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HARMONIC MAPS ON HYPERBOLIC SPACES WITH SINGULAR BOUNDARY VALUE

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0. Introduction

In [10]–[12], it was proved that a C^1 map $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ with nowhere vanishing energy density can be extended to a proper harmonic map u from one hyperbolic space $\mathbb{H}^m$ to another one $\mathbb{H}^n$. For the case that $m = n = 2$ and f is C^4, the result was also proved by Akutagawa [1] independently. Here we identify the geometric boundaries of $\mathbb{H}^m$ and $\mathbb{H}^n$ with $\mathbb{S}^{m-1}$ and $\mathbb{S}^{n-1}$ respectively. Moreover, it was proved in [11]–[12] that the constructed u is C^1 up to the boundary $\mathbb{S}^{m-1}$ and is unique with respect to the boundary data within the class of maps which are C^1 up to the boundary. The purpose of this paper is to study the Dirichlet problem at infinity of proper harmonic maps for boundary data $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ which may not satisfy the conditions mentioned above. Namely, f may not be smooth or the energy density of f may vanish somewhere. For simplicity, boundary data of this kind is said to be singular. The set where f fails to satisfy one of these conditions will be called the singular set of f.

One of the motivation for the study of this problem is to understand a conjecture of Schoen [16], which says that: *Given a quasi-symmetric homeomorphism f of $\mathbb{S}^1$ there is a unique quasi-conformal harmonic diffeomorphism from $\mathbb{H}^2$ onto itself with boundary value f*, and its generalization [13] : *Every quasi-conformal map from the boundary at infinity of a rank-1 symmetric space M to itself can be extended uniquely*

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to a harmonic rough-isometry from M to itself. The uniqueness part of Schoen's conjecture was proved by Li and the second author [12] and the uniqueness part of the generalized conjecture of Li and Wang was proved by themselves in [13]. As for the existence part, there are not so many results except for the works of [10]–[12] mentioned above and their generalizations to rank-1 symmetric spaces of noncompact type by Donnelly [6], to Damek-Ricci spaces by Ueno [20], and to Carnot spaces by Nishikawa and Ueno [14]. All these results are under the assumptions that the boundary map is smooth (at least C^1) such that its energy density is nowhere zero. For singular boundary maps, it was proved by the second and third authors in [18] that if one can solve the boundary value problem for harmonic map for a particular quasi-symmetric function f on $\mathbb{S}^1$, then one can solve similar problem for near by (with respect to the Teichmüller metric on the universal Teichmüller space) quasi-symmetric functions. Its generalization to higher dimensional hyperbolic spaces has been obtained in [8], [19] and [23].

On the other hand, by studying the Gauss maps of rotationally symmetric constant mean curvature cuts in Minkowski three space, Choi and Treibergs [5] had constructed some interesting harmonic diffeomorphisms from $\mathbb{H}^2$ onto itself by solving an ordinary differential equation. It turns out that if we identify $\mathbb{H}^2$ with the upper half plane equipped with the Poincaré metric, then the boundary value of their maps are given by

$$f(t) = |t|^{\alpha-1}t, \tag{0.1}$$

where $\alpha > 0$ is a constant. In fact, one can solve this problem directly without using the idea of constant mean curvature cuts. That was done by Wang in [22] by solving an ordinary differential equation again. Based on this result, Li and Wang [13] were able to construct harmonic maps on $\mathbb{H}^2$ with boundary value $f : \mathbb{S}^1 \to \mathbb{S}^1$ which is C^1 with nowhere vanishing energy density except at finitely many points and near each of these points f behaves like (0.1). Namely, near such a point, after some transformations, $f(t) = |t|^{\alpha-1}h(t)$, where h is C^1 and $h'(0) \neq 0$. For higher dimensional hyperbolic spaces, it was proved in [22] that if $f : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ is a map with singular set consisting of only finitely many points and if near each singular point p, after conformal transformations of the domain and the range, f is given by $f(x) = |x|^{\alpha-1}x$, then one can extend the map to be harmonic by solving a more complicated ordinary differential equation.

Observe that the boundary value (0.1) is a special case of

$$f(t) = \begin{cases} |t|^{\alpha-1}t & \text{if } t \geq 0, \\ C|t|^{\alpha-1}t & \text{if } t \leq 0, \end{cases} \tag{0.2}$$

where $C > 0$ is a constant. In this case $f(t) = |t|^{\alpha-1}h(t)$, where $h(t)$ is only Lipschitz, $h(0) = 0$ and h' is bounded away from zero. It is unknown up to now whether such a map f can be extended to a quasi-conformal harmonic diffeomorphism on $\mathbb{H}^2$. In this paper we will prove a more general result: *Suppose $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is a map with singular set Σ which is a disjoint union of embedded submanifolds of $\mathbb{S}^{m-1}$, such that f is C^3 on $\mathbb{S}^{m-1} \setminus \Sigma$. Suppose near Σ, if f is considered as a map from $\mathbb{R}^{m-1}$ to $\mathbb{R}^{n-1}$, each component of f will behave like (0.2) (see §2 for more details). Then one can extend f to a proper harmonic map with bounded energy density.* The idea of proof is to use an observation of Bando [2] that if one can extend the boundary map to a map v such that the Poisson equation $\Delta g = -||\tau(v)||$ has a bounded solution, where $||\tau(v)||$ is the norm of the tension field of v, then one can extend the boundary map to a harmonic map. We will also prove that if f is a smooth map from $\mathbb{S}^{m-1}$ to $\mathbb{S}^{n-1}$ such that the zero set of the energy density $e(f)$ of f is a disjoint union of embedded submanifolds and $e(f)$ behaves well near the zero set (see §2 for more details), then f can be extended to a harmonic map. As a consequence, one can show that if f is a nonconstant analytic map from $\mathbb{S}^1$ to $\mathbb{S}^{m-1}$, then f can be extended to a proper harmonic map from $\mathbb{H}^2$ to $\mathbb{H}^m$. This generalizes the result in [13], where the case $m = 2$ was proved.

From the above mentioned existence results, it is not hard to prove that the boundary map (0.2) can be extended to a quasi-conformal harmonic diffeomorphism on $\mathbb{H}^2$. In fact, one can find explicit solutions for the particular boundary data (0.2) and compute the Hopf differentials explicitly. From this, one can show that if f is a homeomorphism from $\mathbb{S}^1$ onto $\mathbb{S}^1$ with only finitely many singular points such that near each of them, f behaves like (0.2) and if h is another quasi-symmetric homeomorphism on $\mathbb{S}^1$ which can be extended to a quasi-conformal harmonic map on $\mathbb{H}^2$, then so does $h \circ f$. As in [18], one can then prove that each element in the closure of the set of such f in the universal Teichmüller space has a quasi-conformal harmonic extension.

Using the method of ordinary differential equations, other harmonic diffeomorphisms on $\mathbb{H}^2$ can be obtained [11]. One may wonder whether one can construct more harmonic maps on $\mathbb{H}^2$ by this method. In order to reduce the harmonic map equations to ordinary differential equations, usually we have to assume the solutions are invariant under groups acting on the solutions space. In this paper, we will find all group-invariant harmonic maps from a domain in $\mathbb{R}^2$ into $\mathbb{H}^2$. The first step is to find the infinitesimal generators (see [15]), and then we will prove the following:

Theorem 5.2. *The Lie point symmetry group of the system of harmonic map equations from a domain in $\mathbb{R}^2$ into $\mathbb{H}^2$ is the product of the local group of local conformal transformations of the domain and the isometry group of $\mathbb{H}^2$.*

From this, we will find all group-invariant harmonic maps from a domain in $\mathbb{R}^2$ into $\mathbb{H}^2$ up to a conformal transformation of the domain and an isometry on $\mathbb{H}^2$. They are invariant either under the group generated by a translation on $\mathbb{R}^2$ and a translation on the upper-half plane model of $\mathbb{H}^2$, invariant under the group generated by a translation on $\mathbb{R}^2$ and a dilation on the upper-half plane model of $\mathbb{H}^2$, or invariant under the group generated by a translation on $\mathbb{R}^2$ and a rotation on the unit disc model of $\mathbb{H}^2$. Not all of the harmonic maps obtained are diffeomorphisms. Among the group-invariant harmonic diffeomorphisms only those with boundary value (0.2) are quasi-conformal.

We will also discuss the problem of uniqueness on the Dirichlet problem at infinity for the class of singular boundary maps. As mentioned before, it was proved in [11] that if f has no singular point, then any two harmonic extensions of f which are C^1 up to the boundary must be the same. However, even if the boundary map is very nice, for example, the identity map of $\mathbb{S}^{m-1}$, there are examples of non-uniqueness [11]–[12], [7]. In fact, given k points on $\mathbb{S}^{m-1}$, one can construct a k-parameter family of harmonic maps u from $\mathbb{H}^m$ to itself with $u|_{\mathbb{S}^{m-1}}$ being the identity map; see [12]. Each member u in the family is C^1 up to the boundary except possibly at those k points. That is, u fails to satisfy the conditions of the uniqueness theorem in [11] only at finitely many points. If we take two different such maps u and v, then $d_{\mathbb{H}^m}(u, v)$ will be zero at the geometric boundary $\partial\mathbb{H}^m$ of $\mathbb{H}^m$ except possibly at finitely many points. Near at least one of the points, $d_{\mathbb{H}^m}(u, v)$ will grow like $\exp((m-1)r)$, where r is the distance function from a fixed point in $\mathbb{H}^m$. Hence it seems that the reason for $u \neq v$ is that $d_{\mathbb{H}^m}(u, v)$ grows too fast. In this work, we will prove the following.

Theorem 3.1. *Let Σ be a closed subset of $\mathbb{S}^{m-1}$. Let u and v be two harmonic maps from $\mathbb{H}^m$ to $\mathbb{H}^n$ such that the function $dist_{\mathbb{H}^n}(u(p), v(p))$*

satisfies:

(i) $\lim_{p\to x} dist_{\mathbb{H}^n}(u(p), v(p)) = 0$ *for any boundary point* $x \in \mathbb{S}^{m-1}\backslash\Sigma$*; and*

(ii) either (a) $\mathcal{H}^{m-2}(\Sigma) < \infty$ *and* $dist_{\mathbb{H}^n}(u(p), v(p)) = o(\exp(r(p))$ *as* $p \to \infty$*, or (b)* $H^{m-2}(\Sigma) = 0$ *and* $dist_{\mathbb{H}^n}(u(p), v(p)) = O(\exp(r(p))$ *as* $p \to \infty$*.*

Here $\mathcal{H}^{m-2}$ *is the* $(m-2)$*-dimensional Hausdorff measure on* $\mathbb{S}^{m-1}$ *and* $r(p)$ *is the distance function in* $\mathbb{H}^m$ *from a fixed point* o*.*

Then $u \equiv v$*.*

Note that the growth condition will be satisfied if both u and v have uniformly bounded energy density. Hence, one may wish to compare Theorem 3.1 with the following result: if u is a harmonic diffeomorphism on $\mathbb{H}^2$ with bounded energy density so that $u|_{\mathbb{S}^1}$ is the identity map, then u is quasi-conformal by [21], and hence u must be the identity by the uniqueness theorem in [12]. As a consequence, if the boundary map is analytic and if u and v have bounded energy density such that u and v are C^1 up to the boundary portion which is the complement of the zero set of $e(f)$, then $u \equiv v$.

The structure of this paper is as follows. In §1, we will give an estimate for solutions of the Poisson's equation in $\mathbb{H}^m$. In §2, we will prove the existence theorems. In §3, we will prove some uniqueness results. In §4, we will give an explicit solution for (0.2), using the method of ordinary differential equations. We will also give some applications of the result to the theory of universal Teichmüller space. In §5, we will find all group-invariant harmonic maps from a domain in $\mathbb{R}^2$ into $\mathbb{H}^2$.

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1. Estimates of solutions of the Poisson's equation

We will use the Poincaré unit ball model for $\mathbb{H}^m$ and identify the geometric boundary of $\mathbb{H}^m$ with the unit sphere $\mathbb{S}^{m-1}$. Let o be the center of the unit ball $\mathbb{B}^m(1)$. Let $\tilde{\Sigma}$ be a compact manifold without boundary of dimension $m-1-k$ with $k \geq 1$ and let $\iota : \tilde{\Sigma} \to \mathbb{S}^{m-1}$ be an immersion. Denote $\iota(\tilde{\Sigma})$ by Σ. Denote the distance of a point

η in $\mathbb{S}^{m-1}$ to Σ by $d_\Sigma(\eta)$. Using geodesic polar coordinates at o, the coordinates of a point $x \in \mathbb{H}^m$ are $(\eta(x), r(x))$, where $\eta(x) \in \mathbb{S}^{m-1}$ is the point of infinity of the geodesic ray from o to x and $r(x)$ is the distance between o and x in $\mathbb{H}^m$. The main purpose of this section is to prove the following lemma.

Lemma 1.1. *Let f be a nonnegative function on $\mathbb{H}^m$ such that*

$$f(x) \leq \frac{C_1 e^{-r(x)}}{d_\Sigma(\eta(x)) + e^{-r(x)}}$$

for some constant C_1, then there is a constant C_2 such that

$$\int_{\mathbb{H}^m} G(x,y) f(y) dy \leq C_2$$

for all $x \in \mathbb{H}^m$. Here $G(x,y)$ is the minimal positive Green's function of $\mathbb{H}^m$.

