'} - \mathrm{tr}_I)(\mathrm{Ric}' - \mathrm{Ric}) - (\mathrm{tr}_I + \mathrm{tr}_{I'})\, C(\varDelta \wedge \varDelta)]*1 \ .$$

Proof. With the assumption that I and I' induce the same volume element we may take the integral formula of Theorem 2

$$0 = \int [\mathrm{tr}_I \,\mathrm{Ric}' - \mathrm{tr}_I \,\mathrm{Ric} + \mathrm{tr}_I \, C(\varDelta \wedge \varDelta)]*1$$

and the integral formula resulting from reversing the order of I and I'

$$0 = \int [\mathrm{tr}_{I'} \mathrm{Ric} - \mathrm{tr}_{I'} \mathrm{Ric}' + \mathrm{tr}_{I'} \, C(\varDelta \wedge \varDelta)]*'1 \ ,$$

and add them to get

$$0 = -\int (\mathrm{tr}_{I'} - \mathrm{tr}_I)(\mathrm{Ric}' - \mathrm{Ric}) *1 + \int (\mathrm{tr}_I + \mathrm{tr}_{I'})\, C(\varDelta \wedge \varDelta)*1$$

as desired.

We note that in local components this last integral formula reads

$$(7) \qquad 0 = \int [(g^{\sigma\mu\prime} - g^{\sigma\mu})(R_{\sigma\mu}{}' - R_{\sigma\mu}) - (g^{\sigma\mu\prime} + g^{\sigma\mu}) K^{\lambda}_{\alpha\sigma} K^{\alpha}_{\lambda\mu}]*1 \ .$$

4. Einstein metrics

We recall that a metric I is said to be Einstein if $\mathrm{Ric} = \lambda I$, and that if I is defined on a connected manifold of dimension ≥ 3, then λ is necessarily a constant. We will say that M is a positive Einstein manifold if it admits an Einstein metric with λ a positive constant.

We now prove a purely algebraic Lemma which will be needed in the next Theorem.

Lemma 5. *Let A be a positive definite symmetric real matrix. Then*

$$\mathrm{tr}\, A + \mathrm{tr}\, A^{-1} \geq 2m \ ,$$

with equality holding if and only if A is the identity matrix.

Proof. Given any positive real number μ

$$\mu + 1/\mu \geq 2$$

with equality holding if and only if $\mu = 1$. Therefore applying this result m times with A diagonal gives the result.

Theorem 6. *Let M be a compact oriented manifold without boundary. Then a positive Einstein metric and an Einstein metric with arbitrary constant factor, which are subscalar and induce the same element of volume, are isometric.*

Proof. By Theorem 2 with I' the positive Einstein metric

$$0 = \int [\mathrm{tr}_I \mathrm{Ric}' - \mathrm{tr}_I \mathrm{Ric} + \mathrm{tr}_I C(\Delta \wedge \Delta)]^*1$$
$$= \int [\lambda' \mathrm{tr}_I I' - \lambda m + \mathrm{tr}_I (C(\Delta \wedge \Delta)]^*1 \ ,$$

but by Newton's inequality and the hypothesis that I and I' induce the same element of volume

$$\mathrm{tr}_I I' \geq m(\det_I I')^{1/m} = m \ .$$

Hence

$$\int \mathrm{tr}_I C(\Delta \wedge \Delta)^*1 \leq m(\lambda - \lambda') \int {}^*1 \ ,$$

and the hypothesis of subscalar implies $\lambda' \leq \lambda$ and proves that I is also positive Einsteinian. Since the hypothesis on the two Einstein metrics is now symmetric, we have the reverse inequality $\lambda \leq \lambda'$ and hence the equality of the Einstein constants.

As such Theorem 4 now gives

$$\lambda \int (\mathrm{tr}_{I'} - \mathrm{tr}_I)(I' - I)^*1 = \int (\mathrm{tr}_I + \mathrm{tr}_{I'})\, C(\Delta \wedge \Delta)^*1$$

or, using the hypothesis of subscalar,

$$\lambda \int (2m - \mathrm{tr}_I I' - \mathrm{tr}_{I'} I)^*1 \geq 0 \ ,$$

but $\lambda > 0$ and by Lemma 5 the integrand is nonpositive, hence it must vanish identically which by Lemma 5 again implies that $I = I'$ as claimed.

5. The second fundamental form geometry

Let $X\colon M_m \to R^{m+1}$ be an immersion of an m-manifold M_m with the property that the normal vector e_{m+1} may be chosen so that the second fundamental form

$$II = -dX \cdot de_{m+1}$$

is nondegenerate, and hence so that $-II$ defines an abstract pseudo-Riemannian structure on M_m.

We will now study the pseudo-Riemannian geometry of this metric and the natural geometric problem of finding conditions under which two hypersurfaces inducing the same nondegenerate second fundamental form differ by a motion, a program previously only considered for $m = 2$ and II negative definite (see [2], [4], [6]), which is the case of convex surfaces in R^3.

We fix the range of indices so that the capital Latin letters run from 1 to $m+1$ and small Greek letters run from 1 to m, and study the family of local coframes $\{\omega^\alpha\}$ on M_m satisfying

$$-II = \Sigma\varepsilon_\alpha(\omega^\alpha)^2 ,$$

where $\varepsilon_\alpha = \pm 1$. The induced metric $I = dX \cdot dX$ is then expressible in the form

$$I = \Sigma g_{\alpha\beta}\omega^\alpha\omega^\beta .$$

We analyze the geometry of the metric $-II$ by studying the local liftings into the space of affine frames on R^{m+1} having the last leg e_{m+1} normal to the image and having $e_1, \cdots, e_m$ the images of the local frames dual to $\{\omega^\alpha\}$. As a result for the euclidean dot product we have

$$e_\alpha \cdot e_\beta = g_{\alpha\beta} .$$

The space of affine frames on R^{m+1} supports linear differential forms ω^A, ω_A^B obtained from right invariant forms on the affine group of R^{m+1} via an idedtification unique up to right translation.

