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NONDEGENERATE CURVES ON A RIEMANNIAN MANIFOLD

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1. Introduction

Let X be a connected Riemannian manifold of dimension $n \geq 3$. By a nondegenerate curve we mean a C^2 immersion γ of the interval I or the circle C into X, such that the square of the geodesic curvature $k_g(\gamma)^2$ never vanishes. By forcing the geodesic curvature to be positive we are able to associate with γ a moving orthonormal 2-frame $(t(\gamma)(t), n(\gamma)(t))$, $t(\gamma)(t), n(\gamma)(t) \in T(X)_{\gamma(t)}$ along γ, where $t(\gamma)(t)$ is the unit tangent to γ, and $n(\gamma)(t)$ is the principal normal; these all will be discussed in more detail in the next section. We can also associate with γ the continuous positive function $k_g(\gamma)(t)$ given by the geodesic curvature. Let $\pi_0 \colon V_2(X) \to X$ be the Stiefel bundle of orthonormal two frames constructed from $T(X)$. Thus, we can associate with γ, a curve $\varphi(\gamma)(t) = (\gamma(t), t(\gamma)(t), n(\gamma)(t), k_g(\gamma)(t))$ in the bundle $\pi \colon V \to X$ where $V = V_2(X) \times R^+$ (R^+ being the positive reals) which is a cross-section over γ. Let us pick $\theta_0 \in C$, and $v_0 = (x_0, t_0, n_0, k_0) \in V_{x_0}$. Let N_0 be the nondegenerate immersions γ of the circle C into X, such that $\varphi(\gamma)(\theta_0) = v_0$. Our main theorem states that φ, which associates with each $\gamma \in N_0$ a loop $\varphi(\gamma)$ in V based at v_0, in a weak homotopy equivalence, and hence by Whitehead's theorem a homotopy equivalence (provided N_0 has a suitable topology). Hence we see that the arc-components of N_0 (nondegenerate regular homotopy classes) are in a one-one correspondence with the elements of $\pi_1(V_2(X) \times R^+, v_0) \cong \pi_1(v_2(X), (x_0, t_0, n_0))$. In the case where $X = R^3$, with the Euclidean (flat) metric we recover the main theorem of [3].

2. Definitions and an outline of the paper

Let X be a Riemannian manifold of dimension ≥ 3, g its Riemann metric, and D the Riemannian connection (covarient derivative) induced by g (see [6]). Let $\gamma \colon I \to X$ be an immersion, t parametrize the interval $[a, b] = I$, $\gamma(t)$ be the parametrized curve, and $\dot\gamma(t) = d\gamma/dt|_t = d\gamma(d/dt) \in T(X)_{\gamma(t)}$ be the tangent vector of the parametrized curve $\gamma(t)$. The square of the geodesic curvature is given by the formula $k_g(\gamma)(t)^2 = |\dot\gamma(t)|_{\gamma(t)}^{-2} |D_{\dot\gamma(t)} t(\gamma)(t)|_{\gamma(t)}^2$ where $t(\gamma)(t) = \dot\gamma(t)/|\dot\gamma(t)|_{\gamma(t)}$

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is the unit tangent vector of γ at $\gamma(t)$, and $|v|_{\gamma(t)} = g(\gamma(t))(v, v)^{1/2}$ where $v \in T(X)_{\gamma(t)}$. It is easy to see by a direct calculation that this number is independent of the orientation and parametrization chosen for I. Let us fix once and for all, an orientation for I. If γ is nondegerate, we can define a unique principal normal vector by the formula

$$n(\gamma)(t) = [D_{\dot{\gamma}(t)}(t(\gamma)(t))]\,|\dot{\gamma}(t)|^{-1}\,(+\sqrt{(k_g(\gamma)(t))^2})^{-1} .$$

We will always follow this convention. It is again easily seen that $n(\gamma)$ is independent of the choice of parameter on I. (It does depend upon the orientation which we have fixed.) Finally, we set $k_g(\gamma)(t) = +\sqrt{k_g(\gamma)(t)^2}$. We note that $k_g(\gamma)$ and $n(\gamma)$ are of class C^{k-2}, and $t(\gamma)$ is of class C^{k-1}, whenever γ is of class C^k.

Let π_0: $V_2(X) \to X$ be the Stiefel bundle of 2-frames in n-space associated with the tangent bundle $T(X)$. By this we mean for each $x \in X$, the fiber $\pi_0^{-1}(x) = V_2(X)_x$ is the Stiefel manifold of orthonormal 2-frames in the Euclidean vector space $(T(X)_x, g(x))$. We recall that $V_2(X)_x$ is compact, and can be viewed as a closed submanifold of $S_x \times S_x$ where S_x is the unit sphere in $T(X)_x$. In fact most of the time we will view $V_2(X)_x$ as a closed bounded subset of $T(X)_x \times T(X)_x$, i.e., $V_2(X) = \{(v, \omega) \in T(X)_x \times T(X)_x, |v|_x = |\omega|_x = 1$, and $g(x)(v, \omega) = 0\}$. Finally, let $V = V_2(X) \times R^+$, where R^+ denotes the strictly positive real numbers, and let π: $V \to X$ be the composition of the projection onto the first factor followed by π_0.

Let us fix an orientation for the circle C, and let $I = [0, 2]$. Let us set $E(I, X) = \{f\colon [0, 2] \to X \,|\, f$ is C^2, and f is a nondegenerate immersion$\}$. Let $E(C, X)$ be those elements of $E(I, X)$ which can be extended to a C^2 periodic map of period 2 and principal domain of definition $[0, 2]$. Let us endow these sets with the C^2-topology. (The two possible choices of C^2-topology agree because I and C are both compact, [2], [8]. In fact, these are open subsets of the function spaces consisting of all mappings $C^2(I, X)$ and $C^2(C, X)$.) The elements of $E(I, X)$ and $E(C, X)$ are the parametrized non-degenerate curves. Let $ND(I, X)$ and $ND(C, X)$ denote respectively the set of equivalence classes of elements of $E(I, X)$ and $E(C, X)$, where we identify f and g if and only if they differ by an orientation preserving C^2 reparametrization of I or C. If we identify an element of $E(I, X)$ which is parametrized proportional to arc length with the corresponding unique element of $ND(I, X)$, we can view $ND(I, X)$ as a subspace of $E(I, X)$. Let us define R: $E(I, X) \times [0, 1] \to ND(I, X)$ by the formula $R(\gamma, u)(t) = \gamma((1 - u)t + us_\gamma(t))$ where s_γ is the parameter proportional to arc length, and t is the given parameter. R is continuous and defines a deformation retract of $E(I, X)$ onto $ND(I, X)$, and therefore these spaces have the same homotopy type. Let $C^k(I, M)$ denote the C^k functions from I into a manifold M with the C^k topology.

If $\gamma \in ND(I, X)$ or $E(I, X)$, let $t(\gamma) \in C^1(I, T(X))$ denote the map $t(\gamma)(t) =$ unit tangent vector to γ at $\gamma(t)$. The induced map $t\colon E(I, X) \to C^1(I, T(X))$ is clearly continuous. Similarly we can define continuous maps $n\colon E(I, X) \to C^0(I, T(X))$ and $k_g\colon E(I, X) \to C^0(I, R^+)$ by the formulas $n(\gamma)(t) =$ principal normal to γ at $\gamma(t)$, and $k_g(\gamma)(t) =$ geodesic curvature of γ at $\gamma(t)$. We can also define $v\colon E(I, X) \to C^0(I, V)$ by $v(\gamma)(t) = (\gamma(t), t(\gamma)(t), n(\gamma)(t), k_g(\gamma)(t))$. When we replace I by C all the same statements hold true. Let us pick $v_0 = (x_0, t_0, n_0, k_0) \in V$, let $E_0 = \{\gamma \in E(I, X) \mid v(\gamma)(0) = v_0\}$, and give E_0 the induced topology. We can now state precisely our main theorem.

Theorem A. *Let $p\colon E_0 \to V$ be defined by $p(\gamma) = v(\gamma)(1)$; p is clearly a continuous map. Let us pick a base point $\gamma_0 \in p^{-1}(v_0)$, and let $p_*\colon \pi_k(E_0, p^{-1}(v_0); \gamma_0) \to \pi_k(V, v_0)$ be the usual induced map on homotopy groups (and sets). Then p_* is an isomorphism for all $k \geq 2$, and a bijection for $k = 1$.*

We prove this by showing that the triple $p\colon E_0 \to V$ satisfies enough of a homotopy lifting property to imply p_* is a bijection. We define and discuss this property in some detail in § 3, and show among other things that it is a local property.

Pick a point $\theta_0 \in C$, and let $N_0 = \{\gamma \in ND(C, X) \mid v(\gamma)(\theta_0) = v_0\}$. Thus the deformation retract defined by R gives us a homotopy equivalence between the spaces $p^{-1}(v_0), p^{-1}(v_0) \cap E_0(C, X)$ and N_0. We show in § 7 that $\pi_i(E_0, \gamma_0) = 0$ for all i. Therefore the homotopy sequence implies that $\pi_i(N_0, \gamma_0) \cong \pi_{i+1}(V, v_0) \cong \pi_{i+1}(V_2(X), (x_0, t_0, n_0))$, assuming γ_0 is parametrized proportional to arc length. If we set $i = 0$, we can classify the arc-components of N_0, i.e., the based nondegenerate regular homotopy classes, by looking at $\pi_1(V_2(X), (x_0, t_0, n_0))$. Let $\Omega_0 = \{\gamma \in C^0(C, V) \mid \gamma(\theta_0) = v_0\}$, where Ω_0 has the C^0 (compact-open) topology. Let $\varphi\colon N_0 \to \Omega_0$ be defined by $\varphi(\gamma)(t) = (\gamma(t), t(\gamma)(t), n(\gamma)(t), k_g(\gamma)(t))$; φ is continuous and by our theorem a weak homotopy equivalence. Both N_0 and Ω_0 carry the structure of paracompact Banach manifold [10]. Hence by theorems of Palais [9] these spaces satisfy the hypotheses of the Whitehead theorem. Thus $\varphi\colon N_0 \to \Omega_0$ is a homotopy equivalence.

We will close this section by outlining the remainder of this paper. § 3 as mentioned deals with a local lifting property which will imply Theorem A. In § 4 we compare locally the case of an arbitrary metric and the flat metric induced by taking Riemann normal coordinates as orthonormal coordinates of a flat space. We can then reduce the "curved" space problem to a slightly more involved "flat" problem. The crucial lemma of this paper is Lemma 5.1. It is a generalization of the proposition in [5]; also see [3, 2.1]. The idea is as follows. Let $\lambda\colon [0, 1] \to S^{n-1}$ be an immersion, and $\rho(t) > 0$ a C^1 function. Then $\gamma(t) = \int_0^t \lambda(\tau)\rho(\tau)d\tau$ is nondegenerate, $t(\gamma)(1) = \lambda(1), n(\gamma)(1) = t(\lambda)(1)$, and $k_g(\gamma)(1) = 1/\rho(1)$. If we use Proposition 4.1 to reduce the problem to a Euclidean one, we can then try to apply Smales immersion theorem [11], to curves on the sphere, and then try to construct the desired nondegenerate curves γ by picking the appropriate weighting function ρ. However, in our lifting problem we must be able to construct ρ such that $\gamma(1) = x$, x being some relatively arbitrary point near 0. In § 5.1 we see how arbitrary x can be, provided λ has some nice properties. In § 6 we prove some technical lemmas which enable us to apply Lemma 5.1 by insuring that our λ's have the desired properties. § 7, entitled odds and ends, contains a technical reparametrization, Lemma 7.1, and the proof that E_0 is weakly contractable, Corollary 7.2. In § 8 we reduce the proof of Theorem A to an abstract Theorem 8.2, which we prove in § 9. In § 8 we have to introduce certain Sobolev spaces. Anything we need can be found in [1, pp. 165–168].

3. Abstract topology

Let $I^n =$ the n-cube$=\{(x_1, \cdots, x_n) \mid 0 \leq x_i \leq 1, 1 \leq i \leq n\} \subseteq R^n$, $I^{n-1}_{k,i} = \{x \in I^n \mid x_k = i\}$, $i = 0, 1$, $F^{n-1} = \bigcup_{k=1}^{n} I^{n-1}_{k,1}$ $\partial I^n = \bigcup_{(k,i)} I^{n-1}_{k,i}$, and $J^{n-1} = \{x \in \partial I^n \mid x \notin \operatorname{Int} I^{n-1}_{n,1}\}$.

Definition 3.1. A one parameter family of maps $h_t\colon I^n \to I^n, 0 \leq t \leq 1$, is said to be an *admissible deformation* of I^n if:

i) the induced map $H\colon I \times I^n \to I^n$ defined by $H(t, x) = h_t(x)$ is continuous,

ii) $h_0 = \mathrm{id}, h_t \mid F^{n-1} = \mathrm{id}$ for all $t \in [0, 1]$, and

iii) $h_t(\partial I^n) \subseteq \partial I^n$ for all $t \in [0, 1]$.