Proof. Note that $G(x,y)$ is a function of $r(x,y)$, where $r(x,y)$ is the distance between x and y. Using the fact that f is bounded, we have

$$\begin{aligned} \int_{\mathbb{H}^m} G(x,y) f(y) dy &= \int_{r(x,y)\leq 1} G(x,y) f(y) dy + \int_{r(x,y)\geq 1} G(x,y) f(y) dy \\ &\leq C_3 (1 + \int_{\mathbb{H}^m} e^{-(m-1)r(x,y)} f(y) dy) \\ &= C_3 \left(1 + \left(\int_{r(y)\leq 1} + \int_{r(y)\geq 1} \right) e^{-(m-1)r(x,y)} f(y) dy \right) \qquad (1.1) \\ &\leq C_4 \left(1 + \int_{r(y)\geq 1} e^{-(m-1)r(x,y)} \frac{e^{-r(y)}}{d_\Sigma(\eta(y)) + e^{-r(y)}} dy \right) \end{aligned}$$

for some constants C_3 and C_4 which are independent of x. We have the following relation:

$$\begin{aligned} e^{r(x,y)} &\geq \cosh r(x,y) \\ &= \cosh r(x) \cosh r(y) - \sinh r(x) \sinh r(y) \cos\theta \\ &= \cosh r(x) \cosh r(y) \left(1 - \tanh r(x) \tanh r(y) \cos\theta\right) \\ &\geq \frac{1}{4} e^{r(x)+r(y)} \left(1 - \tanh r(x) \tanh r(y) \cos\theta\right), \end{aligned}$$

where θ is the angle between the geodesics at o from o to x and y. Note that $\cos\theta =< \eta(x), \eta(y) >$. Hence

$$
\begin{aligned}
&\int_{r(y)\geq 1} e^{-(m-1)r(x,y)} \frac{e^{-r(y)}}{d_\Sigma(\eta(y)) + e^{-r(y)}} dy \\
&\qquad \leq C_5 e^{-(m-1)r(x)} \left\{ \int_1^\infty e^{-r} \left(\int_{\mathbb{S}^{m-1}} h(\eta, r) d\eta \right) dr \right\},
\end{aligned}
\tag{1.2}
$$

where C_5 is a constant depending only on m and

$$h(\eta, r) = \frac{(1- < \eta, \eta(x) > \tanh r(x) \tanh r)^{-m+1}}{d_\Sigma(\eta) + e^{-r}}.$$

By the definition of Σ, there are finitely many open sets $\tilde{U}_j$ of $\tilde{\Sigma}$ such that $\bigcup_j \tilde{U}_j = \tilde{\Sigma}$, and there are $\eta_j \in \mathbb{S}^{m-1}$ and $\rho_j > 0$ such that $\iota(\tilde{U}_j) \subset B_{\eta_j}(\rho_j)$. Here $B_{\eta_j}(\rho_j)$ is the geodesic ball of radius ρ_j with center at η_j in $\mathbb{S}^{m-1}$. Moreover there are diffeomorphisms G_j such that the following hold:

(i) G_j maps $B_{\eta_j}(4\rho_j)$ into $\{|z| < 2\} \subset \mathbb{R}^{m-1}$, $z = (z_1, \ldots, z_{m-1})$, with $G_j\left(\iota(\tilde{U}_j) \cap B_{\eta_j}(2\rho_j)\right) \subset \{(z_1, \ldots, z_{m-1}) |\, z_1 = \cdots = z_k = 0\}$, where $k \geq 1$ is the codimension of Σ and $G_j\left(B_{\eta_j}(2\rho_j)\right) \subset \{|z| < 1\}$;

(ii) There is a $\delta_0 > 0$ such that for any j, (a) if $\eta \in B_{\eta_j}(2\rho_j)$ and $\eta' \notin B_{\eta_j}(3\rho_j)$, then $| < \eta, \eta' > | \leq 1 - \delta_0$, (b) if $\eta \notin B_{\eta_j}(2\rho_j)$, then $d_{\Sigma_j}(\eta) \geq \delta_0$; and (c) there is a constant $C > 0$ such that for each j, if $\eta \in B_{\eta_j}(2\rho_j)$ then $d_{\Sigma_j}(\eta) \geq C\left(\sum_{i=1}^k (z_i)^2\right)^{\frac{1}{2}}$,

where $\Sigma_j = \iota(\tilde{U}_j) \cap B_{\eta_j}(\rho_j)$ and $d_{\Sigma_j}(\eta)$ is the distance of a point η in $\mathbb{S}^{m-1}$ to Σ_j. For each point $\eta \in \mathbb{S}^{m-1}$, $d_\Sigma(\eta) = d_{\Sigma_j}(\eta)$ for some j. Hence

$$h(\eta, r) \leq \sum_j h_j(\eta, r), \tag{1.3}$$

where

$$h_j(\eta, r) = \frac{(1- < \eta, \eta(x) > \tanh r(x) \tanh r)^{-m+1}}{d_{\Sigma_j}(\eta(y)) + e^{-r}}.$$

Let us first assume that $r(x) \geq 1$. For each j,

$$\begin{aligned}\int_{\mathbb{S}^{m-1}} h_j(\eta,r)d\eta &= \int_{B_{\eta_j}(2\rho_j)} h_j(\eta,r)d\eta \\ &\quad + \int_{\mathbb{S}^{m-1}\setminus B_{\eta_j}(2\rho_j)} h_j(\eta,r)d\eta = I + II.\end{aligned} \tag{1.4}$$

By (ii),

$$II \leq (\delta_0 + e^{-r})^{-1} \int_{\mathbb{S}^{m-1}} (1- < \eta, \eta(x) > \tanh r(x) \tanh r)^{-m+1} d\eta.$$

$\mathbb{S}^{m-1}$ is the sphere $\sum_{i=1}^m (x_i)^2 = 1$. To estimate II, we may assume that $\eta(x) = (1, 0, \dots, 0)$. If $\eta = (x_1, \dots, x_m)$, then $< \eta, \eta(x) >= x_1$. Let

$$\epsilon = \tanh r(x) \tanh r.$$

By the co-area formula for $m \geq 3$, we obtain

$$\begin{aligned} II &\leq \left(\delta_0 + e^{-r}\right)^{-1} \int_{-1}^{1} (1 - \epsilon x_1)^{-m+1} (1 - (x_1)^2)^{\frac{m}{2}-1} \\ &\qquad \cdot |\nabla_{\mathbb{S}^{m-1}} x_1|^{-1} dx_1 \\ &\leq C_6 \left(\delta_0 + e^{-r}\right)^{-1} \int_{-1}^{1} (1 - \epsilon x_1)^{-\frac{m}{2}-\frac{1}{2}} dx_1 \\ &\leq C_7 \left(\delta_0 + e^{-r}\right)^{-1} (1-\epsilon)^{-\frac{m-1}{2}} \end{aligned} \tag{1.5}$$

for some constants C_6 and C_7 which are independent of x. Here we have used the fact that the area of the sphere $\sum_{i=2}^m (x_i)^2 = 1 - (x_1)^2$ for fixed x_1 is $(1-(x_1)^2)^{\frac{m}{2}-1}$ times the area of the unit sphere in $\mathbb{R}^{m-1}$, $|\nabla_{\mathbb{S}^{m-1}} x_1| = (1-(x_1)^2)^{\frac{1}{2}}$ and $1 - |x_1| \leq 1 - \epsilon x_1$ because $0 \leq \epsilon < 1$. We have also used the fact that $r(x) \geq 1$ and $r \geq 1$. For $m = 2$, direct computation shows that (1.5) is still true.

To estimate I, let $\delta = e^{-r(x)}$. For each j,

$$\begin{aligned} &\int_{B_{\eta_j(2\rho_j)}} h_j(\eta,r)d\eta \\ &\qquad = \int_{\eta \in B_{\eta_j}(2\rho_j),\ d_{\Sigma_j}(\eta) \geq \delta} h_j(\eta,r)d\eta \\ &\qquad + \int_{\eta \in B_{\eta_j}(2\rho_j),\ d_{\Sigma_j}(\eta) \leq \delta} h_j(\eta,r)d\eta \\ &\qquad = III + IV. \end{aligned}$$

As in the estimate of II, there is a constant C_8 independent of x such that

$$III \leq C_8 \left(\delta + e^{-r}\right)^{-1} (1-\epsilon)^{-\frac{m-1}{2}}. \tag{1.6}$$

Suppose $\eta(x) \notin B_{\eta_j}(3\rho_j)$. Since $\eta \in B_{\eta_j}(2\rho_j)$, by (ii) we have

$$1- <\eta, \eta(x)> \tanh r(x) \tanh r \geq 1-(1-\delta_0) \geq \delta_0.$$

Hence

$$\begin{aligned} IV &\leq \int_{B_{\eta_j}(2\rho_j)} h_j(\eta, r) d\eta \\ &\leq C_9 \delta_0^{-m+1} \int_{B_{\eta_j}(2\rho_j)} \frac{1}{d_{\Sigma_j}(\eta) + e^{-r}} d\eta \\ &\leq C_{10} \int_{|z|<1} \frac{1}{|z_1| + e^{-r}} dz_1 \cdots dz_{m-1} \\ &\leq C_{11} r \end{aligned} \tag{1.7}$$

for some constants C_9, C_{10}, C_{11} independent of x. Suppose $\eta(x) \in B_{\eta_j}(3\rho_j)$. Using the fact that $<\eta, \eta'>= 1 - \frac{1}{2}||\eta - \eta'||^2$ and (iic), we see that if $m \geq 3$,

(1.8)

$$\begin{aligned} IV &\leq C_{12} \int_{\{|z|<1,\ \left(\sum_{i=1}^k (z_i)^2\right)^{\frac{1}{2}} \leq C_{13}\delta\}} \frac{\left(1-\epsilon+|z-z_0|^2\right)^{-m+1}}{\left(\sum_{i=1}^k (z_i)^2\right)^{\frac{1}{2}} + e^{-r}} dz_1 \cdots dz_{m-1} \\ &\leq C_{14} \int_{\{|z|<1,\ \left(\sum_{i=1}^k (z_i)^2\right)^{\frac{1}{2}} \leq C_{13}\delta\}} \frac{\left(1-\epsilon+|z-z_0|^2\right)^{-m+1}}{|z_1| + e^{-r}} dz_1 \cdots dz_{m-1} \\ &\leq C_{15} \left(\int_{|z'|<3} \left(1-\epsilon+|z'|^2\right)^{-m+1} dz_2 \cdots dz_{m-1} \right) \\ &\qquad \cdot \int_{|z_1| \leq C_{13}\delta} \frac{1}{|z_1| + e^{-r}} dz_1 \\ &\leq C_{16} \left(\int_0^3 (\sqrt{1-\epsilon} + r)^{(-2m+2)} r^{m-3} dr \right) \log\left(\frac{\delta + e^{-r}}{e^{-r}} \right) \\ &\leq C_{16} \log\left(\frac{\delta + e^{-r}}{e^{-r}} \right) \left(\int_0^3 (\sqrt{1-\epsilon} + r)^{-m-1} dr \right) \\ &= C_{17} \log\left(1 + e^{r-r(x)}\right) (1-\epsilon)^{-\frac{m}{2}}, \end{aligned}$$

where $z_0 = G_j(\eta(x))$, $z' = (z_2, z_3, \dots, z_{m-1})$ and C_{12}–C_{17} are constants independent of x. Here we have used the fact that $|z_0| \le 2$. If $m = 2$, then

$$IV \le C_{18}(1-\epsilon)^{-1} \int_{|z_1|<C_{13}\delta} \frac{1}{|z_1| + e^{-r}} dz_1,$$

and we see that the final result of (1.8) is still true.

By the definition of ϵ and the fact that $\tanh t \le 1$ for all t, we have

$$1 - \epsilon = 1 - \tanh r(x) \tanh r \ge 1 - \tanh r(x) \ge e^{-2r(x)}. \tag{1.9}$$

Similarly,

$$1 - \epsilon \ge e^{-2r}. \tag{1.10}$$

From (1.2)–(1.8) it follows that

$$\begin{aligned} &\int_{r(y)\ge 1} e^{-(m-1)r(x,y)} \frac{e^{-r(y)}}{d_\Sigma(\eta(y)) + e^{-r(y)}} dy \\ &\quad \le C_{19} e^{-(m-1)r(x)} \left\{ \int_1^\infty e^{-r}(\delta_0 + e^{-r})^{-1}(1-\epsilon)^{-\frac{m-1}{2}} dr \right. \\ &\qquad + \int_1^\infty e^{-r}(\delta + e^{-r})^{-1}(1-\epsilon)^{-\frac{m-1}{2}} dr \\ &\qquad \left. + \int_1^\infty e^{-r} r\, dr + \int_1^\infty e^{-r} \log(1 + e^{r - r(x)})(1-\epsilon)^{-\frac{m}{2}} dr \right\}. \end{aligned} \tag{1.11}$$

By (1.9) we get

$$\begin{aligned} e^{-(m-1)r(x)} &\int_1^\infty e^{-r}(\delta_0 + e^{-r})^{-1}(1-\epsilon)^{-\frac{m-1}{2}} dr \\ &\le C_{20} e^{-(m-1)r(x) + (m-1)r(x)} \int_1^\infty e^{-r} dr \\ &\le C_{21}. \end{aligned} \tag{1.12}$$

If $r \ge r(x)$, then we use (1.9) and $\delta + e^{-r} \ge \delta = e^{-r(x)}$, and if $r \le r(x)$, then we use (1.10) and $\delta + e^{-r} \geq e^{-r}$ to conclude that

$$
\begin{aligned}
& e^{-(m-1)r(x)} \int_1^\infty e^{-r}(\delta + e^{-r})^{-1}(1-\epsilon)^{-\frac{m-1}{2}} dr \\
& \quad \leq e^{-(m-1)r(x)} \Big(\int_1^{r(x)} e^{-r+r+(m-1)r} dr \\
& \qquad + \int_{r(x)}^\infty e^{-r+r(x)+(m-1)r(x)} dr \Big) \\
& \quad \leq C_{22}.
\end{aligned} \tag{1.13}
$$

Using (1.9) for $r \geq r(x)$ and (1.10) for $r \leq r(x)$ respectively, we have

$$
\begin{aligned}
& e^{-(m-1)r(x)} \int_1^\infty e^{-r} \log(1 + e^{r-r(x)})(1-\epsilon)^{-\frac{m}{2}} dr \\
& \quad \leq e^{-(m-1)r(x)} \Big(\int_1^{r(x)} e^{-r+mr} dr \\
& \qquad + \int_{r(x)}^\infty e^{-r+mr(x)} \left(r - r(x) + 1\right) dr \Big) \\
& \quad \leq C_{23},
\end{aligned} \tag{1.14}
$$

where we have used the fact that $\log(1+e^{r-r(x)}) \leq \log 2$ if $r \leq r(x)$, and $\log(1+e^{r-r(x)}) \leq \log 2 + r - r(x)$ if $r \geq r(x)$. Combining (1.11)–(1.14) yields that there is a constant C_{24} independent of x such that

$$
\int_{r(y)\geq 1} e^{-(m-1)r(x,y)} \frac{e^{-r(y)}}{d_\Sigma(\eta(y)) + e^{-r(y)}} dy \leq C_{24}.
$$

Putting this back to (1.1), we have

$$
\int_{\mathbb{H}^m} G(x,y) f(y) dy \leq C_2 \tag{1.15}
$$

for some constant C_2 for all x with $r(x) \geq 1$. If $r(x) \leq 1$, then $\tanh r(x) \leq 1 - \delta_1$ for some $\delta_1 > 0$. As in the estimate in (1.7), it is not hard to show that (1.15) is still true with a possibly larger constant. q.e.d.