In particular, if (X, e_A) denotes an affine frame with X the base point and $\{e_A\}$ the $(m + 1)$ legs, then we have the structure equations

$$\begin{aligned} dX &= \Sigma\omega^A e_A , & d\omega^A &= \Sigma\omega^B \wedge \omega_B^A , \\ de_A &= \Sigma\omega_A^B e_B , & d\omega_A^C &= \Sigma\omega_A^B \wedge \omega_B^C . \end{aligned}$$

The confusion in notation between the local coframes on M_m and the coframes on the space of affine frames is intentional, since the local liftings are so chosen that the restriction of ω^α to the affine frames of a local lifting equals the ω^α on M_m, and as a result we can suppress notation indicating pullbacks of local liftings without fear of confusion.

Matters being so the pullbacks of our local liftings satisfy

$$(8) \qquad \omega^{m+1} = 0 ,$$

and

$$(9) \qquad -II = \Sigma\varepsilon_\alpha(\omega^\alpha)^2 .$$

Taking the exterior derivative of (8) gives

$$0 = d\omega^{m+1} = \Sigma\omega^\alpha \wedge \omega_\alpha^{m+1} ,$$

which by Cartan's lemma implies

$$\omega_\alpha^{m+1} = \Sigma h_{\alpha\beta}\omega^\beta h_{\alpha\beta} = h_{\beta\alpha} ,$$

but

$$-II = dx \cdot de_{m+1} = \Sigma\omega^\alpha\omega_{m+1}^\gamma e_\alpha \cdot e_\gamma = \Sigma g_{\alpha\gamma}\omega^\alpha\omega_{m+1}^\gamma ,$$

and

$$0 = d(e_{m+1} \cdot e_\gamma) = \Sigma\omega_{m+1}^\beta g_{\beta\gamma} + \omega_\gamma^{m+1} ;$$

hence

$$-II = -\Sigma\omega^\alpha\omega_\alpha^{m+1} = -\Sigma h_{\alpha\beta}\omega^\alpha\omega^\beta ,$$

which by comparison with (9) gives

$$-h_{\alpha\beta} = \varepsilon_\alpha\delta_{\alpha\beta} ,$$

and results in

$$\omega_\alpha^{m+1} = -\varepsilon_\alpha\omega^\alpha . \tag{10}$$

Now differentiation of (10) gives

$$\begin{aligned} \varepsilon_\alpha d\omega^\alpha &= -d\omega_\alpha^{m+1} = -\Sigma\omega_\alpha^\beta \wedge \omega_\beta^{m+1} \\ &= \Sigma\omega_\alpha^\beta \wedge \varepsilon_\beta\omega^\beta = -\Sigma\omega^\beta \wedge \varepsilon_\beta\omega_\alpha^\beta , \end{aligned} \tag{11}$$

but the structure equations give

$$d\omega^\alpha = \Sigma\omega^\beta \wedge \omega_\beta^\alpha . \tag{12}$$

Therefore letting

$$\varphi_\beta^\alpha = \tfrac{1}{2}(\omega_\beta^\alpha - \varepsilon_\alpha\varepsilon_\beta\omega_\alpha^\beta)$$

and adding ε_α times (11) and (12) we get

$$d\omega^\beta = \Sigma\omega^\alpha \wedge \varphi_\beta^\alpha, \quad \text{and} \quad \varepsilon_\alpha\varphi_\beta^\alpha + \varepsilon_\beta\varphi_\alpha^\beta = 0$$

which implies (see [11]) that the Levi-Civita connection of $-II$ in these coframes is given by φ_β^α.

In the same way it follows from (12) and

$$dg_{\alpha\beta} = \Sigma g_{\alpha\gamma}\omega_\beta^\gamma + \omega_\alpha^\gamma g_{\gamma\beta} \tag{13}$$

that the Levi-Civita connection of I in these coframes is given by ω^α_β.

As such it follows by definition that the difference tensor $\Delta = \Sigma K^\beta_{\alpha\gamma}\omega^\gamma$ between the Levi-Civita connections of I and $-II$ is given by

$$\Delta = \omega^\gamma_\alpha - \varphi^\gamma_\alpha = \tfrac{1}{2}(\omega^\gamma_\alpha + \varepsilon_\alpha\varepsilon_\gamma\omega^\alpha_\gamma) .$$

We note for future reference that this computation immediately implies the symmetry

$$\varepsilon_\beta\varepsilon_\alpha K^\alpha_{\beta\gamma} = K^\beta_{\alpha\gamma} . \tag{14}$$

Next let us recall that the mean curvature H and the Gauss-Kronecker curvature K are defined by

$$mH = \mathrm{trace}_I\, II , \qquad K = \det{}_I\, II ; \tag{15}$$

hence

$$mH = -\Sigma\varepsilon_\alpha g^{\alpha\alpha} , \tag{16}$$

$$K = (-1)^m\pi(\varepsilon_\alpha)/(\det g_{\alpha\beta}) . \tag{17}$$

It will also be convenient to recall that the third fundamental form III of a hypersurface is the metric induced from the Gauss map

$$e_{m+1}\colon M_n \to R^{m+1} .$$

Thus

$$III = de_{m+1}\cdot de_{m+1} = \Sigma g_{\alpha\gamma}\omega^\alpha_{m+1}\omega^\gamma_{m+1} = \Sigma\varepsilon_\sigma\varepsilon_\lambda g^{\sigma\lambda}\omega^\sigma\omega^\lambda , \tag{18}$$

where $g^{\alpha\beta}$ is the matrix inverse to $g_{\alpha\beta}$.