Remark. Let h_t be an admissible deformation of I^n, and $K\colon (I^n, F^{n-1}) \to (I^n, J^{n-1})$ a homeomorphism mapping F^{n-1} homeomorphically onto J^{n-1}. Let $\bar{h}_t = K \circ h_t \circ K^{-1}$ and let $\bar{H}\colon I \times I^n \to I^n$ be the induced map defined by $\bar{H}(t, x) = \bar{h}_t(x) = K \circ H(t, K^{-1}x)$. Then $\bar{H}$ is continuous, $\bar{h}_t(x) = x$ for all $x \in J^{n-1}$, $t \in [0, 1]$, $\bar{h}_0 = \mathrm{id}$ and $\bar{h}_t(\partial I^n) \subseteq \partial I^n$. Hence, if we replace F^{n-1} by J^{n-1} in Definition 3.1 we get a completely equivalent notion.

Definition 3.2. Let $h_t\colon I^n \to I^n, 0 \leq t \leq 1$, be an admissible deformation of I^n. We say h_t is a *strong admissible deformation* if $h_t(I^{n-1}_{k,1}) \subseteq I^{n-1}_{k,1}$ for $1 \leq k \leq n$.

Definition 3.3. Let $\pi\colon E \to B$ be a triple where E and B are topological spaces, and π is a continuous map.

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ I^n \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi_0"'] & B \end{tikzcd}$$

We say $\{\pi\colon E \to B\}$ has (*strong*) *property P*, if for each n and each pair of continuous maps $\varphi_0\colon I^n \to B$ and $\psi\colon F^{n-1} \to E$ such that $\pi \circ \psi = \varphi_0 \mid F^{n-1}$, we can find a (strong) admissible deformation h_t of I^n and an extension Ψ of ψ to all I^n such that $\pi \circ \Psi = \varphi_0 \circ h_1$.

Let us note that there is a notion exactly equivalent to Definition 3.3 if we replace F^{n-1} by J^{n-1}. In fact, $\{\pi\colon E \to B\}$ has property P if and only if it has property P with J^{n-1} replacing F^{n-1} in Definition 3.3. If we use this remark, and then apply the usual proof in the case where $\pi\colon E \to B$ is a Serre fibration (see [7]) we get the following important proposition.

Proposition 3.4. *Let $\pi\colon E \to B$ be a triple consisting of two topological spaces and a continuous map which satisfies property P. Pick $b_0 \in B$, and $y_0 \in \pi^{-1}(b_0) = F$. Then the canonical map $\pi_*\colon \pi_n(E, F; y_0) \to \pi_n(B, b_0)$ is a bijection* (1 — 1 and onto).

The following elementary proposition follows immediately from the definitions.

Proposition 3.5. *Let $\pi\colon E_1 \to B$, and $p\colon E_2 \to E_1$ have* (*strong*) *property P. Then $\pi \circ p\colon E_2 \to B$ satisfies* (*strong*) *property P.*

Definition 3.6. Let E and B be topological spaces, and $\pi\colon E \to B$ a continuous map. Let $\varphi\colon I^n \to B$, and $\phi\colon F^{n-1} \to E$ be continuous maps such that $\pi \circ \phi = \varphi | F^{n-1}$. By a *deformation* of (φ, ϕ) we mean a continuous map $\tilde{\phi}\colon F^{n-1} \times I \to E$ such that $\pi \circ \phi_t = \varphi$ on F^{n-1} and $\phi_0 = \phi$ where $\phi_t = \tilde{\phi} | F^{n-1} \times \{t\}$.

Proposition 3.7. *Let $\pi\colon E \to B$ be as above. Then π has* (*strong*) *property P if and only if, for each n and each pair of continuous maps $\varphi\colon I^n \to B$ and $\phi\colon F^{n-1}$ such that $\pi \circ \phi = \varphi | F^{n-1}$, we can find*:

i) *a deformation ϕ_t of (φ, ϕ),*

ii) *a* (*strong*) *admissible deformation h_t of I^n, and*

iii) *an extension Ψ of ϕ_1 to I^n such that $\pi \circ \Psi = \varphi \circ h_1$.*

Proof. If $\pi\colon E \to B$ has (strong) property P, this is a triviality. Let $\varphi\colon I^n \to B$ and $\phi\colon F^{n-1} \to E$ be a pair of continuous maps such that $\pi \circ \phi = \varphi | F^{n-1}$. We want to find a (strong) deformation $\bar{h}_t$ of I^n and an extension Ψ of ϕ to I^n such that $\pi \circ \Psi = \varphi \circ \bar{h}_1$. Let us define a (strong) admissible deformation $\bar{h}_t$ of I^n as follows. Let h_t be the (strong) admissible deformation given by ii) in the hypotheses. Let $C_t = (x \in I^n \mid t/2 \le x_k \le 1)$ for $0 \le t \le 1$, and $T^{(t)}_{k,0} = \{(x_1, \cdots, x_n) \mid x \in I^n,$ $x_k = st/2, st/2 \le x_l \le 1$ for $l \ne k, 0 \le s \le 1\}$. Then $I^n = \bigcup_{k=1}^{n} T^{(t)}_{k,0} \cup C_t$ for each fixed t. Let us introduce the following shorthand if $v = (x, \cdots, x_n) \in R^n$ and $a \in R$, by $x - a$ we mean $(x_1 - a, \cdots, x_n - a)$. We will now define $\bar{h}_t$. If $x \in C_t$, then define $(\bar{h}_t(x))_k = \left(h_t\left(\frac{x - t/2}{1 - t/2}\right)\right)_k$, and if $x \in T^{(t)}_{k,0}$, then $x_k = st/2$, $0 \le s \le 1$, and set $(\bar{h}_t(x))_k = 0$ and $(\bar{h}_t(x))_l = (x_l - st/2)/(1 - st/2)$. We then see by direct calculation that $\bar{h}_0 = \mathrm{id}$, $\bar{h}_t | F^{n-1} = \mathrm{id}$, and $\bar{h}_t$ is well-defined and is a (strong) admissible deformation of I^n. Let $\tilde{\Psi}$ be the extension of ϕ_1 to I^n given by i) and iii). We define the desired Ψ on C_1 by $\Psi(x) = \tilde{\Psi}\left(\frac{x - 1/2}{1 - 1/2}\right)$. If $x \in T^{(1)}_{k,0}$, then $0 \le x_k \le 1/2$, say $x_k = s/2, 0 \le s \le 1$, and therefore $s/2 \le x_l \le 1$ for $l \ne k$. We then set $\Psi(x) = \phi_s(\bar{h}_1(x))$. We can then check directly that Ψ extends ϕ, and Ψ is continuous and well-defined, and that $\pi \circ \Psi = \varphi \circ \bar{h}_1$. This completes this proof.

Definition 3.8. Let E and B be topological spaces, and $\pi\colon E \to B$ a continuous map. We say $\pi\colon E \to B$ has *strong local property P* if for each $x \in B$ there exists a neighborhood U of x such that $\pi\colon \pi^{-1}(U) \to U$ has property P.

Theorem 3.9. *If $\pi\colon E \to B$ has strong local property P, then it has strong property P.*

Proof. For each $b \in B$, let U_b be an open neighborhood of b such that $\pi\colon \pi^{-1}(U_b) \to U_b$ has property P. Let $\varphi\colon I^n \to B$, and $\phi\colon F^{n-1} \to E$ be continuous maps such that $\pi \circ \phi = \varphi$ on F^{n-1}. The sets $\varphi^{-1}(U_b)$ forms an open cover of I^n. Hence by the Lebesgue covering lemma there exists an integer $N > 0$ such that any subcube of I^n, with sides parallel to those of I^n and side of length $1/N$, is contained in one of the sets $\varphi^{-1}(U_b)$. Let $B_I = B_{i_1,\dots,i_n} = \{x \in I^n \mid i_k/N \leq x_k \leq i_k + 1/N\}, 0 \leq i_k \leq N-1, 1 \leq k \leq n$. Set $B_{I,k,0} = \{x \in B_I \mid x_k = i_k/N\}, B_{I,k,1} = \{x \in B_I \mid x_k = i_k + 1/N\}$, and $F_I = \bigcup_{k=1}^{n} B_{I,k,0}$. The B_I's cover I^n, and each B_I is contained in one of the sets $\varphi^{-1}(U_b)$. We will order the $(N)^n$ n-tuples $I = (i_1, \cdots, i_n)$ lexicographically. If I is an n-tuple, let $I+1$ be the n-tuple immediately succeeding I, and $\nu(I)$ the number of n-tuples less than or equal to I. We now construct the continuous extension Ψ of ϕ and the strong admissible deformation h_t of I^n by induction.

Induction step I. Let $C_I = F^{n-1} \cup \bigcup_{I' \leq I} B_{I'}$. Assume there exist a continuous mep $\Psi_I\colon C_I \to E$ extending ϕ, and a continuous function $H_I\colon [0, J(I)] \times I^n \to I^n$ where $J(I) = \nu(I)/N^n$ with the following properties. Set $h_{I,t}(x) = H_I(t,x)$. Then $h_{I,0} = \text{id}, h_{I,t} \mid F^{n-1} = \text{id}$ for $0 \leq t \leq J(I), h_{I,t}(I^{n-1}_{k,1}) \subseteq I^{n-1}_{k,1}$ and $\pi \circ \Psi_I = \varphi \circ h_{I,J(I)}$.

We will now prove our theorem by showing that step I implies step I + 1, and noting that step 0 is trivially true, and step $(N)^n$ is the desired result. Look at B_{I+1} and note that $F_{I+1} = B_{I+1} \cap C_I$. Let $f = \varphi \circ h_{I,J(I)} \mid B_{I+1}$, and $p = \Psi_I \mid F_{I+1}$. But we know that B_{I+1} is contained in one of the $\varphi^{-1}(U_b)$. Hence we can find continuous maps $K\colon [J(I), J(I+1)] \times B_{I+1} \to B_{I+1}$, and $\underline{P}\colon B_{I+1} \to E$ extending p with the following properties. $K(J(I), x) = x, K(t,x) = x$ for $x \in F_{I+1}$ and $t \in [J(I), J(I+1)], K(t,x) \in B_{I+1,k,1}$ for $x \in B_{I+1,k,1}$ and $t \in [J(I), J(I+1)]$, and $\pi \circ \underline{P}(x) = f(K(J(I+1), x))$ for $x \in B_{I+1}$. Define $\Psi_{I+1}\colon C_{I+1} = C_I \cup B_{I+1} \to E$ by $\Psi_{I+1} \mid C_I = \Psi_I$ and $\Psi_{I+1} \mid B_{I+1} = \underline{P}$. Ψ_{I+1} is clearly a well-defined continuous extension of ϕ. We now extend $K\colon [J(I), J(I+1)] \times B_{I+1} \to B_{I+1}$ to a map $K\colon [J(I), J(I+1)] \times I^n \to I^n$ as follows. If for some $k, x_k \leq i_k/N^n$, then we set $K(t, (x_1, \cdots, x_n)) = (x_1, \cdots, x_n)$. We are left with the case where $x_k \geq i_k/N^n$ for all k. We then set $K(t,x)_k = x_k$ provided $x_k \geq (i_{k+1})/N^n$. We define $\bar{x}$ by the formula $(\bar{x})_l = x_l$ if $i_l/N^n \leq x_l \leq (i_{l+1})/N^n$ for some index l, and by $(\bar{x})_k = (i_{k+1})/N^n$ if $x_k \geq (i_{k+1})/N^n$. Then $\bar{x} \in B_{I+1}$, and we set $K(t,x)_l = K(t,\bar{x})_l$ where l is an index such that $i_l/N^n \leq x_l \leq (i_{l+1})/N^n$. Note if we set $k_t(x) = K(t,x)$, the k_t have the following properties. $k_t(x) = x$ for all $x \in C_I, k_{J(I)}(x) = x$ for all $x \in I^n$, and $k_t(I^{n-1}_{k,1}) \subseteq I^{n-1}_{k,1}$ for all t. Let us define

$$h_{I+1,t}=\begin{cases} h_t(x)\ , & 0\le t\le J(I)\ ,\\ h_{J(I)}(k_t(x))\ , & J(I)\le t\le J(I+1)\ ,\end{cases}$$

and set $H_{I+1}(t,x)=h_{I+1,t}(x)$. It is then easy to directly check that H_{I+1} and Ψ_{I+1} have all the desired properties.

In the remainder of this paper we will prove the following theorem.

Theorem A′. *Let $p\colon E_0\to V$ be the triple defined in § 2. Then $p\colon E_0\to V$ has strong local property P.*

By using Proposition 3.4 and Definition 3.8 we see that Theorem A′ implies Theorem A. Let $(x_0, v, k)\in V$, we want to look at neighborhoods U of this point of the form $U=W\times V_2\times(k_0,\infty)$, where $k_0<$ k and W is a sufficiently small neighborhood which is the domain of x_0 centered Riemann normal coordinates $(x_1,\cdots,x_n)$. The exact form of the neighborhood U will be chosen in the next section. However, given $\psi\colon F^{n-1}\to p^{-1}(U)$ and $\varphi\colon I^n\to U$ such that $p\circ\psi=\varphi\,|\,F^{n-1}$ we cannot lift φ immediately because of the nature of our lifting mechanism. We must first "reparametrize" the cube I^n, and preform some preliminary deformations on the curves in ψ. It is because of this that the topological abstractions of this section are needed.