Lemma 1.2. *Under the same assumptions as in Lemma 1.1, suppose $d_\Sigma(\eta(x)) \geq \delta^* > 0$. Then for $r(x)$ large enough we have*

$$
\int_{\mathbb{H}^m} G(x,y) f(y) dy \leq C(\delta^*) r(x) e^{-r(x)},
$$

where $C(\delta^*)$ depends only on C_1, Σ, m and δ^*.

Proof. Note that for all y with $r(x,y) \le 1$, we have $d_\Sigma(\eta(y)) \ge \frac{1}{2}\delta^*$, provided $r(x)$ is large enough. Use the notation as in the proof of Lemma 1.1. Then (1.1) becomes

$$\begin{aligned}(1.16)\quad &\int_{\mathbb{H}^m} G(x,y)f(y)dy \\ &\le C_{25}\left(e^{-r(x)} + \int_{r(y)\ge 1} e^{-(m-1)r(x,y)}\frac{e^{-r(y)}}{d_\Sigma(\eta(y)) + e^{-r(y)}}dy\right).\end{aligned}$$

Here and below C_i will denote constants depending only on C_1, Σ, m and δ^*. Using the methods of obtaining the estimates (1.5) and (1.7), we get

$$\begin{aligned}&\int_{r(y)\ge 1} e^{-(m-1)r(x,y)}\frac{e^{-r(y)}}{d_\Sigma(\eta(y)) + e^{-r(y)}}dy \\ &\quad \le C_{26}e^{-(m-1)r(x)}\left\{\int_1^\infty e^{-r}(\delta_0 + e^{-r})^{-1}(1-\epsilon)^{-\frac{m-1}{2}}dr \right.\\ &\qquad\qquad \left. + \int_1^\infty e^{-r} r dr\right\} \\ &\quad \le C_{27}e^{-(m-1)r(x)}\left\{\left(\int_1^{r(x)} + \int_{r(x)}^\infty\right)\left[e^{-r}(1-\epsilon)^{-\frac{m-1}{2}}\right]dr + 1\right\}.\end{aligned}$$

By (1.9) and (1.10) as before, the lemma follows. q.e.d.

Remark 1.1. It is easy to see that in Lemmas 1.1 and 1.2, $\tilde{\Sigma}$ can be replaced by a finite family of compact manifolds.

By an observation of Bando [2], we have the following:

Theorem 1.3. *Let Σ be a finite union of compact submanifolds of codimension at least 1 without boundary in $\mathbb{S}^{m-1}$. Using the Poincaré unit ball model of $\mathbb{H}^m$, let o be the center of $\mathbb{B}^m(1)$. Let v be a map from $\mathbb{H}^m$ to $\mathbb{H}^n$ such that*

$$||\tau(v)(x)|| \le \frac{Ce^{-r(x)}}{d_\Sigma(\eta(x)) + e^{-r(x)}},$$

where $(\eta(x), r(x))$ is the geodesic polar coordinates at o of x. Then there is a harmonic map $F: \mathbb{H}^m \to \mathbb{H}^n$ such that $d_{\mathbb{H}^n}(F(x), v(x))$ is uniformly *bounded on $\mathbb{H}^m$. Moreover, if v has bounded energy density, then F also has bounded energy density.*

Proof. We will sketch the proof for the sake of completeness. Let $B(R)$ be the geodesic ball of radius R with center at o. Let F_R be the harmonic map from $B(R)$ to $\mathbb{H}^n$ such that $F_R = v$ on $\partial B(R)$. By [17], the assumption on v and Lemma 1.1, we can find a bounded positive function ϕ on $\mathbb{H}^m$ such that

$$\Delta\phi \leq -||\tau(v)|| \leq \Delta d_{\mathbb{H}^m}(F_R, v).$$

Hence $d_{\mathbb{H}^m}(F_R, v) \leq \phi$. Together with the energy density estimate in [4], the results follow. q.e.d.

2. Harmonic maps with singular boundary value

In [10]–[12], it was proved that if $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is a C^1 map with nowhere vanishing energy density, then one can extend f to a proper harmonic map from $\mathbb{H}^m$ into $\mathbb{H}^n$. In this section, we will construct harmonic maps extending a class of boundary maps which may not satisfy the above conditions. Namely, f may not be smooth, or f may have zero energy density somewhere; that is, f is singular. We will show that in some cases one can solve the boundary value problem for harmonic maps provided the singular set of f is small. Let $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be a map. In this section, we always assume that f satisfies the following conditions:

(a) f is C^3 with nowhere vanishing energy density except possibly at $\bigcup_{j=1}^{\ell} \Sigma_j$, where $\{\Sigma_j\}_{j=1}^{\ell}$ is a finite disjoint family of embedded submanifolds without boundary of $\mathbb{S}^{m-1}$ and $\dim(\Sigma_j) < m-1$ for all j;

(b) if we use the upper half space model for $\mathbb{H}^m$, consider Σ_j to be an embedded submanifold in $\mathbb{R}^{m-1}$ and f a map from $\mathbb{R}^{m-1}$ into $\mathbb{R}^{n-1}$. Then near Σ_j, $f = (f^1, \dots, f^{n-1})$ can be expressed as $f^\alpha(x') = d_{\Sigma_j}^{\beta_{\alpha,j}}(x')h_j^\alpha(x')$, with $\beta_{\alpha,j} > -1$, such that:

(i) each h_j^α is bounded and Lipschitz continuous;

(ii) each h_j^α is smooth with uniformly bounded derivatives up to order 3 on $U_j \setminus \Sigma_j$, where U_j is a neighborhood of Σ_j;

(iii) if $\beta_{\alpha,j} \neq 0$, then $h_j^\alpha = 0$ on Σ and $|\nabla_0 h_j^\alpha|^2 \geq C > 0$ in $U_j \setminus \Sigma_j$ for some constant C ;

(iv) if there is $\beta_{\alpha,j}=0$, then $\sum_{\alpha,\beta_{\alpha,j}=0}|\nabla_0 h_j^\alpha|^2 \ge C > 0$ in $U_j \setminus \Sigma_j$ for some constant C .

Here and below, $d_{\Sigma_j}(x')$ is the Euclidean distance from x' to Σ_j in $\mathbb{R}^{m-1}$, ∇_0 is the Euclidean gradient and Δ_0 is the Euclidean Laplacian either in $\mathbb{R}^m$ or $\mathbb{R}^{m-1}$. Let us consider some examples of maps satisfying these conditions.

Example 1. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(t)=|t|^{\alpha-1}t$ for some $\alpha>0$. Then f satisfies the above assumptions. In this case, the singular set consists of the point 0 and the point at infinity. The boundary value problem for f has been solved in [5], [22]. One may relax the condition on f. For example, one may assume that near 0, $f(t)=|t|^{\alpha-1}h(t)$ where $h(t)$ is smooth with nowhere vanishing derivative. The case that h is C^1 with nowhere vanishing derivative has been solved in [13].

Example 2. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(t)=|t|^{\alpha-1}t$ for $t\le 0$ and $f(t)=C|t|^{\alpha-1}t$ for $t\ge 0$. Here $C>0$ and $\alpha>0$ are constants. We will discuss this kind of boundary maps in more details in §4. Note that if we write $f(t)$ as $|t|^{\alpha-1}h(t)$, then $h(t)$ is only Lipschitz at 0.

Example 3. Let $f:\mathbb{R}^{m-1}\to\mathbb{R}^{m-1}$ be the function defined by $f(x)=|x|^{\alpha-1}x$. Then f satisfies the above conditions with singular set consisting of the origin and the point at infinity. In this case, the singular set is of dimension 0. The boundary value problem for maps with similar behavior at finitely many isolated singular points has been solved in [22]

Example 4. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $f(\rho,\theta)=(r,\phi)$ in polar coordinates, where

$$r=\begin{cases}(1-\rho^2)^\alpha\rho^2 & \text{if } \rho\le 1,\\ (1-\rho^{-2})^\alpha\rho^{-2} & \text{if } \rho\ge 1.\end{cases}$$

In this example, the singular set consists of the unit circle, the origin and the point at infinity.

In this section we will prove the following:

Theorem 2.1. *Let $f:\mathbb{S}^{m-1}\to\mathbb{S}^{n-1}$ be a map satisfying conditions (a) and (b). Then there exists a proper harmonic map F from $\mathbb{H}^m$ to*

$\mathbb{H}^n$ with bounded energy density such that $F = f$ at $\mathbb{S}^{m-1}$. Moreover, F is C^1 up to the boundary except possibly at $\bigcup_{j=1}^{\ell} \Sigma_j$.

We begin our proof of the theorem by the following well-known facts; see [9] and [12] for example:

Lemma 2.2. *Let f be a bounded function on $\mathbb{R}^{m-1}$ with compact support and let u be its harmonic extension in the upper half space in $\mathbb{R}^m$. Let Σ be a closed and bounded set in $\mathbb{R}^{m-1}$. Suppose f is C^3 on $\mathbb{R}^{m-1} \setminus \Sigma$ with uniform bounded derivatives up to order 3. Then the following hold.*

(i) There is a constant $C > 0$ such that

$$|\nabla_0 u|(x', x^m) \leq C \cdot \frac{1}{d_\Sigma(x') + x^m}.$$

(ii)

$$\lim_{x^m \to 0} x^m u_i = 0$$

for $1 \leq i \leq m$.

(iii) If, in addition, f is Lipschitz continuous, then there is a constant C depending only on f such that

$$|u_i| \leq C$$

for $1 \leq i \leq m-1$, and that

$$|u_{ij}|(x', x^m) \leq \frac{C}{d_\Sigma(x') + x^m}$$

for $1 \leq i, j \leq m$, where $x' = (x^1, \ldots, x^{m-1})$.

(iv) If f_i is continuous at x', then u_i is also continuous at $(x', 0)$ for $1 \leq i \leq m-1$.

To prove the theorem, we only consider the case that $\ell = 1$, so that the boundary map f is smooth except at an embedded submanifold Σ. The general case can be proved similarly. Identify $\mathbb{H}^m$ ($\mathbb{H}^n$ respectively) with the upper half space in $\mathbb{R}^m$ ($\mathbb{R}^n$ respectively) with the Poincaré metrics. By choosing a point not in Σ as the point at infinity of $\mathbb{R}^{m-1}$, we may assume that Σ is an embedded submanifold of $\mathbb{R}^{m-1}$ and $f = (f^1, \ldots, f^{n-1})$ satisfies conditions in (b). Then we have the following:

Lemma 2.3. *There exist a neighborhood U of Σ in $\mathbb{R}^{m-1}$ and $a > 0$ such that the map f can be extended from U to a map v from $\{(x', x^m)|\ x' \in U,\ 0 < x^m < a\}$ into $\mathbb{H}^n$ with the properties that v has bounded energy density and the norm of its tension field satisfies*

$$||\tau(v)||(x', x^m) \leq C \cdot \frac{x^m}{d_\Sigma(x') + x^m}$$

for some constant C, where v is considered as a map from an open set in $\mathbb{H}^m$ into $\mathbb{H}^n$. Moreover, v is C^1 up to the boundary except at Σ.

Proof. By choosing a tubular neighborhood of Σ, we can find neighborhoods U and V of Σ, with $U \subset\subset V$, such that d_Σ^2 is smooth in V. Using cutoff functions, without changing the value of f in U we may assume that f^α is zero outside V. By the assumptions on f, for $1 \leq \alpha \leq m-1$,

$$f^\alpha(x') = d_\Sigma^{\beta_\alpha}(x') h^\alpha(x')$$

in U with constants $\beta_\alpha > -1$. We may assume that h^α is zero outside V. Moreover, the functions h^α satisfy the following:

(i) each h^α is bounded and Lipschitz continuous;

(ii) each h^α is smooth with uniformly bounded derivatives up to order 3 on $\mathbb{R}^{m-1} \setminus \Sigma$;

(iii) if $\beta_\alpha \neq 0$, then $h^\alpha = 0$ on Σ and $|\nabla_0 h^\alpha|^2 \geq C > 0$ in $U \setminus \Sigma$ for some constant C;

(iv) if there is $\beta_\alpha = 0$, then $\sum_{\alpha,\ \beta_\alpha = 0} |\nabla_0 h^\alpha|^2 \geq C > 0$ in $U \setminus \Sigma$ for some constant C.