The curvature matrix of the metric $-II$ is easily computed to give

$$\Lambda^\gamma_\beta = \Sigma(\omega^\sigma_\beta - \varphi^\sigma_\beta) \wedge (\omega^\gamma_\sigma - \varphi^\gamma_\sigma) + \tfrac{1}{2}\varepsilon_\beta(\omega^\gamma \wedge \omega^\beta_{m+1} - \omega^\beta \wedge \omega^\gamma_{m+1}) ,$$

and as a result the Riemann-Christoffel curvature tensor defined by

$$\Lambda^\gamma_\beta = \tfrac{1}{2}\Sigma S^\gamma_{\beta\lambda\mu}\omega^\lambda \wedge \omega^\mu$$

is

$$\begin{aligned} S^\gamma_{\beta\lambda\mu} = {} & \Sigma(K^\sigma_{\beta\lambda}K^\gamma_{\sigma\mu} - K^\sigma_{\beta\mu}K^\gamma_{\sigma\lambda}) \\ & + \tfrac{1}{2}(\delta_{\gamma\lambda}\varepsilon_\beta\varepsilon_\mu g^{\beta\mu} - \delta_{\beta\lambda}\varepsilon_\beta\varepsilon_\mu g^{\gamma\mu} - \delta_{\gamma\mu}\varepsilon_\beta\varepsilon_\lambda g^{\beta\lambda} + \delta_{\beta\mu}\varepsilon_\beta\varepsilon_\lambda g^{\gamma\lambda}) . \end{aligned} \tag{19}$$

The Ricci tensor defined by $S_{\beta\lambda} = \Sigma S^\gamma_{\beta\gamma\lambda}$ is

$$S_{\beta\lambda} = -\Sigma(K^\sigma_{\beta\lambda}K^\gamma_{\sigma\gamma} - K^\sigma_{\beta\gamma}K^\gamma_{\sigma\lambda}) + \tfrac{1}{2}((m-2)\varepsilon_\beta\varepsilon_\lambda g^{\beta\lambda} + \varepsilon_\beta\delta_{\beta\lambda}\Sigma\varepsilon_\lambda g^{\gamma\gamma}) , \tag{20}$$

or using the notation of § 2 and the above remarks we may express this in an index free way by

$$\mathrm{Ric}_{-II} = C(\varDelta \wedge \varDelta) + \tfrac{1}{2}(m-2)III + \tfrac{1}{2}mHII \ .$$

The scalar curvature defined by

$$\mathrm{tr}_{-II}\,\mathrm{Ric}_{-II} = \Sigma\varepsilon_\beta S_{\beta\beta}$$

is

$$-\Sigma\varepsilon_\beta(K^\sigma_{\beta\beta}K^\tau_{\beta\tau} - K^\sigma_{\beta\tau}K^\tau_{\sigma\beta}) + (m-1)\varepsilon_\beta\Sigma\varepsilon_\tau g^{\tau\tau} \ , \tag{21}$$

which we may express in an index free way by

$$\mathrm{tr}_{-II}\,\mathrm{Ric}_{-II} = \mathrm{tr}_{-II}\,C(\varDelta \wedge \varDelta) - m(m-1)H \ .$$

Finally, it is natural to consider the expressions for the curvature matrix of I and its invariants relative to the local coframes $\{\omega^\alpha\}$. The curvature matrix of the metric I is easily computed to give

$$\Theta^\tau_\alpha = \omega^{m+1}_\alpha \wedge \omega^\tau_{m+1} \ ,$$

and hence the Riemann-Christoffel curvature tensor defined by

$$\Theta^\tau_\alpha = \tfrac{1}{2}\Sigma R^\tau_{\alpha\beta\lambda}\omega^\beta \wedge \omega^\lambda$$

is

$$R^\tau_{\alpha\beta\lambda} = \varepsilon_\alpha\varepsilon_\beta\delta_{\alpha\lambda}g^{\tau\beta} - \varepsilon_\alpha\varepsilon_\lambda\delta_{\alpha\beta}g^{\tau\lambda} \ , \tag{22}$$

and the Ricci tensor is

$$R_{\alpha\lambda} = \varepsilon_\alpha\delta_{\alpha\lambda}\Sigma\varepsilon_\beta g^{\beta\beta} - \varepsilon_\alpha\varepsilon_\lambda\Sigma\delta_{\alpha\beta}g^{\beta\lambda} \ , \tag{23}$$

which we may express in an index free way by

$$\mathrm{Ric}_I = mHII - III \ .$$

The scalar curvature is

$$\mathrm{tr}_I\,\mathrm{Ric}_I = m^2H^2 - S \ ,$$

where $S = \mathrm{tr}_I\, III$ is known as the length of the second fundamental form.

An important result of these calculations is the following characterization of umbilics.

Proposition 7. *Let $x\colon M_m \to R^{m+1}$ be an immersion with a nondegenerate second fundamental form. Then*

$$\mathrm{tr}_I\,\mathrm{Ric}_{-II} - \mathrm{tr}_I\,\mathrm{Ric}_I - \mathrm{tr}_I\,C(\varDelta \wedge \varDelta) \geq 0 \ ,$$

with equality holding if and only if the point is an umbilic.

Proof.

$$\begin{aligned}&\mathrm{tr}_I \,\mathrm{Ric}_{-II} - \mathrm{tr}_I \,\mathrm{Ric}_I - \mathrm{tr}_I \,C(\Delta \wedge \Delta)\\ &\quad = \mathrm{tr}_I \,C(\Delta \wedge \Delta) + \tfrac{1}{2}(m-2)S + \tfrac{1}{2}m^2H^2 - m^2H^2 + S - \mathrm{tr}_I \,C(\Delta \wedge \Delta)\\ &\quad = \tfrac{1}{2}(mS - m^2H^2)\,,\end{aligned}$$

which by the Cauchy inequality is greater than or equal to zero with equality to zero if and only if the point is an umbilic.