4. A local comparison to determine the desired neighborhood

Let (X,g) be the given Riemannian manifold, and let $(\tilde x,\tilde v,\tilde k)\in V$, $\tilde k\in R^+$, $\tilde v=(\tilde t,\tilde n)$, and $\tilde v\in V_2(X)_{\tilde x}$. Let $U=W\times V_2\times(k,\infty)$, where $0<k<\tilde k$, W is the domain of $\tilde x$-centered geodesic coordinates $(x_1,\cdots,x_n)$ and V_2 is the Stiefel manifold of orthonormal 2-frames in n-space. Let the metric tensor g take its usual coordinate form $g(\mathrm{x})=\sum g_{ij}(x)dx^idx^j$ on W. We recall that $g_{ij}(0)=g_{ij}(\tilde x)=\delta_{ij}$, $(\partial g_{ij}/\partial x_k)(0)=0$, and therefore the Christoffel symbols $\Gamma^k_{ij}(0)=0$. If we identify the tangent space $T(X)_x$, $x\in W$, with R^n in the usual way (i.e., $a=(a_1,\cdots,a_n)$ is identified with $\sum a_i(\partial/\partial x_i)(x)$), then we note that as x varies over W, we identify $V_2(X)_x$ with a slightly different subset of $R^n\times R^n$ determined by the variation in the metric. This identification clearly varies smoothly with $x\in W$. We can also define upon W the flat metric $g_F=\sum\delta_{ij}dx^idx^j$. If $\gamma\colon I\to W$ is a nondegenerate immersion with respect to $g(g_F)$ we call it $g(g_F)$-nondegenerate. If $\gamma\colon I\to W$ is $g(g_F)$- degenerate let $t(\gamma), n(\gamma)$, $k_g(\gamma)[t_F(\gamma), n_F(\gamma), k_F(\gamma)]$ denote the unit tangent vector, the principal normal vector, and the geodesic curvature of γ calculated with respect to $g(g_F)$.

Let us pick $(x,v,l)\in W\times V_2\times(k,\infty)$. Furthermore, assume $v=(a,b)\in R^n\times R^n$, where $\sum(a_i)^2=\sum(b_i)^2=1$ and $\sum a_ib_i=0$ (i.e., (a,b) is a 2-frame with respect to the flat metric). Let $t_0\in I$, and $\gamma\colon I\to W$ be a g_F-nondegenerate curve such that $\gamma(t_0)=x$, $t_F(\gamma)(t_0)=a$, $n_F(\gamma)(t_0)=b$, and $k_F(\gamma)(t_0)=l$. We then see $t(\gamma)(t_0)=a/(\sum g_{ij}(x)a_ia_j)^{1/2}$, and $(k_g(\gamma)(t_0))^2=(\sum g_{ij}(x)a_ia_j)^{-3}([\sum g_{ij}(x)a_ia_j]$ $[\sum g_{ij}(x)c_ic_j]-[\sum g_{ij}(x)a_ic_j]^2)$, where $c_i=b_il+\sum_{j,k}\Gamma^i_{jk}(x)a_ja_k$. Hence

$t(\gamma)(t_0)$ and $k_g(\gamma)(t_0)$ depend upon x, a, b and l alone and not on our choice of γ. We can use these formulas to define the functions $t(x, a) = t(\gamma)(t_0)$ and $k_g(x, a, b, l) = k_g(\gamma)(t_0)$. Now $k_g(0, a, b, l)^2 = l^2$, and $\partial(k_g(0, a, b, l)^2)/\partial l = 2l$. Hence because of the compactness of V_2 we can find a neighborhood W_1 of $0, W_1 \subseteq W$ such that $k_g(x, a, b, l)^2 > (2k/3)^2$, if $k < l$ and $x \in W_1$. In that $k_g(\gamma)(t_0) = k_g(x, a, b, l) > 0$ if $x \in W_1$, we can define the principal normal $n(\gamma)(t_0) = k_g(x, a, b, l)^{-1}(\sum g_{ij}(x)a_i a_j)^{-2}[d]$ where $d = c(\sum g_{ij}(x)a_i a_j) - a(\sum g_{ij}(x)a_i c_j), c_j = b_j l + \sum_{i,k} \Gamma^j_{ik}(x)a_i a_k$ and $c = (c_1, \cdots, c_n)$. We see that $n(\gamma)(t_0)$ does not depend on γ but only on $x, v = (a, b)$ and l, and we can then set $n(x, a, b, l) = n(\gamma)(t_0)$. Hence we have defined a smooth 1–1 map α: $W_1 \times V_2 \times (k, \infty) \to W_1 \times V_2 \times (2k/3, \infty)$ by the formula $\alpha(x, (a, b), l) = (x, t(x, a), n(x, a, b, l), k_g(x, a, b, l))$.

Let us pick $(x, v, l) \in W \times V_2 \times (k, \infty)$ where we assume $v = (a, b) \in R^n \times R^n$, $\sum g_{ij}(x)a_i a_j = \sum g_{ij}(x)b_i b_j = 1$, and $\sum g_{ij}(x)a_i b_j = 0$ (i.e., (a, b) is an orthonormal 2-frame in the metric $g(x)$). Let $t_0 \in I$, and let us choose a g-nondegenerate curve $\gamma: I \to W$ such that $\gamma(t_0) = x, t(\gamma)(t_0) = a, n(\gamma)(t_0) = b$ and $k_g(\gamma)(t_0)$. We then see that $t_F(\gamma)(t_0) = a/(\sum (a_i)^2)^{1/2} = t_F(x, a)$. We also see that $k_F(\gamma)(t_0)^2 = (\sum (a_i)^2)^{-3}[(\sum (c_k)^2)(\sum (a_k)^2) - (\sum a_k c_k)^2] = k_F(x, a, b, l)^2$ where $c_k = b_k l - \sum_{i,j} \Gamma^k_{ij}(x)a_i a_j$. Hence $k_F(\gamma)(t_0)^2$ depends only on (x, a, b, l). Furthermore $k_F(0, a, b, l)^2 = l^2$, and $\partial(k_F(0, a, b, l)^2)/\partial l = 2l$. By the compactness of V_2, we can find a neighborhood W_2 of 0, $W_2 \subseteq W$ such that $k_F(x, a, b, l)^2 > (2k/3)^2$ for $l > k$ and $x \in W_2$. Since $k_F(\gamma)(t_0) = k_F(x, a, b, l) > 0$ for $x \in W_2$ $(l > k)$, we can define the principal normal $n_F(\gamma)(t_0) = n_F(x, a, b, l) = k_F(x, a, b, l)^{-1}$ $(\sum (a_k)^2)^{-2}[\sum (a_k)^2 c - (\sum a_k c_k)a]$, where $c_k = b_k l - \sum_{i,j} \Gamma^k_{ij}(x)a_i a_j$. Thus $n_F(\gamma)(t_0)$ depends only upon (x, a, b, l). Then as before, we have defined a smooth 1–1 map β: $W_2 \times V_2 \times (k, \infty) \to W_2 \times V_2 \times (2k/3, \infty)$ by the formula $\beta(x, (a, b), l) = (x, t_F(x, a), n_F(x, a, b, l), k_F(x, a, b, l))$. Finally we note that $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$ whenever these compositions are well-defined. This discussion can be summerized by the following proposition.

Proposition 4.1. *Let us pick $k > 0$. Then we can find a neighborhood W_0 of 0, $W_0 \subseteq W$, which depends only upon our choice of k, with the following properties:*

1) *If $\gamma: I \to W_0$ is g-nondegenerate, and $k_g(\gamma)(t) > k$, then γ is g_F-nondegenerate and $k_F(\gamma)(t) > 2k/3$. Furthermore, if $\gamma: I \to W_0$ is g_F-nondegenerate and $k_F(\gamma)(t) > 2k/3$, then γ is g-nondegenerate and $k_g(\gamma)(t) > k/3$.*

2) *Let us pick $(x, v = (a, b), l) \in W_0 \times V_2 \times (k, \infty)[(x, v = (a, b), l) \in W_0 \times V_2 \times (2k/3, \infty)]$. Pick $t_0 \in I$, and let $\gamma: I \to W_0$ be a $g[g_F]$-nondegenerate curve such that $\gamma(t_0) = x, t(\gamma)(t_0) = a, n(\gamma)(t_0) = b$ and $k_g(\gamma)(t_0) = l[\gamma(t_0) = x, t_F(\gamma)(t_0) = a, n_F(\gamma)(t_0 \doteq b$ and $k_F(\gamma)(t_0) = l]$. Then $t_F(\gamma)(t_0), n_F(\gamma)(t_0)$ and $k_F(\gamma)(t_0)[t(\gamma)(t_0), n(\gamma)(t_0)$ ond $k_g(\gamma)(t_0)]$ are all well-defined and depend only upon (x, a, b, l). We therefore set $t_F(\gamma)(t_0) = t_F(\gamma)(x, a, b, l), n_F(\gamma)(t_0) = n_F(x, a, b, l)$ and $k_F(\gamma)(t_0) = k_F(x, a, b, l)[t(\gamma)(t_0) = t(x, a, b, l), n(\gamma)(t_0) = n(x, a, b, l)$ and*

$k_g(\gamma)(t_0) = k_g(x, a, b, l)$]. *In this way we define smooth* 1–1 *maps* $\alpha\colon W_0 \times V_2 \times (k, \infty) \to W_0 \times V_2 \times (2k/3, \infty)$ *and* $\beta\colon W_0 \times V_2 \times (2k/3, \infty) \to W_0 \times V_2 \times (k/3, \infty)$ *defined by* $\alpha(x, a, b, l) = (x, t_F(x, a, b, l), n_F(x, a, b, l), k_F(x, a, b, l))$ *and* $\beta(x, a, b, l) = (x, t(x, a, b, l), n(x, a, b, l), k_g(x, a, b, l)$. *Finally* $\alpha \circ \beta = id$ *and* $\beta \circ \alpha = id$ *whenever the composition is well-defined.*

5. A generalization of Fenchel's lemma

Let R^n possess its usual Riemann (Euclidean) structure, $S^{n-1} \subseteq R^n$ be the unit sphere with its usual Riemann structure, and $\gamma\colon I \to R^n$ be an immersion. We recall that γ is nondegenerate if and only if $t(\gamma)\colon I \to S^{n-1}$ is an immersion. If we are given an immersion $\lambda\colon [0, 1] \to S^{n-1}$, we want to find a curve $\gamma\colon [0, 1] \to R^n$ such that $t(\gamma) = \lambda$, $\gamma(1) =$ a predetermined point x, and $k(\gamma)(t) > k > 0$, k being some some predetermined number.

Lemma 5.1. *Let $D \subseteq R^n$ be a disc radius R, $0 < R \leq 1$, centered at* 0. *Let $c(n) = 18n/\sqrt{n}$, and $B(n) =$ some number, $B(n) > 1$, which depends only upon n and which we will determine in the next section. Let k be a real number such that $0 < k < [c(n)B(n)]^{-1}$, (t_0, n_0) and (t_1, n_1) be two given orthonormal 2-frames, and $k_i, i = 0, 1,$ be two positive numbers such that $k_i > k, i = 0, 1$. Pick $x \in D$ such that $|x| < R\sqrt{n}/(2n)$. Let $\lambda\colon [0, 1] \to S^{n-1}$ be an immersion such that*

1) *$\lambda(0) = t_0, \lambda(1) = t_1, t(\lambda)(0) = n_0$ and $t(\lambda)(1) = n_1$,*

2) *$\lambda\,|\,[0, 1/2]$ is parametrized proportional to arc length and $|\lambda'(s)| \leq B(n)$ for $s \in [0, 1/2]$,*

3) *the set $\{\lambda(t)\,|\,0 < t < 1/2\}$ contains the 2^n vertices of the inscribed cube.*

Then we can find a C^∞ function $\rho(t), 0 < \rho(t) < 1/k$, such that the curve $\gamma(t) = \int_0^s \lambda(\tau)\rho(\tau)d\tau$ has the following properties:

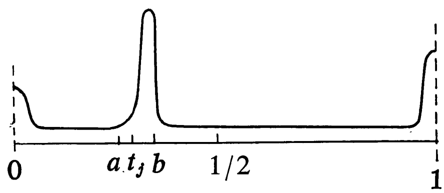
a) *$\gamma(1) = x, t(\gamma)(i) = t_i, n(\gamma)(i) = n_i, k(\gamma)(i) = k_i, i = 0, 1.$*

b) *$|\gamma(t)| < R$, and $k(\gamma)(t) > k$.*

Proof. It is easy to see $k(\gamma)(t) = (\rho(t))^{-1}$. Hence if $0 < \rho(t) < c(n)B(n)$, then $k(\gamma)(t) > k$. Furthermore $t(\gamma)(t) = \lambda(t)$ and $n(\gamma)(t) = t(\lambda)(t)$. Let $K = \left\{y\,|\,y = \int_0^1 \rho(\tau)\lambda(\tau)d\tau\right.$, where $\rho(\tau)$ is smooth, $0 < \rho(\tau) < 1/k, \rho(i) = (k_i)^{-1}$ for $i = 0, 1$, and $\left.\int_0^1 \rho(\tau)d\tau \leq .9R\right\}$. We note that K is a convex set. Let $t_j \in (0,1/2), 1 \leq j \leq 2^n$, be the points such that $\lambda(t_j)$ are the vertices of the inscribed cube. If we can show that each vertex $.9R\lambda(t_j)$ of the inscribed cube in the sphere of radius $.9R$ is within $.9R\sqrt{n}/(3n)$ of K, then we see that $K \supseteq$ open ball about 0 of radius $R\sqrt{n}/(2n)$, which implies that $x \in K$.