Let $\xi(x', x^m) = (d_\Sigma^2(x') + (x^m)^2)^{\frac{1}{2}}$ and let u^α be the harmonic extension of h^α in the upper half space for $\alpha = 1, \ldots, m-1$. Let

$$v^\alpha = \xi^{\beta_\alpha}(u^\alpha - x^m u_m^\alpha).$$

If $\beta_\alpha \neq 0$, then $h^\alpha = 0$ on Σ. Hence near Σ,

$$h^\alpha(x') = < \nabla_0 h^\alpha(x'), \nabla_0 d_\Sigma(x') > d_\Sigma(x') + O(d_\Sigma^2),$$

and there is a constant $C_1 > 0$ such that if we choose U small enough, then in U,

$$\begin{aligned}
C_1 &> |\nabla_0 h^\alpha|^2 + \beta_\alpha^2 d_\Sigma^{-2}(h^\alpha)^2 + 2\beta_\alpha d_\Sigma^{-1} h^\alpha < \nabla_0 h^\alpha, \nabla_0 d_\Sigma > \\
&= |\nabla_0 h^\alpha|^2 + (\beta_\alpha^2 + 2\beta_\alpha)(< \nabla_0 h^\alpha, \nabla_0 d_\Sigma >)^2 + O(d_\Sigma) \\
&= |\nabla_0 h^\alpha|^2 - (< \nabla_0 h^\alpha, \nabla_0 d_\Sigma >)^2 \\
&\quad + (\beta_\alpha + 1)^2(< \nabla_0 h^\alpha, \nabla_0 d_\Sigma >)^2 + O(d_\Sigma) \\
&> C_1^{-1},
\end{aligned}$$

where we have used (iii), and the fact that $|\nabla_0 d_\Sigma| = 1$ and $\beta_\alpha > -1$. Therefore, after choosing a smaller U, we can find a bounded nonnegative function with support in V which is equal to

$$\left(|\nabla_0 h^\alpha|^2 + \beta_\alpha^2 d_\Sigma^{-2}(h^\alpha)^2 + 2\beta_\alpha d_\Sigma^{-1} h^\alpha < \nabla_0 h^\alpha, \nabla_0 d_\Sigma >\right)^{\frac{1}{2}}$$

in U. Let g^α be the harmonic extension of this function in the upper half space. Then, by choosing an even smaller U if necessary, we can find $a > 0$ such that

$$C_1^{-1} \le (g^\alpha)^2(x', x^m) \le C_1 \tag{2.1}$$

for all (x', x^m) such that $x' \in U$ and $0 < x^m < a$. If $\beta_\alpha = 0$, let g^α be the harmonic extension of $|\nabla_0 h^\alpha|$. By (iv), we see that by choosing a smaller a and a larger C_1, we have

$$C_1^{-1} \le \sum_{\alpha, \beta_\alpha \neq 0} (g^\alpha)^2(x', x^m) \le C_1 \tag{2.2}$$

for all (x', x^m) such that $x' \in U$ and $0 < x^m < a$. g^α is well-defined. Let u^n be such that

$$(m-1)(u^n)^2 = \sum_{\alpha=1}^{n-1} \xi^{2\beta_\alpha}(g^\alpha)^2.$$

Define

$$v^n = x^m u^n.$$

In the following computations, we always assume that $x' \in U$ and $0 < a < x^m$. By Lemma 2.2(iii), there is a constant $C_2 > 0$ such that

$$|u_i^\alpha|(x', x^m) \le C_2 \tag{2.3}$$

for $1 \le i \le m-1$ and $1 \le \alpha \le n-1$,

$$|\nabla_0 u_i^\alpha|(x', x^m) \le \frac{C_2}{\xi(x', x^m)} \tag{2.4}$$

for $1 \le i \le m$, $1 \le \alpha \le n-1$ and (by Lemma 2.2(i))

$$|\nabla_0 g^\alpha|(x', x^m) \le \frac{C_2}{\xi(x', x^m)} \tag{2.5}$$

for $1 \le \alpha \le n-1$. By (2.1) and (2.2),

$$C_2 \sum_{\alpha=1}^{n-1} \xi^{2\beta_\alpha} \ge \big(u^n(x', x^m)\big)^2 \ge C_2^{-1} \sum_{\alpha=1}^{n-1} \xi^{2\beta_\alpha}. \tag{2.6}$$

In particular, $u^n > 0$. Note that if there is $\beta_\alpha = 0$, then

$$(u^n)^2 \ge C_2^{-1} > 0.$$

For $1 \le \alpha \le n-1$, by (2.4),

$$|(u^\alpha - x^m u_m^\alpha)_m| = 2x^m |u_{mm}^\alpha| \le \frac{2C_2 x^m}{\xi}, \tag{2.7}$$

and by Lemma 2.2(ii),

$$\lim_{(x', x^m) \to (x_0', 0)} (u^\alpha - x^m u_m^\alpha)(x', x^m) = h^\alpha(x_0'). \tag{2.8}$$

Hence for $\beta_\alpha \ne 0$, by (2.7),

$$|u^\alpha(x', x^m) - x^m u_m^\alpha(x', x^m) - h^\alpha(x')| \le 2C_2 x^m.$$

So, by assumptions (ii) and (iii),

$$|u^\alpha - x^m u_m^\alpha|(x', x^m) \le C_3 \xi(x', x^m). \tag{2.9}$$

Since $|\nabla_0 \xi| = 1$, by (2.1), (2.5) and (2.6),

$$\begin{aligned} u^n |\nabla_0 u^n| &\le C_4 \sum_{\alpha=1}^{n-1} \xi^{2\beta_\alpha - 1} \\ &\le C_5 \frac{(u^n)^2}{\xi} \end{aligned}$$

for some constants C_4 and C_5. Thus

$$|\nabla_0 u^n| \leq C_5 \frac{u^n}{\xi}. \tag{2.10}$$

Similarly, by choosing a larger C_5, we can prove that

$$|\Delta_0 (u^n)^2| \leq C_5 (\frac{u^n}{\xi})^2.$$

Then using (2.10), we conclude that

$$|\Delta_0 u^n| \leq C_6 \frac{u^n}{\xi^2} \tag{2.11}$$

for some constant C_6. Since $\beta_\alpha > -1$, (2.6) implies

$$\lim_{x^m \to 0} v^n = 0.$$

Combining this with (2.8) yields

$$v(x', x^m) = (v^1, \ldots, v^n)(x', x^m) \to (f(x'_0), 0)$$

when $(x', x^m) \to (x'_0, 0)$.

Now we are ready to estimate the norm of the tension field of v. The components of the tension field of v is given by

$$\tau^\alpha(v) = (x^m)^2 \left(\Delta_0 v^\alpha - \frac{m-2}{x^m} v^\alpha_m - \frac{2}{v^n} < \nabla_0 v^\alpha, \nabla_0 v^n > \right)$$

for $1 \leq \alpha \leq n-1$,

$$\tau^n(v) = (x^m)^2 \left(\Delta_0 v^n - \frac{m-2}{x^m} v^n_m + \frac{1}{v^n} \left(\sum_{\alpha=1}^{n-1} |\nabla_0 v^\alpha|^2 - |\nabla_0 v^n|^2 \right) \right)$$

and

$$||\tau(v)||^2 = \frac{1}{(v^n)^2} \sum_{\alpha=1}^{n} |\tau^n(v)|^2.$$

To estimate $\frac{(x^m)^2}{v^n} \Delta_0 v^\alpha$ for $1 \leq \alpha \leq n-1$, we have

$$\begin{aligned}
\Delta_0 v^\alpha = & -2\xi^{\beta_\alpha} u^\alpha_{mm} + \beta_\alpha \xi^{\beta_\alpha - 2} \left(\xi \Delta_0 \xi + (\beta_\alpha - 1) \right) (u^\alpha - x^m u^\alpha_m) \\
& - 2\beta_\alpha \xi^{\beta_\alpha - 1} \left(\sum_{i=1}^{m-1} \xi_i (u^\alpha_i - x^m u^\alpha_{im}) - \xi_m x^m u^\alpha_{mm} \right),
\end{aligned}$$

where we have used the fact that $|\nabla_0 \xi| = 1$. Suppose $\beta_\alpha = 0$. Then by (2.4) and (2.6), we have

$$\left|\frac{(x^m)^2}{v^n}\Delta_0 v^\alpha\right| = \left|\frac{x^m}{u^n}\Delta_0 v^\alpha\right| \le 2C_2\frac{x^m}{\xi}.$$

If $\beta_\alpha \neq 0$, by (2.3), (2.4), (2.6), (2.9) and the fact that $\xi\Delta_0\xi$ is bounded, we conclude that in any case:

$$\left|\frac{(x^m)^2}{v^n}\Delta_0 v^\alpha\right| \le C_7\frac{x^m}{\xi} \tag{2.12}$$

for some constant C_7. To estimate

$$\frac{(x^m)^2}{v^n}\frac{m-2}{x^m}v^\alpha_m = (m-2)\frac{1}{u^n}v^\alpha_m,$$

note that, since

$$\nabla_0 v^\alpha = \xi^{\beta_\alpha}\left(\nabla_0 u^\alpha - x^m\nabla_0 u^\alpha_m - u^\alpha_m\nabla_0 x^m\right) + \beta_\alpha\xi^{\beta_\alpha-1}\nabla_0\xi\left(u^\alpha - x^m u^\alpha_m\right),$$

we have

$$v^\alpha_m = -\xi^{\beta_\alpha}x^m u^\alpha_{mm} + \beta_\alpha\xi^{\beta_\alpha-1}\frac{x^m}{\xi}\left(u^\alpha - x^m u^\alpha_m\right).$$

If $\beta_\alpha = 0$, then from (2.6) and (2.4), it follows that

$$\left|\frac{(x^m)^2}{v^n}\frac{m-2}{x^m}v^\alpha_m\right| \le C_8\frac{x^m}{\xi} \tag{2.13}$$

for some constant C_8. If $\beta_\alpha \neq 0$, then (2.4), (2.6) and (2.9) yield that (2.13) is still true. To estimate

$$\frac{(x^m)^2}{v^n}\frac{1}{v^n} < \nabla_0 v^\alpha, \nabla_0 v^n >= \frac{1}{(u^n)^2} < \nabla_0 v^\alpha, \nabla_0 v^n >,$$

note that

$$\nabla_0 v^n = x^m\nabla_0 u^n + u^n\nabla_0 x^m,$$

and that

$$
\begin{aligned}
&\frac{1}{v^n} < \nabla_0 v^\alpha, \nabla_0 v^n > \\
&= \frac{\xi^{\beta_\alpha - 1}}{v^n} \Big[\xi \Big(x^m < \nabla_0 u^\alpha, \nabla_0 u^n > -(x^m)^2 < \nabla_0 u^\alpha_m, \nabla_0 u^n > \\
&\qquad\qquad - x^m u^\alpha_m u^n_m - x^m u^n u^\alpha_{mm} \Big) \\
&\qquad\quad + \beta_\alpha \left(x^m < \nabla_0 \xi, \nabla_0 u^n > + u^n \cdot \frac{x^m}{\xi} \right) (u^\alpha - x^m u^\alpha_m) \Big] \\
&= \frac{\xi^{\beta_\alpha - 1}}{v^n} \Big[\xi \Big(x^m \sum_{i=1}^{m-1} u^\alpha_i u^n_i - (x^m)^2 < \nabla_0 u^\alpha_m, \nabla_0 u^n > - x^m u^n u^\alpha_{mm} \Big) \\
&\qquad\quad + \beta_\alpha \left(x^m < \nabla_0 \xi, \nabla_0 u^n > + u^n \cdot \frac{x^m}{\xi} \right) (u^\alpha - x^m u^\alpha_m) \Big] \\
&= \frac{\xi^{\beta_\alpha - 1}}{u^n} \Big[\xi \Big(\sum_{i=1}^{m-1} u^\alpha_i u^n_i - x^m < \nabla_0 u^\alpha_m, \nabla_0 u^n > - u^n u^\alpha_{mm} \Big) \\
&\qquad\quad + \beta_\alpha \left(< \nabla_0 \xi, \nabla_0 u^n > + u^n \cdot \frac{1}{\xi} \right) (u^\alpha - x^m u^\alpha_m) \Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{(x^m)^2}{v^n} \frac{1}{v^n} < \nabla_0 v^\alpha, \nabla_0 v^n > \\
&\qquad = \frac{x^m \xi^{\beta_\alpha - 1}}{(u^n)^2} \Big[\xi \Big(\sum_{i=1}^{m-1} u^\alpha_i u^n_i - x^m < \nabla_0 u^\alpha_m, \nabla_0 u^n > - u^n u^\alpha_{mm} \Big) \\
&\qquad\qquad + \beta_\alpha \left(< \nabla_0 \xi, \nabla_0 u^n > + u^n \cdot \frac{1}{\xi} \right) (u^\alpha - x^m u^\alpha_m) \Big].
\end{aligned}
$$

By (2.3), (2.4), (2.6), (2.9) and (2.10) we have

$$
\left| \frac{(x^m)^2}{v^n} \frac{1}{v^n} < \nabla_0 v^\alpha, \nabla_0 v^n > \right| \le C_9 \frac{x^m}{\xi}.
$$