Let $\begin{tikzcd} M_m \arrow[r, "x"] \arrow[dr, "x^{\#}"'] & R^{m+1} \\ & R^{m+1} \end{tikzcd}$ be two immersions of a piece of hypersurface which induce the same nondegenerate second fundamental form. We will make the added convention that geometric quantities computed relative to the second immersion $x^{\#}$ are denoted with a $\#$.

As such we have three possible metrics

$$I = dx \cdot dx, \quad I^{\#} = dx^{\#} \cdot dx, \quad -II = -II^{\#}\,.$$

It is well known (see [8]) that x and $x^{\#}$ differ by a euclidean motion if and only if the local conditions $I = I^{\#}$, $II = II^{\#}$ are satisfied. Our problem is to find other conditions which imply $I = I^{\#}$.

Proposition 8. *Let* $\begin{tikzcd} M_m \arrow[r, "x"] \arrow[dr, "x^{\#}"'] & R^{m+1} \\ & R^{m+1} \end{tikzcd}$ *be two immersions of an m-manifold* M_m *with* $m \geq 3$ *which induce the same nondegenerate second fundamental form and the same quadratic form* $C(\Delta \wedge \Delta)$. *Then* x *and* $x^{\#}$ *differ by a motion.*

Proof. Since $II = II^{\#}$, the Ricci tensors

$$S_{\beta\lambda} = S^{\#}_{\beta\lambda}$$

or writing $C(\Delta \wedge \Delta)$ as $C_{\beta\lambda}$

$$\begin{aligned}&C_{\beta\lambda} - \tfrac{1}{2}((m-2)\varepsilon_\beta\varepsilon_\lambda g^{\beta\lambda} + \varepsilon_\beta\delta_{\beta\lambda}\Sigma\varepsilon_\gamma g^{\gamma\gamma})\\ &\qquad\qquad = C^{\#}_{\beta\lambda} - \tfrac{1}{2}((m-2)\varepsilon_\beta\varepsilon_\lambda g^{\beta\lambda\#} + \varepsilon_\beta\delta_{\beta\lambda}\Sigma\varepsilon_\gamma g^{\gamma\gamma\#})\,,\end{aligned}$$

which since $C_{\beta\lambda} = C^{\#}_{\beta\lambda}$ implies $g^{\beta\lambda} = g^{\beta\lambda\#}$, or equivalently $I = I^{\#}$ as required.

Proposition 9. *Let* $\begin{tikzcd} M_m \arrow[r, "x"] \arrow[dr, "x^{\#}"'] & R^{m+1} \\ & R^{m+1} \end{tikzcd}$ *be two immersions of an m-manifold* M_m *which induce the same nondegenerate second fundamental form and have induced metrics with the same Ricci tensor. Then* x *and* $x^{\#}$ *differ by a motion.*

Proof. Since the induced metrics have the same Ricci tensor, (23) gives

$$-mH\varepsilon_\alpha\delta_{\alpha\lambda} - \varepsilon_\alpha\varepsilon_\lambda g^{\alpha\lambda} = -mH^{\#}\varepsilon_\alpha\delta_{\alpha\lambda} - \varepsilon_\alpha\varepsilon_\lambda g^{\alpha\lambda\#} , \tag{24}$$

and taking the trace with respect to $-I$ we get

$$-m(m-1)H = -m(m-1)H^{\#} ,$$

which together with (24) yields $g^{\alpha\lambda} = g^{\alpha\lambda\#}$ as desired.

As a result it is natural to ask for which pieces of hypersurface does $\mathrm{Ric}_I = \mathrm{Ric}_{-II}$. The question is natural since the last proposition asserts that given a nondegenerate second fundamental form there is up to motions at most one such piece of hypersurface. The present author was unable to settle this as a local problem, but if the hypersurface is required to be compact, it follows from Corollary 15 that the only solutions are spheres.

6. The rank of $\Delta(I, -II)$

Next we derive some algebraic and geometric properties of Δ, the difference tensor of I and $-II$. Let $Z \in T$ be a tangent vector locally defined by $Z = \sum z_\alpha e_\alpha$. Since $\Delta \in (T^* \circledcirc T^*) \otimes T$, we may define $Z\cdot\Delta \in T^* \circledcirc T^*$ to be the symmetric tensor locally defined by

$$Z\cdot\Delta = \Sigma\varepsilon_\gamma K^\gamma_{\alpha\beta} Z_\gamma \omega^\alpha \circledcirc \omega^\beta .$$

Matters being so we may introduce a symmetric bilinear form B in each tangent space defined for $Z, Y \in T$ by

$$B(Z, Y) = \mathrm{tr}_{-II}\, [(Y\cdot\Delta)^t (Z\cdot\Delta)] .$$

Thus for a local frame $\{e_\alpha\}$

$$B(e_\alpha, e_\beta) = \Sigma\varepsilon_\alpha\varepsilon_\beta K^\alpha_{\sigma\mu} K^\beta_{\sigma\mu} .$$

Definition. The rank of Δ is the rank of B as a bilinear form.

Now let us assume this symmetric bilinear form B has rank q at a point p, and choose a local frame $\{e_\alpha\}$ as above but with the additional property that $e_{q+1}, \cdots, e_m$ span the conjugate subspace of B at p. We will call such a local basis an adapted local basis.

Thus, if we fix the ranges of indices

$$1 \leq a, b, c \leq q, \quad q + 1 \leq r, s, t \leq m, \quad 1 \leq \alpha, \beta, \gamma \leq m ,$$

then we have for an adapted local basis at the point p that

$$0 = B(e_r, e_r) = \Sigma(K^r_{\sigma\mu})^2 ,$$

and hence that $0 = K^r_{\sigma\mu}$.

Proposition 10. $\dim\limits_{1\leq\alpha,\beta\leq m} \{\varphi_\alpha^\beta - \omega_\alpha^\beta\} = \operatorname{rank} \varDelta$.