Pick one of the $t_j, 0 < t_j < 1/2$, such that $\lambda(t_j)$ is a vertex of the inscribe cube. Let us pick $\rho_j(t)$ as follows. Let $\rho_j(0) = (k_0)^{-1}, \rho_j(1) = (k_1)^{-1}$,

$\int_0^1 \rho_j(t)dt = .9R, \rho_j(t) > 0$, and $\rho_j(t)$ be smooth.



Pick an interval $[a, b]$ about t_j such that $[a, b] \subseteq (0, 1/2)$ and $b - a = 2(.9R)/(c(n)B(n))$. But $((b - a)/2)c(n)B(n) = .9R$, so we can choose $\rho_j(t)$ to also satisfy the relations $\int_0^a \rho_j(t) < 1/2(.9R\sqrt{n}/(9n))$, $\int_b^1 \rho_j(t)dt < 1/2(.9R\sqrt{n}(9n))$ and $\rho_j(t) < 1/k$. Let $\lambda_j = \int_0^1 \rho_j(t)\lambda(t)dt$. Then $\lambda_j \in K$, and $|\lambda_j - .9R\lambda(t_j)| = \left|\int_0^1 (\lambda(t) - \lambda(t_j))\rho_j(t)dt\right|$ because $.9R\lambda(t_j) = \int_0^1 \rho_j(t)\lambda(t_j)dt$. Therefore $|\lambda_j - .9R\lambda(t_j)| \leq \left|\int_0^a\right| + \left|\int_a^b\right| + \left|\int_b^1\right| < 2 \cdot 2(1/2)(.9R\sqrt{n}/(9n)) + \left|\int_a^b\right|$. But $|\lambda(t) - \lambda(t_j)| \leq |b - a| \sup|\lambda'(t)| \leq |b - a| B(n)$ by Taylor's formula. Therefore $\left|\int_a^b (\lambda(t) - \lambda(t_j))\rho_j(t)dt\right| < |b - a| B(n) \int_a^b \rho_j(t)dt < 1.8R\, c(n)^{-1}(.9R) < (1.8R)c(n)^{-1} = (.9R)(\sqrt{n}/(9n))$. Hence $|\lambda_j - .9R\lambda(t_j)| < .9R\sqrt{n}/(3n)$, which is what we wanted to show.

6. Smashing and stretching

Let us fix some notation for this section. Let $D \subseteq R^n$ be an open disc of radius R centered at 0. Give D its usual Riemann structure, and let (e_1, e_2) be an orthonormal 2-frame. Let $E = \{\gamma\colon [-1, 1] \to D \mid$ a C^2-nondegenerate immersion $\gamma\}$, where we give E the C^2 topology. Let k_0 be some strictly positive real number, and set $E_0(k_0) = \{\gamma \in E \mid \gamma(0) = 0,\ t(\gamma)(0) = e_1,\ n(\gamma)(0) = e_2,\ k(\gamma)(t) > k_0,\ t \in [-1, 1]\}$.

In this section we will prove two main lemmas (6.3, and 6.4) which easily imply the following theorem.

Theorem 6.1. *Let X be a compact set, and $\varphi\colon X \to E_0(k_0)$ a continuous map. Then we can find a continuous deformation $\Phi\colon X \times [0, 1] \to E_0(k_0)$ of φ* (i.e., $\varphi(x) = \Phi(x, 0)$) *with the following properties:*

1) *There exist numbers S and T, $0 < S < T < 1$, such that $\Phi(x, u)(t) = \varphi(x)(t)$ for all $|t| \geq T, x \in X, u \in [0, 1]$, and $\Phi(x, 1) = \Phi(y, 1)(t) = f(t)$ for all $0 \leq |t| \leq S$, $x, y \in X$, and C^∞ $f(t)$.*

2) *The path* $t(f(t)), 0 < t < S$, *passes through each of the* 2^n *vertices of the inscribed cube, and* $\int_0^S k(f)(t)dt < B(n) = 2^{n+5}(80 + (n-1)^{1/2})$.

If we are to employ Lemma 5.1 it is clear that a theorem of this type is needed.

Sublemma 6.2. *Let* X *be a compact set, and* $a: X \to C^2([-1,1]; R)$ *be a continuous map, and assume* $a(x)(0) = a'(x)(0) = a''(x)(0) = 0$. *Then there exist continuous functions* $b_i: X \to \mathscr{C}^0([-1,1], R), i = 0, 1$, *such that*

1) $a(x)(s) = s^2 b_0(x)(s), a'(x)(s) = s b_1(x)(s)$ *and*

2) $b_0(x)(0) = b_1(x)(0) = 0$.

Proof. This is a direct consequence of the fact $a(x)(s) = s\int_0^1 Da(x)(st)dt$ where D denotes differentiation with respect to the variable $v = st$.

Lemma 6.3 (*Smashing lemma*). *Let* X *be a compact set, and* $\varphi: X \to E_0(k_0)$ *a continuous map. Let* $U = [-a, a], 0 < a < 1$, *and assume* $\varphi(x)\,|\,U$ *is parametrized by arc length for all* $x \in X$. *Let us extend* (e_1, e_2) *to an orthonormal basis* $(e_1, \cdots, e_n)$ *of* R^n, *and use these as coordinates. Then we can find two neighborhoods* $V = [-c, c]$ *and* $W = [-b, b]$ *such that* $0 < c < b < a$, *and a continuous deformation* $\Phi: X \times [0,2] \to E_0(k_0)(i.e., \Phi(x, 0) = \varphi(x))$ *of* φ *such that*

1) $\Phi(x, u)(t) = \varphi(x)(t)$ *if* $b \le |t| \le 1, \Phi(x, 2)(t) = \Phi(y, 2)(t) = (t, t^2K/2, \cdots, 0)$ *for* $|t| \le c, x, y \in X, K > \max_{x \in X}(k(x))$ *where* $k(x) = k(\varphi(x))(0)$,

2) $\int_{-b}^{b} k(\Phi(x, u))(t)dt < 1$.

Proof. Step I. Let us restrict ourselves to the interval $[-a, a]$. We see $\varphi(x)(s) = se_1 + (s^2k(x)/2)e_2 + a(x)(s)$, $a(x)(s) = \sum_{i=1}^{n} a_i(x)(s)e_i$, and $a_i(x)(s)$ satisfy the hypotheses of Sublemma 6.2: $0 \le |s| \le a$. Let λ be a C^∞ function so chosen that $\lambda(s) \equiv 1$ on $[-l/2, l/2], 0 \le \lambda(s) \le 1, \lambda(s) \equiv 0$ for $|s| \ge l, l < a$, and that there exists positive constants C_1 and C_2 which are independent of our choice of l, such that $|\lambda'(s)| \le C_1/l$ and $|\lambda''(s)| \le C_2/l^2$. Set $p(x, s) = se_1 + (s^2k(x)/2)e_2$, and let $\Phi(x, u)(s) = p(x, s) + a(x)(s)[1 - \lambda(s)u], 0 \le u \le 1$. $\Phi(x, u)(s) = \varphi(x)(s)$ if $|s| \ge l$. Note that we have not yet chosen l. There exists an $\varepsilon > 0$ such that if $\|\varphi(x) - \Phi(x, u)\|_2 < \varepsilon$ for all $x \in X, u \in [0, 1]$ where $\|\ \|_2$ is the C^2-norm, then $\Phi(x, u) \in E_0$. But $\Phi(x, u)(s) - \varphi(x)(s) = a(x)(s)u\lambda(s), |s| \le l$, and $\Phi(x, u)(s) - \varphi(x)(s) \equiv 0, |s| \ge l$. Hence $|\Phi(x, u)(s) - \varphi(x)(s)| \le \sup_{s \in [-l, l]} |a(x)(s)|$, $|\Phi'(x, u)(s) - \varphi'(x)(s)| \le |\lambda'(s)|\,|a(x)(s)| + |\lambda(s)|\,|a'(x)(s)|$, and $|\Phi''(x, u)(s) - \varphi''(x)(s)| \le |\lambda''(s)|\,|a(x)(s)| + 2\,|\lambda'(s)|\,|a'(x)(s)| + |\lambda(s)|\,|a''(x)(s)|$. Hence by Sublemma 6.2, the compactness of X and the estimates on λ' and λ'', we can find an l so small that $\|\Phi(x, u) - \varphi(x)\|_2 < \varepsilon$ and $\int_{-l}^{l} k(\Phi(x, u))(t)dt < 1/10$.

Set $b = l$ ($b = b$ in the statement of Lemma 6.3).

Step II. Let us limit ourselves to $|s| \leq l/2$. Hence $\Phi(x, 1)(s) = (s, s^2k(x)/2, 0, \cdots, 0)$, and let $\Phi(x)(s) = \Phi(x, 1)(s)$. Let $\psi(s)$ be a C^∞ function so chosen that $\psi(s) \equiv 0$ for $|s| \geq d, \psi(s) \equiv 1$ for $|s| \leq 5d/6$, and $0 \leq \psi(s) \leq 1$, and we can choose positive constants C_1 and C_2 independent of d such that $|\psi'(s)| \leq C_1/d$ and $|\psi''(s)| \leq C_2/d^2$. Let us assume $2d < l/2$. Let $\Phi(x, u)(s) = (s, s^2/2(K\psi(s)u + (1 - u\psi(s))k(x)), u\xi(s), 0, \cdots, 0)$, where $\xi(s)$ is an even ($\xi(s) = \xi(-s)$) C^∞ real-valued function such that $\xi(s) \equiv 0$ for $|s| \leq d/6$ and $|s| \geq 2d$, and where $0 \leq u \leq 1$. By this formula there exist A_1 and B_1 such that if $|\xi(s)| \leq A_1$ and $d \leq B_1$, then $\Phi(x, u)(s) \in D$ for all (x, u, s). $\Phi'(x, u)(s) = (1, sh(x, u, s), u\xi'(s), 0, \cdots, 0)$ where $h(x, u, s) = s[Ku\psi(s) + (1 - u\psi(s))k(x)] + (s^2/2)[K - k(x))u\psi'(s)]$. Pick $\varepsilon < 0$ so small that $k(x)^2/(1 + \varepsilon)^2 > k_0^2$. There exist A_2 and B_2 such that if $|\xi'(s)| < A_2$ and $d \leq B_2$, then $|\Phi'(x, u)(s)|^3 < 1 + \varepsilon$ for all (x, u) and $|s| \leq 2d$. $\Phi''(x, u)(s) = (0, m(x, u, s), u\xi''(s), 0, \cdots, 0)$, where $m(x, u, s) = k(x) + u(K - k(x))\mu(s)$ and $\mu(s) = [\psi(s) + 2s\psi'(s) + (s^2/2)\psi''(s)]$. There exists a positive constant C_3 independent of our choice of d such that $|\mu(s)| \leq C_3, |s| \leq l/2$. If $|s| \leq 5d/6$ or $|s| \geq d$, then $m(x, u, s) \neq 0$ and hence $\Phi(x, u)(s)$ is nondegenerate. We assume $\xi''(s) \neq 0, 5d/6 \leq |s| \leq d$. This implies $\Phi(x, u)(s)$ is everywhere nondegenerate.