From (2.12)–(2.14) it follows that

$$
|\frac{1}{v^n} \tau^\alpha(v)| \le C_{10} \frac{x^m}{\xi}. \tag{2.15}
$$

To estimate
$$\frac{(x^m)^2}{v^n}\Delta_0 v^n = \frac{x^m}{u^n}\left(2u_m^n + x^m\Delta_0 u^n\right),$$
we use (2.10) and (2.11) to get
$$\left|\frac{(x^m)^2}{v^n}\Delta_0 v^n\right| \le C_{11}\frac{x^m}{\xi}. \tag{2.16}$$
Finally, to estimate
$$\frac{(x^m)^2}{v^n}\left[-\frac{m-2}{x^m}v_m^n + \frac{1}{v^n}\left(\sum_{\alpha=1}^{n-1}|\nabla_0 v^\alpha|^2 - |\nabla_0 v^n|^2\right)\right],$$
we note by (2.10) that
$$\begin{aligned}\frac{(x^m)^2}{v^n}\left(-\frac{m-2}{x^m}v_m^n\right) &= \frac{x^m}{u^n}\left(-\frac{m-2}{x^m}u^n - (m-2)u_m^n\right)\\ &= -(m-2) + O(\frac{x^m}{\xi}).\end{aligned} \tag{2.17}$$

Using the fact that $\nabla_0\xi = \frac{1}{\xi}(d_\Sigma\nabla_0 d_\Sigma + x^m\nabla_0 x^m)$, we have
$$\begin{aligned}|\nabla_0 v^\alpha|^2 =&\xi^{2\beta_\alpha}\left|\nabla_0 u^\alpha - x^m\nabla_0 u_m^\alpha - u_m^\alpha\nabla_0 x^m\right|^2\\ &+ \beta_\alpha^2\xi^{2\beta_\alpha-2}(u^\alpha - x^m u_m^\alpha)^2\\ &+ 2\beta_\alpha\xi^{2\beta_\alpha-1}(u^\alpha - x^m u_m^\alpha)\\ &\qquad\cdot < \nabla_0 u^\alpha - x^m\nabla_0 u_m^\alpha - u_m^\alpha\nabla_0 x^m, \nabla_0\xi >\\ =&\xi^{2\beta_\alpha}\Bigg(|\nabla_0 u^\alpha|^2 + (x^m)^2|\nabla_0 u_m^\alpha|^2 + (u_m^\alpha)^2\\ &\qquad - 2x^m < \nabla_0 u^\alpha, \nabla_0 u_m^\alpha >\\ &\qquad - 2(u_m^\alpha)^2 + 2x^m u_m^\alpha u_{mm}^\alpha\\ &\qquad + \beta_\alpha^2\xi^{-2}(u^\alpha - x^m u_m^\alpha)^2\\ &\qquad + 2\beta_\alpha\xi^{-2}(u^\alpha - x^m u_m^\alpha)d_\Sigma < \nabla_0 u^\alpha, \nabla_0 d_\Sigma >\\ &\qquad + O(\frac{x^m}{\xi})\Bigg)\\ =&\xi^{2\beta_\alpha}\left(A^\alpha(x', x^m) + O(\frac{x^m}{\xi})\right),\end{aligned}$$
where we have used (2.3), (2.4), (2.9) and the fact that
$$< \nabla_0 d_\Sigma, \nabla_0 x^m >= 0.$$

Here

$$
\begin{aligned}
A^\alpha = & \sum_{i=1}^{m-1} (u_i^\alpha)^2 + \beta_\alpha^2 \xi^{-2} (u^\alpha - x^m u_m^\alpha)^2 \\
& + 2\beta_\alpha \xi^{-2} (u^\alpha - x^m u_m^\alpha) d_\Sigma < \nabla_0 u^\alpha, \nabla_0 d_\Sigma > .
\end{aligned}
$$

Also, by (2.10)

$$
\begin{aligned}
|\nabla_0 v^n|^2 &= (u^n)^2 + (x^m)^2 |\nabla_0 u^n|^2 + 2x^m u^n u_m^n \\
&= (u^n)^2 \left(1 + O(\frac{x^m}{\xi})\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \frac{(x^m)^2}{v^n} \cdot \frac{1}{v^n} \left(\sum_{\alpha=1}^{n-1} |\nabla_0 v^\alpha|^2 - |\nabla_0 v^n|^2 \right) \\
& \qquad = (m-2) + \frac{1}{(u^n)^2} \left(\sum_{\alpha=1}^{n-1} \xi^{2\beta_\alpha} A^\alpha - (m-1)(u^n)^2 \right) + O(\frac{x^m}{\xi}).
\end{aligned}
$$

Combining this with (2.17) yields

$$
\begin{aligned}
& \frac{(x^m)^2}{v^n} \left[-\frac{m-2}{x^m} v_m^n + \frac{1}{v^n} \left(\sum_{\alpha=1}^{n-1} |\nabla_0 v^\alpha|^2 - |\nabla_0 v^n|^2 \right) \right] \\
& \qquad = \frac{1}{(u^n)^2} \left(\sum_{\alpha=1}^{n-1} \xi^{2\beta_\alpha} (A^\alpha - (g^\alpha)^2) \right) + O(\frac{x^m}{\xi}).
\end{aligned} \tag{2.18}
$$

It remains to estimate $A^\alpha - (g^\alpha)^2$. First from (2.1), (2.2), (2.3), (2.9), it follows that A^α and g^α are bounded. At a point (x', x^m) with $x^m \geq \frac{1}{4} d_\Sigma(x')$, we have

$$
|A^\alpha - (g^\alpha)^2| \leq C_{12} \frac{x^m}{\xi}
$$

for some constant C_{12}. On the other hand, if $x^m \leq \frac{1}{4} d_\Sigma(x')$, then $d_\Sigma(x') > 0$ and x' is a point of continuity of $\nabla_0 h^\alpha$. By the definition of g^α and Lemma 2.2,

$$
A^\alpha(x', 0) = (g^\alpha)^2 (x', 0).
$$

From (2.3), (2.4), (2.5), (2.7) and (2.9) it follows that

$$|\frac{\partial A^\alpha}{\partial x^m} - \frac{\partial (g^\alpha)^2}{\partial x^m}| \le C_{13}\frac{1}{\xi}$$

for some constant C_{13}. Thus we also have

$$|A^\alpha - (g^\alpha)^2| \le C_{12}\frac{x^m}{\xi}$$

by choosing a larger C_{12}. Combining this with (2.16) and (2.18), we can find a constant C_{14} such that

$$|\frac{1}{v^n}\tau^n(v)| \le C_{14}\frac{x^m}{\xi},$$

which together with (2.15) implies that

$$||\tau(v)||(x', x^m) \le C_{15}\frac{x^m}{\xi(x', x^m)}$$

for some constant C_{15}. To estimate the energy density of the map v, by (2.3), (2.4), (2.6) and (2.9), we have for $1 \le \alpha \le n-1$ and $1 \le i \le m-1$,

$$\begin{aligned} |v_i^\alpha| &= \left|\beta_\alpha \xi^{\beta_\alpha - 1}\xi_i(u^\alpha - x^m u_m^\alpha) + \xi^{\beta_\alpha}(u_i^\alpha - x^m u_{mi}^\alpha)\right| \\ &\le C_{16}\xi^{\beta_\alpha - 1}\left(\xi + \xi\left(1 + \frac{x^m}{\xi}\right)\right) \\ &\le 3C_{16}\xi^{\beta_\alpha} \\ &\le C_{17}u^n \end{aligned} \tag{2.19}$$

for some constants C_{16}, C_{17}. For $1 \le \alpha \le n-1$, from (2.6), (2.7) and (2.9) it follows that

$$\begin{aligned} |v_m^\alpha| &= \left|\beta_\alpha \xi^{\beta_\alpha - 1}\xi_m(u^\alpha - x^m u_m^\alpha) - \xi^{\beta_\alpha}x^m u_{mm}^\alpha)\right| \\ &\le C_{17}\xi^{\beta_\alpha} \\ &\le C_{18}u^n \end{aligned} \tag{2.20}$$

for some constants C_{18}, C_{19}. On the other hand, by (2.10) we have

$$\begin{aligned} |\nabla_0 v^n| &= |x^m\nabla_0 u^n + u^n\nabla_0 x^m| \\ &\le x^m|\nabla_0 u^n| + u^n \\ &\le C_{20}u^n. \end{aligned} \tag{2.21}$$

The energy density of the map constructed above is given by

$$\frac{(x^m)^2}{(v^n)^2}\sum_{\alpha=1}^{n}|\nabla_0 v^\alpha|^2 = \frac{1}{(u^n)^2}\sum_{\alpha=1}^{n}|\nabla_0 v^\alpha|^2.$$

Hence by (2.19)-(2.21), the map v has bounded energy density. It is easy to see that v is C^1 up to the boundary except at Σ. q.e.d.

Proof of Theorem 2.1. Near $x^m = 0$ and away from Σ, the map v constructed in the previous lemma has the following properties. v is smooth, $v_i^\alpha \to f_i^\alpha$ as $x^m \to 0$, for $1 \le i \le m-1$ and $1 \le \alpha \le n-1$, $v_n^\alpha \to 0$ as $x^m \to 0$, and $u^n \to \sqrt{e(f)/(m-1)}$ as $x^m \to 0$. By the method as in the proof of Theorem 4.1 in [12], we can extend the map $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ to a map $\tilde{v}$ from $\mathbb{H}^m$ into $\mathbb{H}^n$ such that if we use the geodesic polar coordinates at the origin, then we obtain

$$||\tau(\tilde{v})||(x) \le C\frac{e^{-r(x)}}{d_\Sigma(\eta(x)) + e^{-r(x)}}$$

for some constant C. Moreover, $\tilde{v}$ has bounded energy density. Here we have used the notation as in Lemma 1.1. By Theorem 1.3, there is a harmonic map F from $\mathbb{H}^m$ into $\mathbb{H}^n$ such that

$$d_{\mathbb{H}^n}(F(x), \tilde{v}(x)) \le C$$

for some constant C for all $x \in \mathbb{H}^m$. Hence $F = v = f$ on $\mathbb{S}^{m-1}$, and F as well as $\tilde{v}$ also has bounded energy density. Using Lemma 1.2 the method in [12], one can show that F is C^1 up to the boundary portion $\partial\mathbb{H}^m \setminus \Sigma$. q.e.d.

By [21] and Theorem 2.1, we have the following:

Corollary 2.4. *Suppose $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a piecewise smooth map with nowhere vanishing energy density. Then there is a proper harmonic map F from $\mathbb{H}^2$ to $\mathbb{H}^2$ with boundary value f and bounded energy density. If, in addition, f is a homeomorphism, then F is quasi-conformal.*

We will also consider another kind of boundary maps. Let $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be a smooth map such that the zero set of the energy density $e(f)$ of f is a disjoint union of finitely many embedded submanifolds Σ_j for $1 \le j \le \ell$. Suppose near each Σ_j, $e(f)(x') = d_{\Sigma_j}^{2\beta_j}(x')g_j(x')$ for some positive integer $\beta_j > 0$ and some smooth function $g_j > 0$. Using similar method we can prove:

Theorem 2.5. *Let $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be a smooth map satisfying the above conditions. Then there exists a proper harmonic map F from $\mathbb{H}^m$ to $\mathbb{H}^n$ with bounded energy density such that $F = f$ at $\mathbb{S}^{m-1}$. Moreover, F is C^1 up to the boundary except possibly at $\bigcup_{j=1}^{\ell} \Sigma_j$.*

Proof. The proof is similar to that of Theorem 2.1 and is more simple. We only sketch the ideas. For simplicity, we assume that $\ell = 1$ and Σ is the zero set of $e(f)$. Moreover, using the upper half space models for $\mathbb{H}^m$ and $\mathbb{H}^n$, we may assume that Σ is an embedded submanifold of $\mathbb{R}^{m-1}$. Near Σ, $e(f)(x') = d_\Sigma^{2\beta}(x')g(x')$ for some positive integer $\beta > 0$ and some smooth function $g > 0$. Write $f = (f^1, \ldots, f^{n-1})$. Let ϕ be the harmonic extension of g to the upper half space. Then $\phi > 0$ near Σ. Let

$$u^\alpha(x', x^m) = f^\alpha(x') \quad \text{for } 1 \leq \alpha \leq n-1,$$

and let

$$u_n(x', x^m) = x^m \xi^\beta(x', x^m)\phi(x', x^m).$$

Here as before, $\xi(x', x^m) = (d_\Sigma^2(x') + (x^m)^2)^{\frac{1}{2}}$. Using the fact that $|\nabla_0 f^\alpha|(x') \leq C d_\Sigma^\beta(x')$ near Σ, we may conclude that $|\nabla_0 f^\alpha|(x') \leq C d_\Sigma^{\beta-1}(x')$ near Σ for some constant C. We can then prove that the tension field of $u = (u^1, \ldots, u^n)$ satisfies

$$||\tau(u)||(x', x^m) \leq C \frac{x^m}{\xi(x', x^m)},$$

and that u has bounded energy density. One can proceed as in the proof of Theorem 2.1. q.e.d.

It was proved in [13] that if f is a nonconstant analytic map from $\mathbb{S}^1$ to itself, then f can be extended to a proper harmonic map from $\mathbb{H}^2$ to itself with bounded energy density. Using Theorem 2.1, or Theorem 2.5, one can show that similar result holds if f is a nonconstant analytic map from $\mathbb{S}^1$ to $\mathbb{S}^{n-1}$.