Proof. This follows immediately from the symmetry $K_{\alpha r}^\beta = \varepsilon_\alpha \varepsilon_\beta K_{\beta r}^\alpha$.

Lemma 11. *Let the rank of $\varDelta$ be q on an open set U. Then in an adapted local basis $\Theta_r^s = \varLambda_r^s$.*

Proof. For an adapted local basis $\omega_\alpha^r = \varphi_\alpha^r$ and $\omega_r^\alpha = \varphi_r^\alpha$; hence

$$\Theta_r^s = d\omega_r^s - \Sigma\omega_r^\alpha \wedge \omega_\alpha^s = d\varphi_r^s - \Sigma\varphi_r^\alpha \wedge \varphi_\alpha^s = \varLambda_r^s \ .$$

Prroposition 12. *Let the rank of $\varDelta$ be q on an open set U. Then*

(1) *U is $(m-q)$-umbilical, and*

(2) *the principal curvature associated to the $(m-q)$-umbilical directions is constant.*

Proof. Let us choose an adapted local basis. Then by the last lemma and equations (19) and (22)

$$\begin{aligned}(K_{r\lambda}^\sigma & K_{\sigma\mu}^s - K_{r\mu}^\sigma K_{\sigma\lambda}^s) \\ & + \tfrac{1}{2}(\varepsilon_r\varepsilon_\mu\delta^{r\mu} - \varepsilon_r\varepsilon_\mu\delta_{r\lambda}g^{\mu s} - \varepsilon_r\varepsilon_\lambda\delta_{s\mu}g^{r\lambda} + \varepsilon_r\varepsilon_\lambda\delta_{r\mu}g^{s\lambda}) \\ & \quad = \varepsilon_r\varepsilon_\lambda\delta_{r\mu}g^{s\lambda} - \varepsilon_r\varepsilon_\mu\delta_{r\lambda}g^{s\mu} \ ,\end{aligned}$$

and taking account of $K_{r\lambda}^\sigma = 0$ we have

$$0 = \varepsilon_\lambda\delta_{r\mu}g^{s\lambda} - \varepsilon_\mu\delta_{r\lambda}g^{s\mu} - \varepsilon_\mu\delta_{s\lambda}g^{r\mu} + \varepsilon_\lambda\delta_{s\mu}g^{r\lambda} \ .$$

As such taking $r = s$ we have

$$0 = \varepsilon_\lambda\delta_{r\mu}g^{r\lambda} - \varepsilon_\mu\delta_{r\lambda}g^{r\mu} \ ,$$

and taking $\lambda \neq r$ and $\mu = r$ we see $0 = g^{r\lambda}$ for $\lambda \neq r$.

Similarly, taking $\lambda = s, \mu = r$ and $s \neq r$ we have

$$\varepsilon_s g^{ss} = \varepsilon_r g^{rr}$$

and since $g^{\alpha\alpha} > 0$, this implies $\varepsilon_s = \varepsilon_r$ and

$$g^{rr} = g^{ss} \qquad \text{for } q+1 \leq r, s \leq m \ .$$

We have now shown that the matrix for the inverse of I in an adapted local basis has the form

$$g^{\alpha\beta} = \left(\begin{array}{c|c} g^{ab} & 0 \\ \hline 0 & \underbrace{\begin{matrix} A & & \\ & \ddots & \\ & & A \end{matrix}}_{m-q} \end{array}\right) \left.\vphantom{\begin{matrix} A \\ \ddots \\ A \end{matrix}}\right\} m-q$$

which implies that the neighborhood U in $(m-q)$-umbilical.

Assuming (1) we have $g^{r\lambda} = A\delta_{r\lambda}$; hence

$$dg^{r\lambda} = -g^{r\mu}\omega_\mu^\lambda - \omega_\gamma^r g^{\gamma\lambda}$$

implies

$$dA\delta_{r\lambda} = -A\omega_r^\lambda - \omega_\gamma^r g^{\gamma\lambda} = \omega_\gamma^r(A\delta_{\gamma\lambda} - g^{\gamma\lambda}) - A(\omega_r^\lambda + \omega_\lambda^r) ,$$

or taking account of (1) again we have

$$AK_{r\mu}^\lambda\omega^\mu = -dA\delta_{r\lambda} + \omega_a^r(A\delta_{a\lambda} - g^{a\lambda}) , \tag{25}$$

which for $\lambda = r$ gives $0 = -dA$ proving that A is constant and establishing (2).

Corollary 13. *Let $\Delta \equiv 0$ on an open set U. Then U is a piece of a hypersphere.*

Proof. Since $\Delta \equiv 0$, the rank of Δ is 0 and the last proposition implies that every point is umbilical. The conclusion is now classical.

7. Uniqueness theorems for convex hypersurfaces

In this section we study compact hypersurfaces with a nondegenerate second fundamental form. The compactness implies that the second fundamental form is actually negative definite since it must be negative definite at the furthest point from an interior origin. As a result the compact hypersurfaces with a nondegenerate second fundamental form are convex hypersurfaces (see [8, Vol. II, p. 41]).

We next investigate an important facet of the quadratic differential form $C(\Delta \wedge \Delta)$ constructed as in § 2 from the difference tensor of the first fundamental form I and the negative of the second fundamental form $-II$, which is the behaviour of its I-trace at umbilical points.

Theorem 14. *Let $X: M_m \to R^{m+1}$ be a compact hypersurface with a negative definite second fundamental form, and let $*1$ denote the volume element induced by X. Then*

$$\int \mathrm{tr}_I\, C(\Delta \wedge \Delta)*1 = (1/4)\int (m^2H^2 - mS)*1 , \tag{26}$$

where H is the mean curvature, and S is the length of II defined in any set of frames by $S = \mathrm{tr}_I\, III$, and as a result

$$\int \mathrm{tr}_I\, C(\Delta \wedge \Delta)*1 \geq 0 \tag{27}$$

if and only if $X(M_m)$ is a euclidean sphere.