$k(\Phi(x, u))(s)^2 = k(x, u)(s)^2 = [1 + h^2 + u^2(\xi')^2]^{-3}[(1 + h^2 + u^2(\xi')^2)(m^2 + u^2(\xi'')^2) - (mh + u^2\xi'\xi'')^2] = [1 + h^2 + u^2(\xi')^2]^{-3}[m^2 + u^2(\xi'')^2 + (mu\xi' - hu\xi'')^2]$. But $m(x, u, s) = k(x) + u[K - k(x)]\mu(s)$, and therefore there exists $u_0 > 0$ such that $m(x, u, s)^2/(1 + \varepsilon)^2 > k_0^2$ for $0 \leq u \leq u_0$;, u_0 is clearly independent of the choice of d and ξ. Set $\xi''(s)^2 = (1 + \varepsilon)^2(k_0^2 + 1)(u_0)^{-2} = \alpha$ for $5d/6 \leq |s| \leq d$, and let $|\xi''(s)|^2 \leq \alpha$ for all other s. Then $u^2\xi''(s)^2/(1 + \varepsilon)^2 > k_0^2$ for $5d/6 \leq |s| \leq d, u_0 \leq u \leq 1$. Hence $|\xi'(s)| \leq 2d\alpha$ and $|\xi(s)| \leq 4d^2\alpha$. $k(x, u)(s)^2 < m(x, u, s)^2 + u^2\xi''(s)^2 < K^2(1 + C_3)^2 + \alpha^2$, and therefore $\int_{-2d}^{2d} k(\Phi(x, u))(s)ds < 4d(K^2(1 + C_3)^2 + \alpha^2)^{1/2}$. Let $d < \min(B_1, B_2, A_2/2\alpha, (A_1/4\alpha)^{1/2}, (.9)(1/4)(K^2(1 + C_3)^2 + \alpha^2)^{-1/2})$. Then $|\xi(s)| < A_1, |\xi'(s)| < A_2, k(\Phi(x, u))(s) > k_0$, and $\int_{-2d}^{2d} k(\Phi(x, u))(s)ds < 9/10$. This proves our lemma if we let $c = d/6$.

Lemma 6.4 (*Stretching lemma*). *Let D be a disc centered at* 0, *and with radius $R < 1$ and the usual metric, etc. Let $0 < A < 1$, and let $\varphi: [-A, A] \to D$ be a nondegenerate immersion such that $\varphi(0) = 0, k(\varphi)(t) > k_0 > 0$, $\int_{-A}^{A} k(\varphi)(t)dt < 1$ and $t(\varphi)(0) = v_0 \in S^{n-1}$. Pick $\omega \in S^{n-1}$ such that the geodesic* (great circular) *distance $d_S(v_0, \omega) < \pi/6$. Then we can find a deformation φ_u of $\varphi, 0 \leq u \leq 3$, such that*

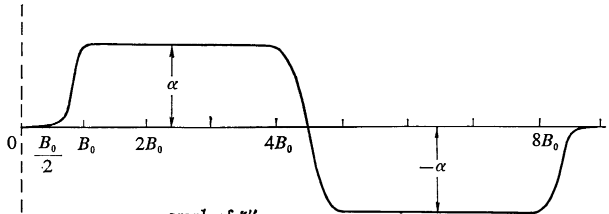
1) $\varphi_u(0) = 0$ *for all* $u, t(\varphi_3)(0) = \omega, \varphi_0(t) = \varphi(t)$,
2) $\varphi_u(t)$ *is nondegenerate for all u and t, and* $k(\varphi_u)(t) > k_0$,
3) *there exists a real numbdr a, $0 < a < A$, such that $\varphi_u(t) = \varphi(t), 0 \leq u \leq 3$, $|t| \geq a$,*

4) φ_u *defines a continuous curve in* $C^2([-A, A], D)$,

5) $\int_{-A}^{A} k(\varphi_3)(t)dt < L(n) = 2^2(80 + \sqrt{n-1})$.

Proof. *Step* 1. Pick coordinates in R^n such that $v_0 = e_1$ and $n(\varphi)(0) = e_2$, and a positive number K such that $K > \sup(1, 2^9k_0, k(\varphi)(0))$. Let us reparametrize φ so that near 0, φ is parametrized by arc length. By applying Lemma 6.3, we can find a deformation $\varphi_u(t)$ of $\varphi, 0 \le u \le 1$, and numbers B and C such that $0<C<B<A, \varphi_1(t)=(t, t^2K/2, 0, \cdots, 0)$ for $|t| \le C$, and $\varphi_u(t) = \varphi(t)$ for $|t| \ge B$. Furthermore choose φ_u so that $k(\varphi_u)(t) > k_0$, $\int_{-B}^{B} k(\varphi_1)(t)dt < 1$, and $|\varphi_1(t)| < R/2$ for $0 \le |t| \le B$. ($|\varphi_u(t) < R$, of course, for all u and $t \in [-A, A]$.)

Step 2. Let us pick a real number $D > 0$ such that $D < \min(C, (2K)^{-1})$. Let λ be a smooth strictly increasing monotone function on $[1, 2]$ such that $\lambda(1) = 0$ and $\lambda(2) = 1$. Set $w = (1, w_2, \cdots, w_n)/(1 + w_2^2 + \cdots + w_n^2)^{1/2}$. But $d_S(v_0, w) < \pi/6$ implies $w_2^2 + \cdots + w_n^2 < 3/4$. Hence $|w_2| < \sqrt{3}/2$. Let $w_2(t)$ be a C^∞ function such that $w_2(0) = 0, w_2'(t) \equiv w_2$ for $0 \le |t| \le B_0$. We can also assume $|w_2'(t)| \le \sqrt{3}/2, w_2(t) \equiv 0$ for $|t| \ge 4B_0, |w_2(t)| < \sqrt{3}\,B_0$ and $|w_2(t)| < (B_0)^{-1}$. Pick B_0 so small that $16B_0 < \min(R/8, D)$. Finally, let us pick m such that $(2^{m+1}K)^{-1} \le B_0 \le (2^mK)^{-1}$; note $m \ge 5$. Let us choose another C^∞ function $\zeta(t)$ such that $\zeta(t) \equiv 0$ for $|t| \le B_0/2, \zeta(t) \equiv 0$ for $|t| \ge 9B_0, \zeta(t)$ is even, $\zeta''(t) = K2^{m-4} = \alpha$ for $B_0 \le |t| \le 4B_0$, and $|\zeta''| \le K2^{m-4}$ elsewhere. Furthermore, we can choose ζ such that $|\zeta'| \le (8B_0)(K)(2^{m-4}) = B_0K2^{m-1} \le 1/2$.



graph of ζ''

We now set $\varphi_u(t) = (t, t^2K/2 + \lambda(u)w_2(t), \lambda(u)\zeta(t), 0, \cdots, 0)$ for $|t| \le D$, $1 \le u \le 2$. For $|t| \ge 9B_0, \varphi_u(t) = \varphi_1(t)$. $\varphi_u'(t) = (1, Kt + \lambda w_2'(t), \lambda(u)\zeta'(t), 0, \cdots, 0)$ and $\varphi_n''(t) = (0, K + \lambda w_2'', \lambda\zeta'', 0, \cdots, 0)$. By our choice of w_2 and ζ we see $\varphi_u(t)$ is nondegenerate. Note that $1 \le |\varphi_u(t)|^2 < 1 + (1/2 + \sqrt{3}/2)^2 + 1 < 4 = 2^2$. $k(\varphi_u)(t)^2 \ge [(K + \lambda w_2'')^2 + (\lambda\zeta'')^2]/|\varphi_u'(t)|^6 > 2^{-6}[(K + \lambda w_2'')^2 + (\lambda\zeta'')^2]$. Now $|w_2''(t)| < 1/B_0 < 2^{m+1}K$. Look at u_0 such that $\lambda(u_0) = 2^{-(m+2)}$. Hence for $1 \le u \le u_0, \lambda(u_0) \le 2^{-(m+2)}$ and therefore $|\lambda(u)w_2''(t)| < K/2$. So $k(\varphi_u)(t)^2 > K^2/2^{10} = (K/2^5)^2 > k_0^2$. If $|t| \le B_0$ or $|t| \ge 4B_0$, then $w_2''(t) = 0$ and $k(\varphi_u)(t)^2 > K^2/2^6 > k_0^2$. Finally, let $B_0 \le |t| \le 4B_0$, and $\lambda(u) \ge (2)^{-(m+2)}$. Then $\lambda(u)\zeta''(s) \ge (2)^{-(m+2)}K2^{m-2} = K2^{-6}$. Therefore $k(\varphi_u)(t)^2 > (K/2^9)^2 > k_0^2$. Let us estimate $J = \int_{-9B_0}^{9B_0} k(\varphi_2)(t)dt$.

$$J \leq \int_{-9B_0}^{9B_0} (|\varphi_2'(t)|^2 |\varphi_2''(t)|^2)^{1/2} dt \leq 2 \int (K + w_2''(t)^2 + \zeta''(t)^2)^{1/2}$$
$$\leq 2 \int ([K + 1/B_0]^2 + K^2 2^{2m-8})^{1/2} \leq 2K \int (1 + (2^{m+1})^2 + (2^{m-4})^2)^{1/2}$$
$$< K2^{m+4} 18B_0 \leq K2^{m+4} 18(K2^m)^{-1} = (9)(2^5) .$$

Note $\varphi_u(t) = \varphi_1(t) + \lambda(u)(0, w_2(t), \zeta(t), 0, \cdots, 0)$ for $|t| \leq D$. $|\varphi_u(t)| \leq R/2 + R/4 < R$, because $16B_0 < \min(R/8, D), |w_2(t)| \leq \sqrt{3}B_0$ and $|\zeta(t)| \leq (8B_0)(1/2)$.

Step 3. Let us restrict ourselves to $\varphi_2(t) = (t, (t^2/2)K + tw_2, 0, \cdots, 0)$ for $|t| \leq B_0/2$. Now $w_2^2 + \cdots + w_n^2 < 3/4$. Let us pick $B_1 = B_0/8$ and let $w_k(t)$, $3 \leq k \leq n$, be C^∞ functions such that $w_k'(t) \equiv w_k$ for $|t| \leq B_1, w_k(t) \equiv 0$ for $|t| \geq 4B_1, w_k(0) = 0, \sum (w_k'(t))^2 < 3/4, |w_k''(t)| < (B_1)^{-1}$, and $\sum_{k=3}^{n} w_k(t)^2 \leq (3/4)(B_0/2) < (R/4)^2$. Let $\lambda(u)$ be a strictly increasing monotone C^∞ function on $[2, 3]$ such that $\lambda(2) = 0$ and $\lambda(3) = 1$. Let $\varphi_u(t) = (t, t^2K/2 + tw_2, \lambda(u)w_3(t), \cdots, w_n(t)\lambda(u))$. Then $\varphi_u'(t) = (1, tK + w_2, \lambda w_3'(t), \cdots, \lambda w_n'(t))$ and $\varphi_u''(t) = (0, K, \lambda w_3'', \cdots, \lambda w_n'')$. Hence $\varphi_u(t)$ is nondegenerate, and $|\varphi_u(t)| < 3R/4 + R/4 = R$. $1 \leq |\varphi_u'(t)|^2 \leq 1 + (Kt + w_2)^2 + 3/4 < 4 = (2)^2$. Hence $k(\varphi_u)(t)^2 > (K/8)^2 > (k_0)^2$. Look at

$$J = \int_{-4B_1}^{4B_1} k(\varphi_3)(t)dt < 2 \int (K^2 + \Sigma\, w_i''(t)^2)^{1/2} < 2 \int (K^2 + (B_1)^{-2}(n-2))^{1/2}$$
$$\leq 2 \int (K^2 + (n-2)K^2(2)^{2m+2})^{1/2} \leq 2K2^{m+1}(n-1)^{1/2}B_0 < 4\sqrt{n-1} .$$

Hence $\int_{-A}^{A} k(\varphi_3)(t)dt < 1 + 1 + 32.9 + 4\sqrt{n-1}$, which completes the proof of this lemma.

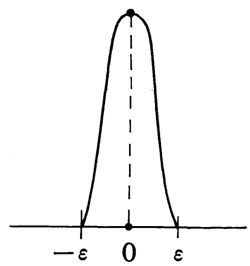
7. Odds and ends

Lemma 7.1. *Let X be a compact set, $\phi: X \to R$ be a continuous function such that $\phi(x) > 0$ for all $x \in X$, and K be a fixed positive number. Then there exists a continuous function $\lambda: X \to \mathscr{C}^\infty(I, I), I = [0, 1]$, such that $\lambda(x)(0) = 0$, $\lambda(x)(1) = 1, \lambda'(x)(t) > 0, \lambda'(x)(0) = \phi(x)/K$, and $\lambda(x)(t) = t$ if $\phi(x) = K$.*

Proof. Let us set $\zeta(x) = \phi(x)/K$. Since X is compact, there exists s_0, $0 < t_0 < 1$, such that $0 < \zeta(x)s_0 < 1$. Set

$$g(x)(s) = \begin{cases} s\zeta(x) , & -1 \leq s \leq s_0 , \\ 1 + (s_0\zeta(x) - 1)(s-1)(s_0-1)^{-1} , & s_0 \leq s \leq 2 . \end{cases}$$

Then $g(x)(0) = 0, g(x)(1) = 1$, and $g(x)$ is continuous and is C^∞ everywhere except at s_0; in fact, $g: X \to \mathscr{C}^0([-1, 2], R)$ is continuous. Extend $g(x)$ to all of R by making it 0 outside $[-1, 2]$. Denote this extension also by $g(x)$, and note that $g(x) \in L^p(R)$, $1 \leq p \leq \infty$, and that $g\colon X \to L^p(R)$ is continuous. Let $0 < \varepsilon < \min(s_0/2, (1 - s_0)/2)$. Let $\varphi_\varepsilon(t) \geq 0$ be the usual C^∞ approximate identity, $\varphi_\varepsilon(t) = \varphi_\varepsilon(-t)$, support $(\varphi_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$ and $\int_{-\infty}^{\infty} \varphi_\varepsilon(t)dt = 1$.