3. Results on uniqueness

In this section, we will discuss the problem of uniqueness for harmonic maps constructed in the previous section. There are examples of non-uniqueness of proper harmonic maps constructed in [11], [7]; see also [12]. It was shown in [12] that given k points on $\mathbb{S}^{m-1}$, there is a k-parameter family of distinct harmonic maps u from $\mathbb{H}^m$ to itself with

$u|_{\mathbb{S}^{m-1}}$ being the identity map. u is C^1 up to the boundary except at those k points where u grows very fast. On the other hand, if u is a harmonic diffeomorphism on $\mathbb{H}^2$ with bounded energy density so that $u|_{\mathbb{S}^1}$ is identity, then u is quasi-conformal by [21] and u must be the identity map by the uniqueness theorem in [12]. Hence one may guess that uniqueness or non-uniqueness depends on the growth rates of the maps to be considered. We will prove that in some cases this is correct. Namely, we have the following:

Theorem 3.1. *Let Σ be a closed subset of $\mathbb{S}^{m-1}$. Let u and v be two harmonic maps from $\mathbb{H}^m$ to $\mathbb{H}^n$ such that the function $dist_{\mathbb{H}^n}(u(p), v(p))$ satisfies:*

(i) $\lim_{p\to x}$ *$dist_{\mathbb{H}^n}(u(p), v(p)) = 0$ for any boundary point $x \in \mathbb{S}^{m-1}\backslash\Sigma$; and*

(ii) either (a) $\mathcal{H}^{m-2}(\Sigma) < \infty$, where $\mathcal{H}^{m-2}$ is the $(m-2)$-dimensional Hausdorff measure on $\mathbb{S}^{m-1}$, and $dist_{\mathbb{H}^n}(u(p), v(p)) = o(\exp(r(p))$ as $p \to \infty$, where $r(p)$ is the distance function in $\mathbb{H}^m$ from a fixed point o, or (b) $H^{m-2}(\Sigma) = 0$ and $dist_{\mathbb{H}^n}(u(p), v(p)) = O(\exp(r(p))$ as $p \to \infty$.

Then $u \equiv v$.

Corollary 3.2. *Let Σ be a closed subset of $\mathbb{S}^{m-1}$. Let f be a map from $\mathbb{S}^{m-1} \setminus \Sigma$ to $\mathbb{S}^{n-1}$ which is C^1 with nowhere vanishing energy density. Let u and v be two harmonic maps from $\mathbb{H}^m$ to $\mathbb{H}^n$, which are C^1 up to the boundary portion $\mathbb{S}^{m-1}\backslash\Sigma$, such that $u|_{\mathbb{S}^{m-1}\backslash\Sigma} = v|_{\mathbb{S}^{m-1}\backslash\Sigma} = f$. Suppose either $\mathcal{H}^{m-2}(\Sigma) < \infty$ and*

$$dist_{\mathbb{H}^n}(u(p), v(p)) = o(\exp(r(p))$$

as $p \to \infty$, or $\mathcal{H}^{m-2}(\Sigma) = 0$ and

$$dist_{\mathbb{H}^n}(u(p), v(p)) = O(\exp(r(p))$$

as $p \to \infty$. Then $u \equiv v$.

Proof. By the proof of Theorem in [11] , for any $x \in \mathbb{S}^{m-1} \setminus \Sigma$,

$$\lim_{p\to x} \mathrm{dist}_{\mathbb{H}^n}(u(p), v(p)) = 0.$$

The corollary follows immediately from Theorem 3.1. q.e.d.

Remark 3.1. The condition $\mathrm{dist}_{\mathbb{H}^n}(u(p), v(p)) = o(\exp(r(p))$ will be satisfied if $e(u)(p)+e(v)(p) = O\left(r^{-2-\epsilon}(p)\exp(2r(p))\right)$ for some $\epsilon > 0$. In particular, if u and v both have bounded energy density, the above condition on $\mathrm{dist}_{\mathbb{H}^n}(u(p), v(p))$ will be satisfied. Note that the examples of non-uniqueness for harmonic maps constructed in [11]–[12] and [7] mentioned above grow like $\exp((m-1)r)$ near those k points, where r is the distance function of a fixed point in $\mathbb{H}^m$.

Let $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an analytic map. Then either $f \equiv$constant, or the zero set of the energy density $e(f)$ of f is an analytic set of dimension $m-2$. Note that if f is a constant map, then f cannot be extended to a harmonic map from $\mathbb{H}^m$ to $\mathbb{H}^n$.

Corollary 3.3. *Let $f : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an analytic map and let u, v be harmonic extensions of f from $\mathbb{H}^m$ to $\mathbb{H}^n$ such that u, v have bounded energy density and are C^1 up to the $\mathbb{S}^{m-1} \setminus \Sigma$, where Σ is the zero set of $e(f)$. Then $u \equiv v$.*

In order to prove the theorem we need several facts. Using the upper half space model for $\mathbb{H}^m$, for any $t_0 \in \mathbb{R}^{m-1}$, the function

$$\left(\frac{y}{|x-t_0|^2 + y^2}\right)^{m-1}$$

is harmonic on $\mathbb{H}^m$, where $x \in \mathbb{R}^{m-1}$ and $y > 0$ are standard coordinates on $\mathbb{H}^m$.

Lemma 3.4. *Let $t_0 \in \mathbb{R}^{m-1}$ and $r > 0$ and let $B_{t_0}(r)$ be the Euclidean open ball of radius r with center t_0 in $\mathbb{R}^{m-1}$. Let*

$$g(x,y) = \int_{t \in \partial B_{t_0}(r)} \left(\frac{y}{|x-t|^2 + y^2}\right)^{m-1} d\mathcal{H}^{m-2}(t).$$

Then g is harmonic on $\mathbb{H}^m$. Moreover, there exists a positive constant $C(m)$ depending only on m such that for any $(x_0, y_0) \in \mathbb{H}^m$, if $0 < y_0 \leq r$ and $d_{\mathbb{R}^{m-1}}(x_0, \partial B_{t_0}(r)) \leq y_0$, then

$$g(x_0, y_0) \geq \frac{C(m)}{y_0}.$$

Proof. It is easy to see that g is harmonic on $\mathbb{H}^m$. To obtain the estimate in the lemma, we may assume that $t_0 = 0$ is the origin. Let

$(x_0, y_0) \in \mathbb{H}^m$ be such that $0 < y_0 \le r$ and $d_{\mathbb{R}^{m-1}}(x_0, \partial B_0(r)) \le y_0$. Also let $t_1 \in \partial B_0(r)$ with

$$|x_0 - t_1| = d_{\mathbb{R}^{m-1}}(x_0, \partial B_0(r)) \le y_0.$$

Then for any point t with $|t - t_1| \le y_0$, we have $|x_0 - t| \le 2y_0$. Hence

$$\begin{aligned} g(x_0, y_0) &\ge \int_{t\in\partial B_0(r), |t-t_1|\le y_0} \left(\frac{y_0}{|x_0 - t|^2 + y_0^2}\right)^{m-1} d\mathcal{H}^{m-2}(t) \\ &\ge (5y_0)^{-(m-1)}\mathcal{H}^{m-2}(\{t \in \partial B_0(r),\ |t - t_1| \le y_0\}). \end{aligned} \tag{3.1}$$

If $\xi_1 = t_1/y_0$, then $\xi_1 \in \partial B_0(\frac{r}{y_0})$ and

$$\begin{aligned} \mathcal{H}^{m-2}&(\{t \in \partial B_0(r),\ |t - t_1| \le y_0\}) \\ &= (y_0)^{m-2}\mathcal{H}^{m-2}(\{\xi \in \partial B_0(\frac{r}{y_0}),\ |\xi - \xi_1| \le 1\}) \\ &\ge C(m)(y_0)^{m-2} \end{aligned}$$

for some positive constant $C(m)$ depending only on m. Here we have used the fact that $r \ge y_0$. By (3.1), the lemma follows. q.e.d.

If u and v are harmonic, then $\mathrm{dist}_{\mathbb{H}^n}(u, v)$ is subharmonic. Theorem 3.1 will follow from the following lemma:

Lemma 3.5. *Let Σ be a closed set in $\mathbb{S}^{m-1}$. Let f be a nonnegative subharmonic function on $\mathbb{H}^m$ such that $\lim_{p\to x} f(p) = 0$ for any point $x \in \mathbb{S}^{m-1} \setminus \Sigma$. Suppose either $\mathcal{H}^{m-2}(\Sigma) < \infty$ and $f(p) = o(\exp(r(p))$ as $p \to \infty$, or $\mathcal{H}^{m-2}(\Sigma) = 0$ and $f(p) = O(\exp(r(p))$ as $p \to \infty$. Then $f \equiv 0$.*

Proof. Let us assume that $\mathcal{H}^{m-2}(\Sigma) < \infty$ and $f(p) = o(\exp(r(p))$ as $p \to \infty$. Using the upper half space model for $\mathbb{H}^m$, we may assume that Σ is a compact set in $\mathbb{R}^{m-1}$ with $\mathcal{H}^{m-2}(\Sigma) < \infty$. For any $\lambda > 0$, there exists a family of open balls $B_{t_i}(r_i)$, $i \in J_\lambda$ with $|J_\lambda| < \infty$, such that $\max_{i\in J_\lambda} r_i \le \lambda$, $\bigcup_{i\in J_\lambda} B_{t_i}(r_i) \supset \Sigma$ and

$$\sum_{i\in J_\lambda} (r_i)^{m-2} \le C_1 \tag{3.2}$$

for some constant C_1 which is independent of λ. Let

$$\mathcal{O}_{i,\lambda} = \{(x, y)|\ x \in B_{t_i}(r_i),\ 0 \le y < r_i,\ \text{and } d_{\mathbb{R}^{m-1}}(x, \partial B_{t_i}(r_i)) > y\}.$$

Also let $\mathcal{A}_\lambda = \mathbb{H}^m \setminus \bigcup_{i\in J_\lambda} \mathcal{O}_{i,\lambda}$. Since $\bigcup_{i\in J_\lambda} B_{t_i}(r_i) \supset \Sigma$, for any $x \in \Sigma$ there is $i \in J_\lambda$ such that $d_{\mathbb{R}^{m-1}}(x, \partial B_{t_i}(r_i)) > 0$. So there exists $\epsilon > 0$ such that if $|x - \tilde{x}| < \epsilon$ and $0 \le \tilde{y} < \epsilon$, then $(\tilde{x}, \tilde{y}) \in \mathcal{O}_{i,\lambda}$. By the assumption of f, we see that f is bounded on $\mathcal{A}_\lambda$. Moreover, if q is a point in the boundary of $\mathcal{A}_\lambda$ in $\overline{\mathbb{H}^m}$ and if $q \notin \partial\mathbb{H}^m$, then $q \notin \Sigma$. For $q \notin \partial\mathbb{H}^m$, we must have $q = (x, y)$ for some $x \in \mathbb{R}^{m-1}$ and $y > 0$, which implies that $q \in \overline{\mathcal{O}_{i,\lambda}} \cap \overline{\mathcal{O}^c_{i,\lambda}}$ for some $i \in J_\lambda$, where $\mathcal{O}^c_{i,\lambda}$ is the complement of $\mathcal{O}_{i,\lambda}$ in $\overline{\mathbb{H}^m}$. Then, on the one hand, we have $x \in \overline{B_{t_i}(r_i)}$, $0 \le y \le r_i$ and $d_{\mathbb{R}^{m-1}}(x, \partial B_{t_i}(r_i)) \ge y$. On the other hand, we also have $x \notin B_{t_i}(r_i)$, $y \ge r_i$ or $d_{\mathbb{R}^{m-1}}(x, \partial B_{t_i}(r_i)) \le y$. Hence we can conclude that $0 < y \le r_i$ and $d_{\mathbb{R}^{m-1}}(x, \partial B_{t_i}(r_i)) = y$, which are the assumptions of the second statement of Lemma 3.4. Therefore, we can apply Lemma 3.4 to the function

$$g^\lambda(x, y) = \sum_{i\in J_\lambda} \int_{t\in\partial B_{t_i}(r_i)} \left(\frac{y}{|x-t|^2 + y^2}\right)^{m-1} d\mathcal{H}^{m-2}(t).$$

Firstly, we conclude that g^λ is a harmonic function on $\mathbb{H}^m$. Then, together with the assumption that $f(p) = o(\exp(r(p)))$, we conclude that for any $\epsilon > 0$, there is a $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then $\epsilon g^\lambda \ge f$ on boundary of $\mathcal{A}_\lambda$ in $\overline{\mathbb{H}^m}$. Since f is bounded on $\mathcal{A}_\lambda$, we have

$$\epsilon g^\lambda(x, y) \ge f(x, y) \tag{3.3}$$

on $\mathcal{A}_\lambda$. Let $(x_0, y_0) \in \mathbb{H}^m$ be fixed. Since $y_0 > 0$, we have $(x_0, y_0) \in \mathcal{A}_\lambda$ for all $0 < \lambda < y_0$. By (3.3), we have,

$$\epsilon g^\lambda(x_0, y_0) \ge f(x_0, y_0) \tag{3.4}$$

if $0 < \lambda < \min\{\lambda_0, y_0\}$. For any λ, by (3.2), we have

$$g^\lambda(x_0, y_0) \le \frac{C_2}{(y_0)^{m-1}}, \tag{3.5}$$

where C_2 is a constant independent of λ. Combining this with (3.4), and letting $\epsilon \to 0$, we conclude that $f(x_0, y_0) = 0$. If $\mathcal{H}^{m-2}(\Sigma) = 0$ and $f(p) = O(\exp(r(p))$ as $p \to \infty$, then (3.4) can be replaced by

$$C_3 g^\lambda(x_0, y_0) \ge f(x_0, y_0), \tag{3.6}$$

where C_3 is a constant independent of λ. Moreover, the constant C_1 in (3.2) can be chosen so that $C_1 \to 0$ as $\lambda \to 0$. Hence the constant C_2 in (3.5) also satisfies $C_2 \to 0$ as $\lambda \to 0$. Letting $\lambda \to 0$ in (3.6), the lemma follows. q.e.d.