Proof. By the integral formula of Theorem 2 and Proposition 7

$$0 = \int [\mathrm{tr}_I \, \mathrm{Ric}_{-II} - \mathrm{tr}_I \, \mathrm{Ric}_I - \mathrm{tr}_I \, C(\Delta \wedge \Delta) + 2 \, \mathrm{tr}_I \, C(\Delta \wedge \Delta)]*1$$
$$= \int [\tfrac{1}{2}(mS - m^2H^2) + 2 \, \mathrm{tr}_I \, C(\Delta \wedge \Delta)]_* 1 \ ,$$

which establishes (26).

The second result follows since Cauchy's inequality implies $m^2H^2 - mS \leq 0$ with equality holding only at umbilic points. Therefore assuming (27) every point is an umbilic, and the image is a euclidean sphere.

Corollary 15. *Let $X\colon M_m \to R^{m+1}$ be a compact hypersurface with a negative definite second fundamental form. Then*

$$\int \mathrm{tr}_I \, \mathrm{Ric}_{-II} *1 \leq \int \mathrm{tr}_I \, \mathrm{Ric}_I *1$$

if and only if $X(M_m)$ is a euclidean sphere.

Proof. By Theorem 2

$$\int \mathrm{tr}_I \, C(\Delta \wedge \Delta)*1 = \int (\mathrm{tr}_I \, \mathrm{Ric}_I - \mathrm{tr}_I \, \mathrm{Ric}_{-II})*1 \ ;$$

hence the hypothesis forces

$$\int \mathrm{tr}_I \, C(\Delta \wedge \Delta)*1 \geq 0 \ ,$$

and the result follows by the last theorem.

We will now show that a direct analysis of $\mathrm{tr}_I \, C(\Delta \wedge \Delta)$ leads to an integral formula proof of a theorem on Weingarten hypersurfaces. (For the best results in this direction see [1].)

First let us observe that equations (16) and (17) with

$$dg^{\beta\gamma} = -g^{\beta\mu}\omega^{\gamma}_{\mu} - \omega^{\beta}_{\lambda}g^{\lambda\gamma} \ , \qquad dg_{\alpha\beta} = g_{\alpha\gamma}\omega^{\gamma}_{\beta} + \omega^{\gamma}_{\alpha}g_{\gamma\beta}$$

imply

(28) $$d(mH/2) = -d(\Sigma g^{\alpha\alpha}/2) = \Sigma g^{\alpha\mu}(\omega^{\alpha}_{\mu} + \omega^{\mu}_{\alpha})/2 = \Sigma g^{\alpha\mu}K^{\alpha}_{\mu\beta}\omega^{\beta} \ ,$$

(29) $$d\left(\frac{1}{2}\log\frac{1}{K}\right) = \frac{1}{2}\,\frac{d(\det g_{\alpha\beta})}{\det g_{\alpha\beta}} = \frac{1}{2}\,\mathrm{trace}\,(\Sigma dg_{\alpha\beta}g^{\beta\gamma})$$
$$= \Sigma\omega^{\alpha}_{\alpha} = \Sigma K^{\alpha}_{\alpha\beta}\omega^{\beta} \ .$$

As a result

$$\mathrm{tr}_I \, C(\Delta \wedge \Delta) = \Sigma g^{\alpha\beta}K^{\alpha}_{\sigma\mu}K^{\beta}_{\sigma\mu} - \Sigma g^{\alpha\beta}K^{\beta}_{\alpha\gamma}K^{\lambda}_{\lambda\gamma}$$
$$= \Sigma g^{\alpha\beta}K^{\alpha}_{\sigma\mu}K^{\beta}_{\sigma\mu} - \mathrm{grad}\,(mH/2)\cdot\mathrm{grad}\,(\tfrac{1}{2}\log(1/K)) \ .$$

Theorem 16. *The only compact C^4-hypersurfaces having a nondegenerate second fundamental form and having the mean curvature H and Gauss-Kronecker curvature K satisfying a fnuctional relationship $f(H, K) = 0$, where $K(\partial f/\partial H)(\partial f/\partial K) < 0$, are euclidean spheres.*

Proof. $f(H, K) = 0$ implies

$$\frac{\partial f}{\partial H} \operatorname{grad} H + \frac{\partial f}{\partial K} \operatorname{grad} K = 0 ,$$

or

$$\begin{aligned} \frac{\partial f}{\partial H} \|\operatorname{grad} H\|^2 &= \frac{\partial f}{\partial K}(-\operatorname{grad} K \cdot \operatorname{grad} H) \\ &= \frac{4K}{m} \frac{\partial f}{\partial K} \operatorname{grad} \left(\frac{1}{2} \log (1/K)\right) \cdot \operatorname{grad} (mH/2) . \end{aligned}$$

Therefore the hypothesis $K(\partial f/\partial H)(\partial f/\partial K) < 0$ implies

$$\operatorname{grad} (mH/2) \cdot \operatorname{grad} (\tfrac{1}{2} \log (1/K)) < 0 ,$$

which forces $\operatorname{tr}_I C(\Delta \wedge \Delta) \geq 0$, and the result follows from Theorem 14.

Finally, we give a characterization of the sphere, which generalizes the result of Liebmann [9] that a compact convex hypersurface with constant K is a sphere.