Set $\lambda(x)(t) = (g(x)_* \varphi_\varepsilon)(t) = \int_{-\infty}^{\infty} g(x)(s)\varphi_\varepsilon(t - s)ds$. Then by our choice of g and the usual properties of the convolution, we can see that $\lambda(x)$ is C^∞, it has all the desired properties, and $\lambda\colon X \to \mathscr{C}^k(I, I)$ is continuous for each k (this last λ is $g_* \varphi_\varepsilon \,|\, [0, 1]$).

Corollary 7.2. *Let X be compact, $K > 0$ a real number, and $\phi_i\colon X \to R$, $i = 1, 2$, continuous real valued functions such that $\phi_1(x) > 0$. Then there exists a continuous function $\lambda\colon X \to \mathscr{C}^\infty(I, I)$ such that $\lambda(x)(0) = 0$, $\lambda(x)(1) = 1$, $\lambda'(x)(0) = \phi_1(x)/K$, $\lambda''(x)(0) = \phi_2(x)$, $\lambda'(x)(t) > 0$, and $\lambda(x)(t) = t$ provided $\phi_1(x) = K$ and $\phi_2(x) = 0$.*

Proof. Let $\bar{\lambda}(x)(t)$ be the functions constructed by Lemma 7.1. Set $\lambda(x)(t) = \bar{\lambda}(x)(t) + \varphi(t)t^2/2[\phi_2(x) - \bar{\lambda}(x)''(0)]$ where φ is a C^∞ function, $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\varphi(t) \equiv 0$ for $t \geq \varepsilon$, $\varphi'(0) = 0$, $|\varphi'(t)| < 2/\varepsilon$, and we will choose ε, $0 < \varepsilon < 1$, as follows:

$$\lambda'(x)(t) = \bar{\lambda}(x)'(t) + [t\varphi(t) + (t^2/2)\varphi'(t)](\phi_2(x) - \bar{\lambda}(x)''(0)) .$$

Hence we can find a number $B > 0$ such that if $0 < \varepsilon < B$, then $\lambda'(x)(t) > 0$. Let us choose ε so small that $0 < \varepsilon < B$. Then $\lambda(x)(t)$ is the desired family of curves.

Remark. Let g be a Riemann metric on R^n, X a compact set, $\gamma\colon X \to C^2([0, 1], R^n)$ a continuous map such that $\gamma(x)$ is g-nondegenerate for all $x \in X$, and $f(t)$, $-1 \leq t \leq 0$, be another g-nondegenerate curve. Assume $f(0) = \gamma(x)(0)$, $t(f)(0) = t(\gamma(x)(0)$, $n(\gamma(x))(0) = n(f)(0)$ and $k_g(f)(0) = k_g(\gamma(x))(0)$ for all $x \in X$. By applying Corollary 7.2 we can find $r\colon X \to \mathscr{C}^\infty(I, I)$, $r(x)(t)$, reparametrization of the $\gamma(x)(t)$ with the following properties:

a) $r(x)(t) = t$ if $f'(0) = \gamma(x)'(0)$ and $f''(0) = \gamma(x)''(0)$,

b) $f'(0) = (d\gamma(x)/d\tau(x))(0)$ and $f''(0) = (d^2\gamma(x)/(d\tau(x))^2(0)$, where $\tau(x) = r(x)(t)$ is the new parameter.

Theorem 7.3. *Let E_0 be as in § 2, and pick $e_0 \in E_0$. Then $\pi_k(E_0, e_0) = 0$, $0 \leq k < \infty$.*

Theorem 7.4. *Let X be a compact set, and $f\colon X \to E_0$ a continuous map. Then f is homotopic to a constant map.*

We note that Theorem 7.4 implies Theorem 7.3 so we now prove Theorem 7.4.

Proof. Let W be a neighborhood of x_0, which is the center of geodesic normal coordinates $(x_1, \cdots, x_n)$ so chosen that $e_1 = \partial/\partial x_1(0) = t_0$ and $e_2 = \partial/\partial x_2(0) = n_0$. Let us reparametrize the $f(x)(t)$ such that for $0 \leq t \leq S < 1$, $f(x)(t) \in W$ and $f(x)(t)$ is parametrized by arc length for $0 \leq t \leq S$ $(S > 0)$. Therefore by Taylor's theorem, $f(x)(t) = te_1 + (t^2k_0/2)e_2 + a(x)(t)$, where $k_0 = k_g(f(x))(0)$, $e_1 = t_0 = t(f(x))(0)$, $e_2 = n_0 = n(f(x))(0)$, and $a(x)(t) = \sum_{i=1}^{n} a_i(x)(t)e_i$ where $a_i(x)(t)$ satisfy the hypotheses of Sublemma 6.2 if we set $a_i(x)(-t) = a_i(x)(t)$ [because $a_i(x)(0) = a_i'(x)(0) = a_i''(x)(0) = 0$]. Hence we can find $0 < S_0 \leq S$ such that $te_1 + t^2k_0/2e_2 + ua(x)(t)$ is nondegenerate for all t, $0 \leq t \leq S_0$ and $u, 0 \leq u \leq 1$. Let $\lambda(u)$ be a C^∞ function which is strictly monotone decreasing, $\lambda\colon [0, 1/2] \to R$, such that $\lambda(0) = 1, \lambda(1/2) = S_0/2$. Set $f(x, u)(t)$ $f(x)(t\lambda(u))$. Hence $f(x, 1/2)(t) = f(x)(tS_0/2) = (tS_0/2)e_1 + ((tS_0/2)^2k_0/2)e_2 + a(x)(tS_0/2)$ because $2S_0/2 = S_0$. Let $\lambda\colon [1/2, 1] \to R$ be another smooth monotonically decreasing function such that $\lambda(1/2) = 1$ and $\lambda(1) = 0$. Set $f(x, u)(t) = (tS_0/2)e_1 + (tS_0/2)^2(k_0/2)e_2 + \lambda(u)a(x)(tS_0/2), 1/2 \leq u \leq 1$. This defines the desired homotopy between f and the constant map $f(x, 2)(t) = (tS_0/2)e_1 + (tS_0/2)^2(k_0/2)e_2, 0 \leq t \leq 2$.

8. Proof of the main theorem

Let $(x, v, k) \in V, v = (t, n) \in T(X)_x \times T(X)_x, g(x)(t, t) = g(x)(n, n) = 1$ and $g(x)(t, n) = 0$. Let k_1 be a real number $0 < k_1 < \min(k, C(n)^{-1}B(n)^{-1})$ where $B(n) = 2^{n+5}(80 + \sqrt{n-1})$ and $C(n) = 18n/\sqrt{n}$, W be the domain of x-centered geodesic coordinates $(x_1, \cdots, x_n)$, $g_F = \sum \delta_{ij}dx^idx^j$ be the flat metric on W, and t_F, n_F, and k_F be the unit tangent vector, the principal normal vector, and the geodesic curvature computed with g_F. We adopt the rest of the notation of § 4. Let W_0 be the disc $\sum (x_i)^2 < (2R)^2R < 1$ such that if γ is a g-nondegenerate curve in W_0 and $k_g(\gamma)(t) > k_1$, then γ is g_F-nondegenerate and $k_F(\gamma)(t) > 2k_1/3$. Furthermore, if γ is g_F nondegenerate in W_0 and $k_F(\gamma)(t) > 2k_1/3$, then γ is g-nondegenerate and $k_g(\gamma)(t) > k_1/3$. Let $D = \{x \in W_0 \mid \sum (x_i)^2 < (R/2)^2(\sqrt{n}/2n)^2\}$ and $V_0 = D \times V_2 \times (k_1, \infty)$. V_0 will be the desired neighborhood of (x, v, k). We will now show that $p\colon p^{-1}(V_0) \to V_0$ satisfies strong property P.

Let I^q be a q-cube, $F^{q-1} \subseteq I^q$ the zero faces, and $\varphi\colon I^q \to V_0$ and $\psi\colon F^{q-1} \to p^{-1}(V_0)$ continuous maps such that $p \circ \psi(c) = \varphi(c)$ for all $c \in F^{q-1}$. Let $\varphi(c) = (x(c), t(c), n(c), k(c))$.

If α is the map of Proposition 4.1, then set $\alpha\varphi(c) = (x(c), t_F(c), n_F(c), k_F(c))$. Note that we do not have to "lift" (φ, ψ) but only a deformation of (φ, ψ); see Definition 3.6 and Proposition 3.7.

Step I. Look at $\psi(c)(t), 0 \leq t \leq 2$. We see, by the compactness of F^{q-1}, that there exists a number t_q, $0<t_q<2$, such that $\psi(c)(t) \in D$ and $k_g(\psi(c))(t) > k_1$ for all $t \in [t_q, 2]$ and $c \in F^{q-1}$. By a deformation we can reparametrize $\psi(c)(t)$ so that $t_q = 1/2$, so we can assume $t_q = 1/2$. Since the group $E(n)$ of Euclidean motions is connected, we can find a map $M\colon I^q \to E(n)$ such that $M(c)((\psi(c)(1), t_F(\psi(c))(1), n_F(\psi(c))(1)) = (0, e_1, e_2)$ where $e_1 = (1, 0, \cdots, 0)$ and $e_2 = (0, 1, 0, \cdots, 0)$. Let $m(c)(t) = M(c)(\psi(c)(t)), t \in [1/2, 2]$. Then $|m(c)(t)| < R\sqrt{n}/(2n)$, $m(c)(t)$ is g_F-nondegenerate, and $k_F(m(c))(t) > 2k_1/3$. Applying Theorem 6.1 to the curves $m(c)(t)$(with 1 replacing 0, etc.), we can find a continuous deformation $m_u(c)(t), 0 \leq u \leq 1$, of $m(c)(t)[m_0(c)(t) = m(c)(t)]$ and two numbers S and T, $0 < S < T < 1/2$, such that

1) $|m_u(c)(t)| < R\sqrt{n}/(2n), t_F(m_u(c))(1) = e_1, n_F(m_u(c))(1) = e_2, m_u(c)(1) = 0$, and $k_F(m_u(c))(t) > 2k_1/3$ for $0 \leq u \leq 1, t \in [1/2, 2]$, and $c \in F^{q-1}$,

2) $m_u(c)(t) = m(c)(t)$ for $|t-1| \geq T, 0 \leq u \leq 1, c \in F^{q-1}$,

3) $m_1(c)(t) = m_1(c')(t) = f(t)$ where $f(t)$ is C^∞, for $|t-1| < S$, and $c, c' \in F^{q-1}$, and

4) the path $t_F(f(t)), 1 < t < 1 + S$, passes through each of the 2^n-vertices of the inscribed cube, $k_F(f)(1) > B(n)^{-1}C(n)^{-1}$, and $\int_1^{S+1} k_F(f)(t)dt < B(n)$.

Let $\tau\colon F^{q-1} \to \mathscr{C}^\infty([1/2, 2], [1/2, 2])$ be a continuous map such that $\tau(c)(t) = t, 1/2 \leq t < 1 - T, \tau(c)(1) = 1, \tau(c)(S + 1) = 3/2, \tau(c)(2) = 2, \tau(c)'(t) > 0$.

If $m_1(c)(\tau)$ denotes $m_1(c)$ parametrized by $\tau(c)(t)$, then $t_F(m_1(c))(\tau)$ is parametrized by the reduced arc length for $1 \leq \tau \leq 3/2$. Let $m_{1+u}(c)(t) = m_1(c)(u\tau(c)(t) + (1-u)t), 0 \leq u \leq 1$, and let $m_2(c)(t)$ be $m_1(c)$ parametrized by $\tau(c)$. Hence $m_2(c)(t)$ is defined for $1/2 \leq t \leq 2$, and the curve $t_F(m_2(c)) \mid [1, 3/2]$ is parametrized by the reduced arc length. Let

$$\psi_u(t) = \begin{cases} \psi(c)(t)\,, & 0 \leq t \leq 1/2\,, \\ M(c)^{-1}(m_u(c)(t))\,, & 1/2 \leq t \leq 2, 0 \leq u \leq 2\,. \end{cases}$$

$\psi_u(t)$ defines a continuous deformation of (φ, ψ), and it is $\psi_2(c)(t)$ which we will try to lift.