4. An explicit solution

Consider the following boundary value problem on the infinite strip $\{-\infty < x < \infty \text{ and } 0 < y < \pi\}$

$$
\begin{cases} \Delta_0 f - 2\cot g < \nabla_0 f, \nabla_0 g > & = 0, \\ \Delta_0 g + \cot g \left(|\nabla_0 f|^2 - |\nabla_0 g|^2\right) & = 0, \end{cases} \tag{4.1}
$$

such that $f(x,0) = \alpha x$, $f(x,\pi) = \alpha x + \beta$, $g(x,0) = 0$, $g(x,\pi) = \pi$ and $0 < g(x,y) < \pi$ for $0 < y < \pi$, where $\alpha > 0$ and β are constants. We may assume that $\beta \geq 0$. If the infinite strips $\{-\infty < x < \infty \text{ and } 0 < y < \pi\}$ and $\{-\infty < f < \infty \text{ and } 0 < g < \pi\}$ are equipped with the metrics

$$
\sin^{-2} y \left(dx^2 + dy^2\right) \quad \text{and} \quad \sin^{-2} g \left(df^2 + dg^2\right) \tag{$*$}
$$

respectively, then a solution of the above boundary value problem is a harmonic map $(x,y) \mapsto (f,g)$ from $\mathbb{H}^2$ to $\mathbb{H}^2$. Moreover, if we use a conformal map of the form $z \to e^z$ and transform the infinite strips to the upper half planes, then the boundary value of the map is given by

$$
\phi(t) = \begin{cases} |t|^{\alpha-1} t & \text{if } t \geq 0, \\ C|t|^{\alpha-1} t & \text{if } t \leq 0, \end{cases}
$$

where $C = e^{\beta}$. By Corollary 2.4, we know that (4.1) has a quasi-conformal solution with the given boundary data. For the case $\beta = 0$, the solution can be expressed explicitly as solutions of some ordinary differential equations; see [5], [22]. In this section, we will show that for $\beta > 0$, we can obtain explicit solutions by solving a system of ordinary differential equations. In fact, we will try to find a solution of (4.1) which takes the form $f(x,y) = \alpha x + h(y)$ and $g(x,y) = g(y)$. In this case, (4.1) becomes:

$$
\begin{cases} h'' - 2\cot g \, h' g' & = 0, \\ g'' + \cot g \left(\alpha^2 + (h')^2 - (g')^2\right) & = 0, \end{cases} \tag{4.2}
$$

with conditions $h(0) = 0$, $h(\pi) = \beta$, $g(0) = 0$, $g(\pi) = \pi$ and $0 < g(y) < \pi$ for $0 < y < \pi$. It is not hard to obtain the first integrals of the systems. Since we would like to have solutions with $h' \geq 0$, we have

$$
h' = a^2 \sin^2 g, \quad \text{and} \quad (g')^2 = \alpha^2 + (b^2 + a^4 - \alpha^2)\sin^2 g - a^4 \sin^4 g, \tag{4.3}
$$

where $a = h'(\frac{\pi}{2})$, $b = g'(\frac{\pi}{2})$ are constants to be chosen to satisfy the boundary conditions. Let $z = \cot g$. Then

$$(z')^2 = \alpha^2 z^4 + c^2 z^2 + b^2,$$

where $c^2 = \alpha^2 + b^2 + a^4$. Hence we let $z(y)$ be the function defined by

$$\int_0^{z(y)} \frac{dz}{\sqrt{\alpha^2 z^4 + c^2 z^2 + b^2}} = \frac{\pi}{2} - y. \tag{4.4}$$

Lemma 4.1. *For any $a \geq 0$, there is a unique constant $b_a > 0$ such that*

$$\int_0^{\infty} \frac{dz}{\sqrt{\alpha^2 z^4 + c_a^2 z^2 + b_a^2}} = \frac{\pi}{2},$$

where $c_a^2 = \alpha^2 + b_a^2 + a^4$. Moreover, $b_a \leq \max\{2\alpha, \alpha^{-1}\}$ and b_a depends continuously on a.

Proof. Let

$$I_1(b) = \int_0^{\infty} \frac{dz}{\sqrt{\alpha^2 z^4 + c^2 z^2 + b^2}},$$

where $c^2 = \alpha^2 + b^2 + a^4$. It is easy to see that $\lim_{b \to 0} I(b) = \infty$. Suppose $b > \max\{2\alpha, \alpha^{-1}\}$. Then

$$\begin{aligned} I_1(b) &\leq \int_0^{\infty} \frac{dz}{\sqrt{\alpha^2 z^4 + b^2 z^2 + b^2}} \\ &\leq \int_0^{\infty} \frac{dz}{\sqrt{\alpha^2 z^4 + 2\alpha b z^2 + b^2}} \\ &= \int_0^{\infty} \frac{dz}{\alpha z^2 + b} \\ &= (b\alpha)^{-\frac{1}{2}} \cdot \frac{\pi}{2} \\ &< \frac{\pi}{2}. \end{aligned}$$

Since $I_1(b)$ is decreasing and continuous in b, the lemma follows easily. q.e.d.

Lemma 4.2. *Let $0 \leq a < \infty$ and let $b_a > 0$ be the constants as in Lemma 4.1, which depend continuously on a. Given $\alpha > 0$ and $\beta \geq 0$, there is $a = a_{\alpha,\beta} \geq 0$ such that*

$$a^2 \int_0^{\infty} \frac{dz}{(1 + z^2)\sqrt{\alpha^2 z^4 + c_a^2 z^2 + b_a^2}} = \frac{\beta}{2},$$

where $c_a^2 = \alpha^2 + b_a^2 + a^4$.

Proof. Let

$$I_2(a) = a^2 \int_0^\infty \frac{dz}{(1+z^2)\sqrt{\alpha^2 z^4 + c_a^2 z^2 + b_a^2}}.$$

Let $y = y(z)$ be such that

$$\int_0^z \frac{dz}{\sqrt{\alpha^2 z^4 + c_a^2 z^2 + b_a^2}} = \frac{\pi}{2} - y.$$

Then, by the choice of b, z can be expressed as a function of y on $0 < y \leq \frac{\pi}{2}$ and we have

$$I_2(a) = a^2 \int_0^{\frac{\pi}{2}} \frac{dy}{1+y^2}.$$

Hence $\lim_{a\to 0} I_2(a) = 0$. In particular, if $\beta = 0$, we simply pick $a = 0$. Since $b_a \leq \max\{2\alpha, \alpha^{-1}\}$, there is a constant $C(\alpha) > 0$ depending only on α such that

$$\begin{aligned} I_2(a) &\geq C(\alpha) \cdot a^2 \int_0^\infty \frac{dz}{(z^4 + a^4 z^2 + 1)^{\frac{1}{2}} (1+z^2)} \\ &\geq \frac{C(\alpha)}{2} \cdot a^2 \int_0^1 \frac{dz}{(a^4 z^2 + 2)^{\frac{1}{2}}} \\ &= \frac{C(\alpha)}{2} \int_0^1 \frac{dz}{\left(z^2 + \frac{2}{a^4}\right)^{\frac{1}{2}}} \\ &\to \infty, \end{aligned}$$

as $a \to \infty$. Since b_a depends continuously on a, $I_2(a)$ is a continuous function of a. Hence the lemma follows. q.e.d.

Let $\alpha > 0$ and $\beta \geq 0$ be given. Let $a = a_{\alpha,\beta} \geq 0$ be the constant obtained in Lemma 4.2, and b_a be the constant in Lemma 4.1 corresponding to a. Let z be the function defined in (4.4), and define

$$g = \cot^{-1} z \quad \text{and} \quad h(y) = a^2 \int_0^y \sin^2 g(\tau) d\tau. \tag{4.5}$$

It is easy to see that $g : [0, \frac{\pi}{2}] \to [0, \frac{\pi}{2}]$, $h : [0, \frac{\pi}{2}] \to [0, \frac{\beta}{2}]$, $g(0) = h(0) = 0$, $g(\frac{\pi}{2}) = \frac{\pi}{2}$ and $h(\frac{\pi}{2}) = \frac{\beta}{2}$. Extend g and h to $[0, \pi]$ such that

$$h(t) = \beta - h(\pi - t), \quad g(t) = \pi - g(\pi - t). \tag{4.6}$$

Let

$$f(x,y) = \alpha x + h(y) \quad \text{and} \quad g(x,y) = g(y). \tag{4.7}$$

Theorem 4.3. *Let $u = (f,g)$, where f and g are given by Lemma 4.1, 4.2, (4.4)-(4.7). Then u is a quasi-conformal harmonic diffeomorphism from $\mathbb{H}^2$ onto itself. Here we identify the domain and the range to be infinite strips with conformal metrics (*). u satisfies the boundary conditions $u(x,0) = (\alpha x, 0)$ and $u(x,\pi) = (\alpha x + \beta, 0)$. Moreover, the Hopf differential of u is equal to $(c_1 + ic_2)dz^2$, where $c_1 = -\frac{1}{4}(b^2 + a^4 - \alpha^2)$ and $c_2 = -\frac{1}{2}\alpha a^2$, which are constants depending on α and β.*

Proof. It is not hard to see that u is harmonic and a diffeomorphism. Since $g' > 0$ and $g'(0) = \alpha$, $g'(\frac{\pi}{2}) = b > 0$ and g' is bounded away from zero. Hence

$$\left|\frac{u_{\bar z}}{u_z}\right| = \left|\frac{\alpha - g' + ih'}{\alpha + g' - ih'}\right| \le 1 - \epsilon$$

for some constant $\epsilon > 0$. We conclude that u is quasi-conformal. By direct computations, using (4.3), the Hopf differential is given by $(c_1 + ic_2)dz^2$ as claimed. q.e.d.

Let us compute the norm of the Hopf differential of the harmonic map constructed in Theorem 4.3. Using w instead of z in the theorem, the Hopf differential is $\Phi = c\,dw^2$, where c is a constant. Let $w = \log z$. Then w is a conformal map from the upper half space into the strip $0 < \mathrm{Im}(w) < \pi$. Hence $\Phi(z) = cz^{-2}dz^2$, and its norm with respect to the hyperbolic metric is given by

$$||\Phi||(z) = \frac{|c|y^2}{x^2 + y^2},$$

where $z = x + iy$. For $|x| > 1$,

$$||\Phi||(z) \le |c|y^2, \tag{4.5}$$

and for $|x| \le 1$,

$$||\Phi||(z) \le \frac{2|c|y}{|x| + y}. \tag{4.6}$$

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism satisfying the conditions in Theorem 2.1. Namely, f is C^3 with nonvanishing energy density except possibly for finitely many points $p_1, \ldots, p_k$. Near each p_i, if we consider

f as a map from $\mathbb{R}$ to $\mathbb{R}$ such that p_i corresponds to 0 and $f(0) = 0$, then $f(t) = |t|^{\alpha-1}(h_1(t) + h_2(t))$, where $h_2(t) = O(t^2)$ as $t \to 0$, is C^3 smooth away from 0 and is piecewise C^3 smooth at 0, and

$$h_1(t) = \begin{cases} t & \text{if } t \geq 0, \\ Ct & \text{if } t < 0 \end{cases}$$

for some constants $\alpha > 0$ and $C > 0$.

Theorem 4.4. *Let f be as above and let h be a quasi-symmetric function from $\mathbb{S}^1$ onto itself such that h has a quasi-conformal harmonic extension on $\mathbb{H}^2$. Then $h \circ f$ also has a quasi-conformal harmonic extension.*

Proof. Let us consider the special case that f is given by (*) at the beginning of this section and let F be the quasi-conformal harmonic diffeomorphism constructed in Theorem 4.3. Suppose $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasi-symmetric function such that h has a quasi-conformal harmonic extension H on $\mathbb{H}^2$. Let Φ_F and Φ_H be the Hopf differentials of F and H respectively. Then by Lemma 3.2 in [18], the norm of the tension field of $H \circ F$ satisfies

$$(4.7) \qquad ||\tau(H \circ F)||(z) \leq C_1 ||\Phi_H||_{QD} \left(1 + ||\Phi_H||_{QD}\right)^{\frac{1}{2}} ||\Phi_F||(z),$$

where C_1 is an absolute constant, and $||\Phi_H||_{QD} = \sup_{z \in \mathbb{H}^2} ||\Phi_H||(z)$ which is finite by [21]. Moreover, $H \circ F$ has bounded energy density. By (4.5), (4.6) and Theorem 1.3, $h \circ f$ can be extended to a harmonic map on $\mathbb{H}^2$ with bounded energy density. It is easy to see that the harmonic map is a quasi-conformal diffeomorphism.