By equation (28)

$$d(mH/2) = \Sigma g^{\alpha\mu} K^{\alpha}_{\mu\beta} \omega^{\beta} ;$$

hence the Laplacian with respect to the $-II$ metric is computed from

$$\begin{aligned} &d\Sigma(g^{\beta\gamma} K^{\gamma}_{\beta\alpha}) - \Sigma g^{\beta\gamma} K^{\gamma}_{\beta\delta} \varphi^{\sigma}_{\alpha} \\ &\quad = -\Sigma g^{\beta\gamma} \omega^{\sigma}_{\gamma} K^{\sigma}_{\beta\alpha} - \Sigma \omega^{\beta}_{\lambda} g^{\lambda\gamma} K^{\gamma}_{\beta\alpha} + \Sigma g^{\beta\gamma} dK^{\gamma}_{\beta\alpha} - \Sigma g^{\beta\gamma} K^{\gamma}_{\beta\sigma} \varphi^{\sigma}_{\alpha} \\ &\quad = \Sigma g^{\beta\gamma} (K^{\gamma}_{\beta\alpha;\lambda} - 2K^{\mu}_{\gamma\lambda} K^{\mu}_{\beta\alpha}) \omega^{\lambda} , \end{aligned}$$

and results in the integral formula

$$(30) \qquad 0 = \int \operatorname{Lap}_{-II}(mH/2) dA_{-II} = \int g^{\beta\gamma} (K^{\gamma}_{\beta\alpha;\alpha} - 2K^{\beta}_{\mu\alpha} K^{\gamma}_{\mu\alpha}) dA_{-II} .$$

Lemma 17. *Let $-II$ be positive definite. Then*

$$(31) \qquad K^{\gamma}_{\alpha\beta;\lambda} - K^{\gamma}_{\alpha\lambda;\beta} = \tfrac{1}{2}(g^{\lambda\gamma}\delta_{\alpha\beta} - g^{\gamma\beta}\delta_{\alpha\lambda} + g^{\alpha\lambda}\delta_{\beta\gamma} - g^{\alpha\beta}\delta_{\gamma\lambda}) ,$$

where ; *denotes covariant differentiation relative to $-II$.*

Proof. We differentiate

$$\tfrac{1}{2}(\omega^\gamma_\alpha + \omega^\alpha_\gamma) = \Sigma K^\gamma_{\alpha\beta}\omega^\beta$$

to get

$$\begin{aligned}\Sigma(dK^\gamma_{\alpha\beta} - K^\gamma_{\sigma\beta}\varphi^\sigma_\alpha + K^\sigma_{\alpha\beta}\varphi^\gamma_\sigma - K^\gamma_{\alpha\sigma}\varphi^\sigma_\beta) \wedge \omega^\beta \\ = \tfrac{1}{2}\omega^{m+1}_\alpha \wedge \omega^\gamma_{m+1} + \tfrac{1}{2}\omega^{m+1}_\gamma \wedge \omega^\alpha_{m+1} \\ = \Sigma(-\tfrac{1}{2}g^{\gamma\sigma}\omega^\alpha \wedge \omega^\sigma - \tfrac{1}{2}g^{\alpha\sigma}\omega^\gamma \wedge \omega^\sigma) ,\end{aligned}$$

and the result follows.

Now utilizing equation (31), (30) becomes

$$(32) \qquad 0 = \int [2\Sigma g^{\beta\gamma}K^\beta_{\alpha\alpha;\gamma} - 4\Sigma g^{\beta\gamma}K^\beta_{\mu\alpha}K^\gamma_{\mu\alpha} - m\Sigma(g^{\alpha\beta})^2 + (\Sigma g^{\alpha\alpha})^2]dA_{-II} .$$

Theorem 18. *Let $X: M_m \to R^{m+1}$ be a convex immersion of a compact manifold M_m with*

$$\Sigma g^{\beta\gamma}K^\beta_{\alpha\alpha;\gamma} \leq 0 .$$

Then $X(M_m)$ is a euclidean sphere.

Proof. By the Cauchy inequality

$$(33) \qquad -m\Sigma(g^{\alpha\beta})^2 + (\Sigma g^{\alpha\alpha})^2 \leq 0$$

with equality holding if and only if there is a change of frames $\{\tau^\alpha\}$ in which

$$-II = \Sigma(\tau^\alpha)^2 , \qquad g^{\alpha\beta} = \lambda\delta_{\alpha\beta} .$$

Now the hypothesis with the observation that

$$\Sigma g^{\beta\gamma}K^\beta_{\mu\alpha}K^\gamma_{\mu\alpha} \geq 0 ,$$

since it is a sum of lengths of vectors, forces the sign of the integrand in (32) to be negative and therefore forces the equality in (33). As a result, $-II = I/\lambda$ which proves that every point is an umbilic. Hence $X(M_m)$ is a euclidean sphere.

8. The third fundamental form geometry

We will continue to work with the frames in § 5 and study the Gauss map

$$e_{m+1}: rN_n \longrightarrow R^{m+1}$$

of a hypersurface with a negative definite second fundamental form.

As we have already noted, the induced metric of the Gauss map is called the third fundamental form and is given by

$$III = de_{m+1} \cdot de_{m+1} = \Sigma g^{\alpha\beta}\omega^\alpha\omega^\beta .$$

Now

$$d\omega^\gamma = \Sigma\omega^\beta \wedge \omega_\beta^\gamma = \Sigma\omega^\beta \wedge (-\omega_\gamma^\beta) - \Sigma\omega^\beta \wedge K_{\beta\alpha}^\gamma\omega^\alpha ,$$

which by the symmetry of $K_{\beta\alpha}^\gamma$ in β, α gives

$$d\omega^\gamma = \Sigma\omega^\beta \wedge (-\omega_\gamma^\beta) ,$$

and

$$dg^{\alpha\beta} = g^{\alpha\gamma}(-\omega_\gamma^\beta) + (-\omega_\gamma^\alpha)g^{\gamma\beta}$$

imply that the Levi-Civita connection for III is

$$\eta_\alpha^\beta = -\omega_\beta^\alpha .$$

The curvature is given by

$$d\eta_\alpha^\beta - \Sigma\eta_\alpha^\gamma \wedge \eta_\gamma^\beta = \Sigma T_{\alpha\lambda\mu}^\beta\omega^\lambda \wedge \omega^\mu ,$$

and we see

$$T_{\alpha\lambda\mu}^\beta = -R_{\beta\lambda\mu}^\alpha .$$

As a result the Ricci tensor

$$T_{\alpha\lambda} = -\Sigma R_{\beta\beta\lambda}^\alpha = (m-1)g^{\alpha\lambda} ,$$

or invariantly $\mathrm{Ric}_{III} = (m-1)III$, which implies that III is an Einstein metric.