Step II. Let $T_0(S^{n-1})$ be the unit tangent bundle over the unit sphere $S^{n-1} \subseteq R^n$. Recall $T_0(S^{n-1})$ is diffeomorphic to the Stiefel manifold V_2 by reviewing the point $x \in S^{n-1}$ as the first vector of a 2-frame and $v \in T_0(S^{n-1})_x$ as the second vector. Let $\tilde{E}_0 = [\lambda\colon [3/2, 2] \to S^{n-1} \mid \lambda$ is an immersion, $\lambda(3/2) = t_F(f)(3/2), t(\lambda)(3/2) = n_F(f)(3/2)]$ where $f(t) = m_2(c)(t), 1 \leq t \leq 3/2$. Define $\pi_0\colon \tilde{E}_0 \to T_0(S^{n-1})$ by $\pi_0(\lambda) = (\lambda(2), t(\lambda)(2))$. Let $\psi_S\colon F^{q-1} \to \tilde{E}_0$ be a continuous map defined by $\psi_S(c)(t) = t_F(m_2(c))(t), 3/2 \leq t \leq 2$, and $\varphi_S\colon I^q \to V_2$ be the continuous map defined by $\varphi_S(c) = (M(c)t_F(c), M(c)n_F(c))$. We see that $\pi_0 \circ \psi_S(c) = \varphi_S(c)$ for $c \in F^{q-1}$. By Smale's theorem [11], we can find Ψ_S extending ψ_S to all I^q such that $\pi_0 \circ \Psi_S = \varphi_S$. We now apply Lemma 7.1, and reparametrize $\Psi_S(c)(t), 3/2 \leq t \leq 2$, so that we can assume $(dt_F(f)(t)/dt)(3/2) = (d\Psi_S(c)(t)/dt)(3/2)$, and we can do this in such a way that we need not reparametrize $\psi(c)(t)$ at all if $c \in F^{q-1}$. Let us define $\lambda(c)(t) = t_F(f)(t)$ for $1 \leq t \leq 3/2$, and $\lambda(c)(t) =$ the reparametrized $\psi(c)(t)$ for $3/2 \leq t \leq 2$. Then $\lambda(c)(t) = t_F(m_2(c))(t)$, $c \in F^{q-1}$, $1 \leq t \leq 2$, $\lambda(c)(t)$ is an immersion $c \in I^q$, $1 \leq t \leq 2$, $\lambda\colon I^q \to \mathscr{C}^1([1,2]; S^{n-1})$ is continuous, $\lambda(c)(1) = e_1, t(\lambda(c))(1) = e_2, \lambda(c)(2) = M(c)t_F(c)$, and $t(\lambda(c))(2) = M(c)n_F(c)$. We want to set $\gamma(c)(t+1) = \int_0^t \rho(c)(\tau)\lambda(c)(\tau+1)d\tau$ where $\rho(c)(\tau)$ is $C^1, 0 < \rho(c)(\tau) < 3/(2k_1)$ for $0 \leq \tau \leq 1$, $\rho(c)(1) = k_F(c)^{-1}$, $\rho(c)(0) = k_F(f)(1)^{-1}$, $\int_0^1 \rho(c)(t)dt \leq R$, and $\gamma(c)(2) = M(c)x(c)$. If we can find such function $\rho(c)(\tau)$ and they depend continuously on $c \in I^q$, and $\rho(c)(t) = \left|\frac{d}{dt}(m_2(c))(t+1)\right|, 0 \leq t \leq 1$, for $c \in F^{q-1}$ we would have our problem solved, by reparametrizing the γ's so the end points match up and then translating back by $M(c)^{-1}$.

Step III. Let $\mathscr{C}^1(S^1, R)$ be the C^1-periodic functions from R to R with period 2π. Then $\mathscr{C} = \mathscr{C}^1(S^1, R)$ is a Banach space in the norm $\|\varphi\|_1 = \sup_{0\leq t\leq 2\pi} |\varphi(t)| + \sup_{0\leq t\leq 2\pi} |\varphi'(t)|$. Let $H^2(S^1, R)$ be the Sobolev space of square integrable periodic functions of period 2π, which possess square integrable weak derivatives f' and f''. Then $H^2(S^1, R) = H$ is a Hilbert space with inner product

$$(f, g) = \int_0^{2\pi} f(t)g(t)dt + \int_0^{2\pi} f'(t)g'(t)dt + \int_0^{2\pi} f''(t)g''(t)dt\ .$$

By Sobolov's lemma (in this case an easy proposition about the absolute convergence of the Fourier series of f')[1, pp. 165–168] we have a continuous linear injection $i\colon H \to \mathscr{C}$. Furthermore $i(H)$ is dense in $\mathscr{C}$. Let $i^*\colon \mathscr{C}^* \to H^*$ be the formal adjoint, pick $c \in I^q$, and define the following linear functionals on $\mathscr{C}$: $\alpha(c) = \rho(0), \omega(c) = \rho(1), \mu_i(c) = i$-th coordinate of $\int_0^1 \rho(t)\lambda(c)(t+1)dt$. It is easy to see that $\alpha(c), \omega(c), \mu_i(c), 1 \leq i \leq n \in C^*\colon I^q \to C^*$ are all continuous, that $\alpha(c), \omega(c)$ and $\mu_i(c), 1 \leq i \leq n$, are linearly independent for each fixed $c \in I^q$, and that $i^*\alpha(c), i^*\omega(c)$ and $i^*\mu_j(c), 1 \leq j \leq n$, are also linearly independent for each $c \in I^q$. Define $n+2$ continuous real valued functions on I^q by: $y_j(c) = j$-th coordinate of $M(c)x(c); 1 \leq j \leq n, A(c) = k_F(f)(1)^{-1}$, and $\Omega(c) = k_F(c)^{-1}$. Let $P = \left\{\rho \in H \mid 0 < \rho(t) < 3/(2k_1) \text{ for } 0 \leq t \leq 1, \int_0^1 \rho(t)dt < R\right\}$.

Then P is an open convex set. We now apply Lemma 5.1 and find for each $c \in I^q$ an element $\rho_c \in P$ such that $y_i(c) = \mu_i(c)(\rho), 1 \leq i \leq n, \alpha(c) = A(c)(\rho)$ and $\omega(c) = \Omega(c)(\rho)$. Therefore the curve $\gamma(c)(t+1) = \int_0^t \rho_c(\tau)\lambda(c)(\tau+1)d\tau$ has the following properties: $\gamma(c)(2) = M(c)x(c)$, $t_F(\gamma(c))(2) = M(c)t_F(c)$, $n_F(\gamma(c))(2) = M(c)n_F(c), k_F(\gamma(c))(t) > (2/3)k_1, 1 \leq t \leq 2, k_F(\gamma(c))(2) = k_F(c)$, $t_F(\gamma(c))(1) = t_F(f)(1), n_F(\gamma(c))(1) = n_F(f)(1), \gamma(c)(1) = f(1) = 0, k_F(\gamma(c))(1) = k_F(f)(1)$. Let $P_c = \left\{\rho \in C \,|\, 0 < \rho(t) < (2/3)k_1, t \in [0,1]; \left|\int_0^\tau \rho(t)\lambda(c)(t+1)dt\right| \leq R, \tau \in [0,1]\right\}$. P_c is convex, and $P \subseteq P_c$. For each $c \in F^{q-1}$ let $p(c)(t) = |m_2(c)'(t+1)|, 0 \leq t \leq 1$. Then $p\colon F^{q-1} \to \mathscr{C}^1([0,1], R)$ is continuous, and $m_2(c)(t+1) = \int_0^t p(c)(\tau)\lambda(c)(\tau+1)d\tau$. We want to extend each $p(c)(t)$ to S^1 (i.e., to $[0, 2\pi]$ so that it is C^1-periodic). It is clear that this can easily be done. Hence assume we have defined a continuous map $p\colon F^{q-1} \to \mathscr{C}^1(S^1, R) = \mathscr{C}$ such that $p(c)(t) = |m_2(c)'(t+1)|, 0 \leq t \leq 1$.

We will now quote two facts; the first, Lemma 8.1 is a restatement of the Gram-Schmidt process, and its proof follows word for word the usual proof, the second, Theorem 8.2 is our main abstract analytic lemma, which we prove in § 9.

Lemma 8.1. *Let H and C be respectively a Hilbert space and a Banach space, $i\colon H \to C$ be a continuous linear injection, $i^*\colon C^* \to H^*$ be its formal adjoint, X be a topological space, $\varphi_i\colon X \to C^*, 1 \leq i \leq k$, be k continuous maps such that $\varphi_i(x), \cdots, \varphi_k(x)$ and $i^*\varphi_i(x), \cdots, i^*\varphi_k(x)$ are linearly independent for each $x \in X, P\colon H^* \to H$ be the duality isomorphism, and $y_i\colon X \to R$ be k continuous real valued functions. Then we can find $\Phi_i\colon X \to C^*, Y_i\colon X \to R$, $1 \leq i \leq k$, continuous functions with the following properties*:

a) *$\Phi_1(x), \cdots, \Phi_l(x)$ for each $x \in X$ span the same subspace of C^* as $\varphi_1(x), \cdots, \varphi_l(x)$ for each $l, 0 < l \leq k$.*

b) *If $F_i(x) = P(i^*(\Phi_i(x))$, then $\langle F_j(x), F_k(x)\rangle = \delta_{jk}$ for all x.*

c) *$\varphi_i(x)(\rho) = y_i(x), 1 \leq i \leq k$, if and only if $\Phi_i(x)\rho = Y_i(x), 1 \leq i \leq k$.*

Theorem 8.2. *Let H be a Hilbert space, C a Banach space, $i\colon H \to C$ a continuous linear inclusion, $i^*\colon C^* \to H^*$ its formal adjoint, $D\colon H^* \to H$ the duality map, $P \subseteq H$ an open convex set, I^n the n-cube, and F^{n-1} the union of zero faces.*

a) *Let $v_j^*\colon I^n \to C^*$ be continuous maps $1 \leq j \leq k$, set $v_j = D(i^*(v_j^*))$, and assume $\langle v_i(x), v_j(x)\rangle = \delta_{ij}, x \in I^n$.*

b) *Let $h_j\colon I^n \to R, 1 \leq j \leq k$, be continuous real valued functions.*

c) *For each $x \in I^n$ a convex set $P_x \subseteq C$ is given such that $P \subseteq P_x$. Assume there exists $p_x \in P$ such that $\langle p_x, v_j(x)\rangle = h_j(x), 1 \leq j \leq k$.*

d) *Let* $p\colon F^{n-1} \to C$ *be a continuous map such that* $p(x) \in P_x$ *and* $v_j^*(x)(p(x)) = h_j(x)$ *for each* $x \in F^{n-1}$ *and* $1 \leq j \leq k$.
Then we can find a strong admissible deformation φ_t *of* I^n *and a continuous map* $\rho\colon I^n \to C$ *extending* $p^n\colon F^{n-1} \to C$ *with the following properties*:

i) $\rho(x) \in P\varphi_1(x)$ *for all* $x \in I^n$.

ii) $v_j^*(\varphi_1(x))(\rho(x)) = h_j(\varphi_1(x)), x \in I^n, 1 \leq j \leq k$.

We apply this to the case where $H = H^2(S^1, R), C = C^1(S^1, R), i =$ the Sobolev inclusion, and $P, P_x(P_c)$ and $p\colon F^{q-1} \to C$ are defined as in the discussion preceeding Lemma 8.1. We take $\alpha, \beta, \mu_j, 1 \leq j \leq n$, as our families of linear functionals, and $A, \Omega, y_j, 1 \leq j \leq n$, as our families of continuous functions. Hence we find a strong admissible deformation ν_t of I^q and an extension ρ of $p\colon F^{q-1} \to C$ with the following properties: Set

$$\gamma_0(c)(t+1) = \int_0^t \rho(c)(\tau)\lambda(\nu_1(c))(\tau+1)d\tau.$$

Then $t_F(\gamma_0(c))(1) = t_F(f)(1)$, $n_F(\gamma_0(c))(1) = n_F(f)(1)$, $k_F(\gamma_0(c))(1) = k_F(f)(1)$, $t_F(\gamma_0(c))(2) = M(\nu_1(c))$, $t_F(\nu_1(c))$, $n_F(\gamma_0(c))(2) = M(\nu_1(c))$, $n_F(\nu_1(c))$, $k_F(\gamma_0(c))(2) = k_F(\nu_1(c))$, $\gamma_0(2) = M(\nu_1(c))x(\nu_1(c))$, $k_F(\gamma_0(c))(t) > 2k_1/3, t \in [1,2]$, and $|\gamma_0(c)(t)| < R$. We now apply Corollary 7.2 in order to reparametrize $\gamma_0(c)(t)$ so that $\gamma_1(c)'(1) = f'(1)$ and $\gamma_1(c)''(1) = f''(1)$, where $\gamma_1(c)(t), t \in [1,2]$, are the reparametrized $\gamma_0(c)$, and we do not reparametrize $\gamma_0(c)(t)$ at all if $\gamma_0(c)'(1) = f'(1)$ and $\gamma_0(c)''(1) = f''(1)$. Let $\gamma_1(c)(t)$ denote the suitably reparametrized $\gamma_0(c)(t)$. Pick a retract $\Omega\colon I^q \to F^{q-1}$, and define

$$\gamma_2(c)(t) = \begin{cases} m_2(\Omega(c))(t)\,, & 1/2 \leq t \leq 1\,, \\ \gamma_1(c)(t)\,, & 1 \leq t \leq 2\,. \end{cases}$$

Then set $\gamma_3(c)(t) = M(\nu_1(c))^{-1}\gamma_2(c)(t)$. Finally set

$$\Psi(c)(t) = \begin{cases} \phi(\Omega(c))(t)\,, & 0 \leq t \leq 1/2\,, \\ \gamma_3(c)(t)\,, & 1/2 \leq t \leq 2\,. \end{cases}$$

Note that $|\gamma_3(c)(t)| < 2R$, $k_F(\gamma_3(c))(t) > 2k_1/3$, $t_F(\gamma_3(c))(2) = t_F(\nu_1(c))$, $n_F(\gamma_3(c))(2) = n_F(\nu_1(c))$, $k_F(\gamma_3(c))(2) = k_F(\nu_1(c))$, and $\gamma_3(c)(2) = x(\nu_1(c))$. Hence $\gamma_3(c)$ is g-nondegenerate and has the correct terminal data. $\Psi\colon I^q \to E_0$ is continuous, $\Psi | F^{q-1} = \phi_2$ (see end of Step I), and $p \circ \Psi = \varphi \circ \nu_1$.

9. Proof of Theorem 8.2

Step I. For each $x \in I^n$, pick $p_x \in P$ such that $\langle p_x, v_j(x)\rangle = h_j(x), 1 \leq j \leq k$. Look at the expression

$$p_{x'}(x) = p_{x'} - \sum_{i=1}^{k} (\langle p_{x'}, v_i(x)\rangle - h_i(x))v_i(x)\,.$$

$p_{x'}(x)$ is continuous in x, and there exists $\varepsilon_{x'} > 0$ such that if $|x - x'| < \varepsilon_{x'}$, then $p_{x'}(x) \in P$, because P is open. Then $\langle p_{x'}(x), v_j(x)\rangle = h_j(x), 1 \leq j \leq k$, and therefore $p_{x'}(x)$ has all the desired properties in a neighborhood of x'. Since these $\varepsilon_{x'}$ neighborhoods about x' form an open covering of the cube I^n, by the Lebesgue covering lemma we can find an integer $N > 1$ such that any cube with side of length $= 1/N$ must lie in one of the $\varepsilon_{x'}$ balls. Let $B_{i_1,\dots,i_n} = \{(x_1, \cdots, x_n \mid i_k/N \leq x_k \leq i_k + 1/N\}, 0 \leq i_k \leq N-1$. On each of the $B_{i_1,\dots,i_n}$ we have one of the $p_{x'}(x)$ defined, call it $p_{i_1,\dots,i_n}(x)$. Hence we have N^n boxes, and N^n "good" functions.

Step II. Let us construct the φ_t: $I^n \to I^n$ as follows. Let $\varphi_t(x_1, \cdots, x_n)_k$ denote the k-th coordinate of $\varphi_t(x)$.

a) If $t/3 \leq x_k \leq 1 - t/3$ for all $k, 1 \leq k \leq n$, then we set

$$\varphi_t(x_1, \cdots, x_n)_k = i_k/N$$

for

$$t/3 + i_k(1 - 2t/3)/N - t/(9N) \leq x_k \leq t/3 + i_k(1 - 2t/3)/N + t/(9N) \ ,$$

and

$$\varphi_t(x_1, \cdots, x_n)_k = i_k/N + \{x_k - [t/3 + i_k(1-2t/3)/N + t/(9N)]\}[9/(9-8t)]$$

for

$$t/3 + i_k(1-2t/3)/N + t/(9N) \leq x_k \leq t/3 + (i_k + 1)(1-2t/3)/N - t/(9N) \ .$$

A direct calculation shows $\varphi_0 = \text{id}$, and φ_t is continuous and well-defined on the inside cube $C_t = \{(x_1, \cdots, x_n \mid t/3 \leq x_k \leq 1 - t/3\}$.

b) Let us fix t. Let $T_{k,0,t} = \{(x_1, \cdots, x_n) \mid x_k = ts/3, 0 \leq s \leq 1$, and $ts/3 \leq x_l \leq 1 - ts/3, 0 \leq s \leq 1$, for $l \neq k\}$, and $T_{k,1,t} = \{(x_1, \cdots, x_n) \mid x_k = 1 - ts/3, 0 \leq s \leq 1$, and $ts/3 \leq x_l \leq 1 - ts/3$ for $l \neq k, 0 \leq s \leq 1\}$. The cube I^n is broken up into the inner cube C_t and the $2n$ "trapazoids" $T_{k,i,t}$, $i = 0, 1$. We now define φ_t on $T_{k,0,t}$. If $x \in T_{k,0,t}$, set $\varphi_t(x_1, \cdots, x_n)_k = 0$. Let $x_k = St/3$. Then $\varphi_t(x_1, \cdots, x_n)_j = i_j/N$ if

$$\begin{aligned} St/N + (i_j/N)(1 - 2St/3) - St/(9N) \\ \leq x_j \leq St/3 + (i_j/N)(1 - 2St/3) + St/N \ , \end{aligned}$$

and

$$\begin{aligned} \varphi_t(x_1, \cdots, x_n)_j = i_j/N + [x_j - St/3 + i_j(1 - 2St/3)/N \\ + St/(9N)][9/(9 - 8St)] \end{aligned}$$

if

$$\begin{aligned} St/3 + (i_j/N)(1 - 2St/3) + St(9N) \\ \leq x_j \leq St/3 + (i_j + 1)(1 - 2St/3)/N - St/(9N) \end{aligned}$$

for $j \neq k$. It is easy to see that $\varphi_0 = \text{id}$, and φ_t is well-defined and continuous on $C_t \cup T_{1,0,t} \cup \cdots \cup T_{n,0,t}$.

c) We will now extend φ_t to $T_{l,1,t}, 1 \leq l \leq n$. Let $x \in T_{k,1,t}$. Then $x_k = 1 - St/3$ for some $S, 0 \leq S \leq 1$, and $St/3 \leq x_j \leq 1 - St/3$ for $j \neq k$. Let

$$\varphi_t(x_1, \cdots, x_n) = \varphi_t\left(\left[(x_1, \cdots, x_n) - \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)\right]\left[\frac{3-2t}{3-2St}\right] + \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)\right),$$

where the φ_t on the right is the φ_t defined on C_t. Again a direct calculation shows that this formula makes sense. A further check shows that $(\varphi_t), 0 \leq t \leq 1$, define a strong admissible deformation of I^n.

Step III. Note that $\varphi_1(T_{k,0,1}) = I^{n-1}_{k,0}$. We define ρ on $\bigcup_{k=1}^{n} T_{k,0,1}$ by $\rho(x) = p(\varphi_1(x))$. We immediately see $\rho | F^{n-1} = p$. Let us look at the cubes

$$C_{i_1,\dots,i_n} = \{(x_1, \cdots, x_n) \mid 1/3 + i_k/(3N) + 1/(9N) \leq x_k \leq 1/3 + (i_k + 1)/(3N) - 1/(9N)\}.$$

Since φ_1 maps $C_{i_1,\dots,i_n}$ homeomorphically onto $B_{i_1,\dots,i_n}$, we can define ρ on $C_{i_1,\dots,i_n}$ by the formula $\rho(x) = p_{i_1,\dots,i_n}(\varphi_1(x))$ for $x \in C_{i_1,\dots,i_n}$. We will now extend ρ to all C_1 by the following induction hypothesis.

Hypothesis $l - 1$. We assume ρ is defined for all $(x_1, \cdots, x_n) \in C_1$ such that $1/3 \leq x_k \leq 2/3$ for $k = 1, \cdots, l-1$, and $1/3 + i_k/(3N) + 1/(9N) \leq x_k \leq 1/3 + (i_k + 1)/(3N) - 1/(9N)$ for $k = l, \cdots, n$. Assume ρ satisfies i) and ii) of the statement of Theorem 8.2 wherever ρ is defined. To show $(l-1) \Rightarrow (l)$, pick $x = (x_1, \cdots, x_n)$ such that $1/3 \leq x_k \leq 2/3$ for $k = 1, \cdots, l$, and $1/3 + i_k/(3N) + 1/(9N) \leq x_k \leq 1/3 + i_k + 1/(3N) - 1/(9N)$ for $k = l+1, \cdots, n$. If $1/3 + i_l/(3N) + 1/(9N) \leq x_l \leq 1/3 + i_l + 1/(3N) - 1/(9N)$, then ρ is already defined on x. If $1/3 \leq x_l \leq 1/3 + 1/(9N)$, we see that φ_1 is constant along the line $(x_1, \cdots, x_{l-1}, 1/3 + t/(9N), x_{l+1}, \cdots, x_n), 0 \leq t \leq 1$. Hence we can define ρ along this line by the formula

$$\begin{aligned}\rho(x_1, \cdots, x_{l-1}, 1/3 + t/(9N), x_{l+1}, \cdots, x_n) &= (1-t)\rho(x_1, \cdots, x_{l-1}, 1/3, x_{l+1}, \cdots, x_n) \\ &\quad + t\rho(x_1, \cdots, x_{l-1}, 1/3 + 1/(9N), x_{l+1}, \cdots, x_n).\end{aligned}$$

ρ is continuous in x and t, and has all the desired properties due to the convexity of the P_x. Set

$$C_{i_l,i_{l+1},\dots,i_n} = \{(x_1, \cdots, x_n) \mid 1/3 \leq x_i \leq 2/3, 1 \leq i \leq l-1, 1/3 + i_k/(3N) + 1/(9N) \leq x_k \leq 1/3 + (i_k + 1)/(3N) - 1/(9N) \text{ for } k = l, l+1, \cdots, n\}.$$

If $1/3 + i_l/(3N) - 1/(9N) \leq x_l \leq 1/3 + i_l/(3N) + 1/(9N), 1 \leq i_l \leq N-1$, we look at the line $(x_1, \cdots, x_{l-1}, 1/3 + (i_l - 1)/(3N) + 2/(9N) + 2t/(9N),$

$x_{l+1}, \cdots, x_n), 0 \le t \le 1$, which joins $(x_1, \cdots, x_{l-1}, 1/3 + i_l/(3N) - 1/(9N), x_{l+1}, \cdots, x_n) \in C_{i_{l-1}, i_{l+1}, \cdots, i_n}$ to $(x_1, \cdots, x_{l-1}, 1/3 + i_l/(3N) + 1/(9N), x_{l+1}, \cdots, x_n) \in C_{i_l, i_{l+1}, \cdots, i_n}$. φ_1 is a constant along this line, and hence we can set

$$\begin{aligned}\rho(x_1, \cdots, x_{l-1}, 1/3 + (i_l - 1)/(3N) + 2/(9N) + 2t/(9N), x_{l+1}, \cdots, x_n) \\ = (1-t)\rho(x_1, \cdots, x_{l-1}, 1/3 + (i_l - 1)/(3N) + 2/(9N), x_{l+1}, \cdots, x_n) \\ + t\rho(x_1, \cdots, x_{l-1}, 1/3 + i_l/(3N) + 1/(9N), x_{l+1}, \cdots, x_n) .\end{aligned}$$

If $2/3 - 1/(9N) \le x_l \le 2/3$, we again note that φ_1 is constant along the line $(x_1, \cdots, x_{l-1}, 2/3 - 1/(9N) + t/(9N), x_{l+1}, \cdots, x_n), 0 \le t \le 1$. Set

$$\begin{aligned}\rho(x_1, \cdots, x_{l-1}, 2/3 - 1/(9N) + t/(9N), x_{l+1}, \cdots, x_n) \\ = \rho(x_1, \cdots, x_{l-1}, 2/3 - 1/(9N), x_{l+1}, \cdots, x_n) .\end{aligned}$$

It is easy to see that we have now constructed by induction ρ with the desired properties on $C_1 \cup T_{1,0,1} \cup \cdots \cup T_{n,0,1}$. $T_{k,1,1} = \{(x_1, \cdots, x_n) \mid x_k = 1 - S/3, 0 \le S \le 1, S/3 \le x_j \le 1 - S/3, j \ne k\}$. For $x \in T_{k,1,1}$ we see $x_k = 1 - S/3$ for some k. Set $\lambda_k(x) = [(x_1, \cdots, x_n) - (1/2, \cdots, 1/2)][1/3 - 2S] + (1/2, \cdots, 1/2)$. Then λ_k defines a retraction of $T_{k,1,1}$ onto $T_{k,1,1} \cap C_1$. The λ_k's agree on the overlaps, so they define a retraction $\lambda\colon \bigcup_{k=1}^{n} T_{k,1,1} \to \left(\bigcup_{k=1}^{n} T_{k,1,1}\right) \cap C_1$. We see immediately that $\varphi_1(x) = \varphi_1(\lambda(x))$ for $x \in \bigcup T_{k,1,1}$. Hence we can extend ρ to $T_{k,1,1}, 1 \le k \le n$, by setting $\rho(x) = \rho(\lambda(x))$.

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