Next we note that the difference tensor of III and $-II$ is given by

$$-\omega_\beta^\alpha - \tfrac{1}{2}(\omega_\alpha^\beta - \omega_\beta^\alpha) = -\tfrac{1}{2}(\omega_\alpha^\beta + \omega_\beta^\alpha) = -\varDelta ,$$

where as usual $\varDelta$ is the difference tensor of I and $-II$.

As a result of these calculations we have a characterization of umbilics which is similar to Proposition 7.

Proposition 19. *Let $X: M_m \to R^{m+1}$ be an immersion with a negative definite second fundamental form. Then*

$$\mathrm{tr}_{III}\,\mathrm{Ric}_{-II} - \mathrm{tr}_{III}\,\mathrm{Ric}_{III} - \mathrm{tr}_{III}\,C(\varDelta \wedge \varDelta) \geq 0$$

with equality holding if and only if the point is an umbilic.

Proof.
$$\mathrm{tr}_{III}\,\mathrm{Ric}_{-II} - \mathrm{tr}_{III}\,\mathrm{Ric}_{III} - \mathrm{tr}_{III}\,C(\varDelta \wedge \varDelta) = -m^2/2 + \tfrac{1}{2}(\Sigma g^{\alpha\alpha})(\Sigma g_{\alpha\alpha}) ,$$

but the Arithmetic-Geometric mean inequality implies that positive definite symmetric matrices A satisfy

$$\mathrm{tr}\,A\ \mathrm{tr}\,A^{-1} \geq m(\det A)^{1/m} m(\det A^{-1})^{1/m} \geq m^2$$

with equality holding if and only if there is a change of basis with A equal to a multiple of the identity. Hence letting $A = II^{-1}I$ gives the result.

Theorem 20. *If*
$$\begin{tikzcd} M_m \arrow[r, "X"] \arrow[dr, "X'"'] & R^{m+1} \\ & R^{m+1} \end{tikzcd}$$
are two imbeddings of a compact manifold without boundary inducing the same negative definite second fundamental forms and the same Gauss-Kronecker curvatures, then they differ by a euclidean motion.

Proof. Now the volume element of the Gauss metric or third fundamental form is given by

$$(\det g^{\alpha\beta})^{1/2} w^1 \wedge \cdots \wedge w^m ,$$

and the Gauss-Kronecker curvature is defined by

$$K = \det{}_I II = (-1)^m 1/\det{}_{-II} I = (-1)^m \det g^{\alpha\beta} .$$

Therefore, if III and III' are the third fundamental forms of a pair of convex immersions having the same second fundamental forms and the same Gauss-Kronecker curvatures, then they are Einstein metrics which induce the same volume element.

The difference tensor J of III and $III^{\#}$ is given by

$$J = \Sigma J^{\gamma}_{\alpha\beta}\omega^{\beta} = \Sigma(K^{\gamma\#}_{\alpha\beta} - K^{\gamma}_{\alpha\beta})\omega^{\beta} ,$$

which is symmetric in all three indices. Since III and $III^{\#}$ induce the same element of volume, and $J^{\gamma}_{\alpha\beta}$ is symmetric in all indices,

$$C(J \wedge J) = \Sigma J^{\beta}_{\alpha\gamma} J^{\beta}_{\alpha\lambda}$$

is positive semi-definite proving that III and $III^{\#}$ are subscalar. The result now follows from Theorem 6.

Corollary 21. *If*
$$\begin{tikzcd} M_m \arrow[r, "X"] \arrow[dr, "X^{\#}"'] & R^{m+1} \\ & R^{m+1} \end{tikzcd}$$
are two imbeddings of a compact manifold without boundary inducing the same negative definite second fundamental forms and the same volume element, then X and $X^{\#}$ differ by a motion.

Proof. X and $X^{\#}$ induce the same volume element if and only if

$$\det{}_{-II}\, g_{\alpha\beta} = \det{}_{-II}\, g^{\#}_{\alpha\beta} ,$$

and hence by (17) if and only if $K = K^{\#}$. Now we may apply Theorem 20 and the result follows.

Utilizing Proposition 19 we may prove the following results in exact analogy to Theorem 14 and Corollary 15.

Theorem 22. *Let $X: M_m \to R^{m+1}$ be compact hypersurface with a negative definite second fundamental form. Then*

$$\int \mathrm{tr}_{III}\, C(\varDelta \wedge \varDelta)*1 \geq 0 \ ,$$

with $*1$ the volume element of III, if and only if $X(M_m)$ is a euclidean sphere.

Corollary 23. *Let $X: M_m \to R^{m+1}$ be a compact hypersurface with a negasive definite second fundamental form. Then*

$$\int \mathrm{tr}_{III}\, \mathrm{Ric}_{-II} *1 \leq m(m-1) \int *1 \ ,$$

where $*1$ is the volume element of III, if and only if $X(M_m)$ is a euclidean sphere.

Finally, if we let $P_1 = H/K$ and $P_m = 1/K$, then we may prove the following in exact analogy to Theorem 16.

Theorem 24. *The only compact C^4-hypersurfaces having a nondegenerate second fundamental form and satisfying a functional relationship $f(P_1, P_m) = 0$, where $P_m(\partial f/\partial P_1)(\partial f/\partial P_m) < 0$, are euclidean spheres.*

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