In general, suppose f satisfies the conditions in the theorem. We remark that for each $\phi : \mathbb{R} \to \mathbb{R}$ which is of the form as in (*), we can modify ϕ to another map ψ such that $\psi = \phi$ outside a compact set, ψ is C^∞ on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi' \neq 0$ on $\mathbb{R}$. Hence we can find $\phi_j : \mathbb{S}^1 \to \mathbb{S}^1$ and ψ such that, for each j with $1 \leq j \leq k-1$, ϕ_j is a composition of a map of the form (*) with conformal maps in the domain and the range, ψ is of the form (*) outside a compact set and is C^∞ with $\psi(0) = 0$ and $\psi' \neq 0$ on $\mathbb{R}$, and

$$f = f_1 \circ \psi \circ \phi_1 \circ \cdots \circ \phi_{k-1},$$

where f_1 is C^1 with nonvanishing energy density. Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a quasi-symmetric function such that h has a quasi-conformal harmonic extension on $\mathbb{H}^2$. By the result in [18], $h \circ f_1$ also has a quasi-conformal harmonic extension on $\mathbb{H}^2$. It remains to show that if $h : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasi-symmetric function such that h has a quasi-conformal harmonic extension H on $\mathbb{H}^2$, so is $h \circ \psi$. By the construction of ψ, we can extend ψ to a map G from $\mathbb{H}^2$ to $\mathbb{H}^2$ such that near $\partial\mathbb{H}^2$ and away from 0, G is exactly the harmonic map constructed in Theorem 4.3. Near 0, G is conformal at the boundary and is smooth. Then as in [12], one can show that, as $y \to 0$,

$$||\tau(G)||(z) = O(y) \tag{4.8}$$

and

$$||\Phi_G||(z) = O(y), \tag{4.9}$$

where Φ_G is the dz^2-part of the pull-back metric by G which is the Hopf differential if G is harmonic. We will call Φ_G the Hopf differential of G even G may not be harmonic and the differential may not be holomorphic. Similar calculation as in the proof of Lemma 3.2 in [18] yields

$$||\tau(H \circ G)||(z) \leq C(||\Phi_H||_{QD}) \left(||\Phi_G||(z) + ||\tau(G)||(z)\right), \tag{4.10}$$

where $C(||\Phi_H||_{QD})$ denotes a constant depending only on $||\Phi_H||_{QD}$. So together with (4.5), (4.6), (4.8) and (4.9), we see that the map $H \circ G$ satisfies the conditions in Theorem 1.3. Hence $H \circ G$ can be deformed to a quasi-conformal harmonic diffeomorphism on $\mathbb{H}^2$. q.e.d.

Corollary 4.5. *Let $\mathcal{G}$ be the subspace of the universal Teichmüller space $\mathcal{T}$ consisting of quasi-symmetric functions from $\mathbb{S}^1$ onto $\mathbb{S}^1$ satisfying the assumptions of Theorem 4.4. Then for all f in the closure of $\mathcal{G}$ in $\mathcal{T}$ with respect to the Teichmüller metric, f can be extended to a unique quasi-conformal harmonic diffeomorphism.*

The closure of $\mathcal{G}$ contains the closure $\overline{N}$ of C^1 diffeomorphisms of $\mathbb{S}^1$ as a proper subset. The characterization of the $\overline{N}$ in terms of the Hopf differentials of the quasi-conformal harmonic extensions is given in [18]. It is interesting to understand what is $\overline{\mathcal{G}}$. Suppose that $f \in \overline{\mathcal{G}}$. Then there exists a sequence f_n in $\mathcal{G}$ such that $f_n \to f$ in Teichmüller sense. Let F and F_n be the corresponding quasiconformal harmonic extensions. By the main result in [18], $||\Phi_{F_n} - \Phi_F||_{QD} \to 0$. Since $||\Phi_{F_n}||(z) \to 0$ as $z \to \partial\mathbb{H}^2$ except at finitely many points and $||\cdot||_{QD} = \sup_{z\in\mathbb{H}^2} ||\cdot||(z)$, we see that, for any $\epsilon > 0$, $\{\xi \in \partial\mathbb{H}^2 \,|\, \limsup_{z\to\xi} ||\Phi_F||(z) > \epsilon\}$ is finite.

This implies that for any $f \in \overline{\mathcal{G}}$, there is a quasiconformal extension F of f such that for any $\epsilon > 0$, the set of boundary points ξ with $\limsup_{z \to \xi} |\mu_F(z)| > \epsilon$ is finite, where μ_F is the complex dilatation of F. We still do not know whether this condition is sufficient.

5. Group-invariant harmonic maps

In this section, we will find all the harmonic maps from a domain in $\mathbb{R}^2$ into hyperbolic 2-space which are explicitly solvable via systems of ordinary differential equations. In other words, we will find all group-invariant harmonic maps from a domain in $\mathbb{R}^2$ into $\mathbb{H}^2$ with respect to Lie point symmetries. In terms of the standard coordinates (x, y) on $\mathbb{R}^2$ and (f, g), $g > 0$ on the upper half-plane model of $\mathbb{H}^2$, the harmonic map equation can be written as

$$\begin{cases} \Delta_0 f &= \frac{2}{g} < \nabla_0 f, \nabla_0 g >, \\ \Delta_0 g &= \frac{1}{g} \left(|\nabla_0 g|^2 - |\nabla_0 f|^2 \right). \end{cases} \tag{5.1}$$

Therefore, the harmonic map equation can be considered as a system of partial differential equations with space of independent and dependent variables $M = \mathbb{R}^2 \times \mathbb{H}^2$ which is considered as a subset of $\mathbb{R}^2 \times \mathbb{R}^2$. Let G be a local group of transformations (i.e., a local group of local diffeomorphisms of $\mathbb{R}^2 \times \mathbb{R}^2$) acting on M. Then a G-invariant harmonic map is a solution $(f(x, y), g(x, y))$ of (5.1) whose graph $\Gamma_{(f,g)} = \{(x, y, f(x, y), g(x, y))\} \subset M$ is a locally G-invariant subset of M. There is a standard computational procedure to determine group-invariant solutions of a given system of partial differential equations, and we will follow this procedure to find all the group-invariant harmonic maps. For a detailed discussion of the procedure and the theory behind, we refer the reader to the book [15].

First of all, we need to determine all the infinitesimal generators of the Lie point symmetry group of the harmonic map system, i.e., all vector fields along the solution space of harmonic map equation given by orbits of particular solutions under one-parameter subgroups. This involves a straightforward but tedious computation.

Theorem 5.1. *The infinitesimal generators of the Lie point symmetry group of (5.1) are of the form*

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \left(a(f^2 - g^2) + bf + c\right)\partial_f + (2afg + bg)\,\partial_g,$$

where a, b and c are real constants, and (ξ, η) satisfies the Cauchy-Riemann equations

$$\xi_x - \eta_y = \xi_y + \eta_x = 0.$$

This can be written in complex form

$$X = 2Re\ \left(F(z)\partial_z + (au^2 + bu + c)\partial_u\right),$$

where $z = x + iy$, $u = f + ig$, and $F(z)$ is a holomorphic function in z.

Proof. Let $X = \xi^i(x^j, u^\beta)\partial_i + \phi^\alpha(x^j, u^\beta)\partial_{u^\alpha}$ be an infinitesimal generator of the Lie point symmetry group, where we write (x^1, x^2) for (x, y), (u^1, u^2) for (f, g), and the repeated indices mean summation. X induces a vector field on the 2-jet:

$$pr^{(2)}(X) = \xi^i\partial_i + \phi^\alpha\partial_{u^\alpha} + \phi^{\alpha,i}\partial_{u_i^\alpha} + \phi^{\alpha,ij}\partial_{u_{ij}^\alpha}$$

for some functions $\phi^{\alpha,i}$ and $\phi^{\alpha,ij}$ determined by ξ^i, ϕ^α, u_i^α, and u_{ij}^α.

Applying it to the harmonic map equation (5.1), we have

$$\sum_i \phi^{1,ii} = -\frac{2}{(u^2)^2}\phi^2 < \nabla_0 u^1, \nabla_0 u^2 > + \frac{2}{u^2}\sum_i(\phi^{1,i}u_i^2 + u_i^1\phi^{2,i}), \tag{5.2}$$

$$\begin{aligned} \sum_i \phi^{2,ii} &= -\frac{2}{(u^2)^2}\phi^2(|\nabla_0 u^2|^2 - |\nabla_0 u^1|^2) \\ &\quad + \frac{2}{u^2}\sum_i(\phi^{2,i}u_i^2 - \phi^{1,i}u_i^1). \end{aligned} \tag{5.3}$$

According to the prolongation formula in [15], for any i

$$\phi^{\alpha,i} = \phi_i^\alpha + \phi_{u^\beta}^\alpha u_i^\beta - \xi_i^j u_j^\alpha - \xi_{u^\beta}^j u_i^\beta u_j^\alpha, \tag{5.4}$$

and

$$\begin{aligned} \sum_i \phi^{\alpha,ii} = \sum_i \Big(&\phi_{ii}^\alpha + 2\phi_{iu^\beta}^\alpha u_i^\beta + \phi_{u^\beta u^r}^\alpha u_i^\beta u_i^r + \phi_{u^\beta}^\alpha u_{ii}^\beta \\ &- \xi_{ii}^k u_k^\alpha - 2\xi_{iu^\beta}^k u_i^\beta u_k^\alpha - 2\xi_i^k u_{ki}^\alpha \\ &- \xi_{u^\beta}^k u_k^\alpha u_{ii}^\beta - 2\xi_{u^\beta}^k u_i^\beta u_{ik}^\alpha \\ &- \xi_{u^\beta u^r}^k u_i^\beta u_i^r u_k^\alpha \Big). \end{aligned} \tag{5.5}$$

Using the harmonic map equation, we see that $\Delta_0 u^\beta = \sum_i u^\beta_{ii}$ can be expressed in quadratic terms in u^α_i. So together with the observation that there is no second order derivative terms of u on the right-hand side of equations (5.2) and (5.3), we conclude by comparing the terms involving u^α_{12} in equations (5.2), (5.3) and (5.5) that

$$(\xi^2_1 + \xi^1_2) + \xi^1_{u^\beta} u^\beta_2 + \xi^2_{u^\beta} u^\beta_1 = 0.$$

Hence

$$\xi^2_1 + \xi^1_2 = 0, \tag{5.6}$$

and for any β

$$\xi^i_{u^\beta} = 0. \tag{5.7}$$

The last equation means that ξ^i are independent of u^α. This simplifies the computation a lot. In fact, the formulae (5.4) and (5.5) become

$$\phi^{\alpha,i} = \phi^\alpha_i + \phi^\alpha_{u^\beta} u^\beta_i - \xi^j_i u^\alpha_j \tag{5.8}$$

and

$$\begin{aligned}\sum_i \phi^{\alpha,ii} = \sum_i \left(\phi^\alpha_{ii} + 2\phi^\alpha_{iu^\beta} u^\beta_i + \phi^\alpha_{u^\beta u^r} u^\beta_i u^r_i\right) \\ + \phi^\alpha_{u^\beta} \Delta_0 u^\beta - (\Delta_0 \xi^k) u^\alpha_k - 2(\xi^1_1 u^\alpha_{11} + \xi^2_2 u^\alpha_{22}).\end{aligned} \tag{5.9}$$

The remaining second order terms involve only u^α_{11} and u^α_{22}. Using the harmonic map equation (5.1) again, we see that $u^\alpha_{11} = -u^\alpha_{22} +$ quadratic terms in u^β_i. Therefore, by comparing the second order terms again, we conclude that

$$\xi^1_1 = \xi^2_2. \tag{5.10}$$

The system (5.6) and (5.10) is just the system of Cauchy-Riemann equations which implies that $\Delta_0 \xi^k = 0$. Hence, (5.9) further simplifies to

$$\begin{aligned}\sum_i \phi^{\alpha,ii} = \sum_i \left(\phi^\alpha_{ii} + 2\phi^\alpha_{iu^\beta} u^\beta_i + \phi^\alpha_{u^\beta u^{g^\alpha}} u^\beta_i u^{g^\alpha}_i\right) \\ + \phi^\alpha_{u^\beta} \Delta_0 u^\beta - 2\xi^1_1 \Delta_0 u^\alpha.\end{aligned} \tag{5.11}$$

Now, we need to handle the first derivative terms. Using (5.8) and the Cauchy-Riemann system (5.6) and (5.10), we have

$$
\begin{aligned}
&\text{R.H.S. (right-hand side) of (5.2)}\\
&= \frac{2}{u^2}\sum_i \left(\phi_i^1 u_i^2 + \phi_i^2 u_i^1\right)\\
&\quad + \frac{2}{u^2}\phi^2_{u^1}|\nabla_0 u^1|^2 + \frac{2}{u^2}\phi^1_{u^2}|\nabla_0 u^2|^2\\
&\quad + \left(-\frac{2}{(u^2)^2}\phi^2 + \frac{2}{u^2}\phi^1_{u^1} + \frac{2}{u^2}\phi^2_{u^2} - \frac{4\xi_1^1}{u^2}\right) < \nabla_0 u^1, \nabla_0 u^2 > .
\end{aligned}
\tag{5.12}
$$

On the other hand, using (5.11) with $\alpha = 1$, we have

$$
\begin{aligned}
&\text{L.H.S. (left-hand side) of (5.2)}\\
&= \sum_i \left(\phi^1_{ii} + 2\phi^1_{iu^\beta} u_i^\beta\right)\\
&\quad + \left(\phi^1_{u^1u^1} - \frac{1}{u^2}\phi^1_{u^2}\right)|\nabla_0 u^1|^2\\
&\quad + \left[2\phi^1_{u^1u^2} + \frac{2}{u^2}(\phi^1_{u^1} - 2\xi_1^1)\right] < \nabla_0 u^1, \nabla_0 u^2 >\\
&\quad + (\phi^1_{u^2u^2} + \frac{1}{u^2}\phi^1_{u^2})|\nabla_0 u^2|^2.
\end{aligned}
\tag{5.13}
$$

Therefore, by comparing the corresponding terms in (5.12) and (5.13), we have

$$\Delta_0 \phi^1 = 0, \tag{5.14}$$

$$\frac{2}{u^2}(\phi_i^1 u_i^2 + \phi_i^2 u_i^1) = 2\phi^1_{iu^\beta} u_i^\beta, \tag{5.15}$$

$$\phi^1_{u^1u^1} - \frac{1}{u^2}\phi^1_{u^2} = \frac{2}{u^2}\phi^2_{u^1}, \tag{5.16}$$

$$\phi^1_{u^1u^2} = -\frac{1}{(u^2)^2}\phi^2 + \frac{1}{u^2}\phi^2_{u^2}, \tag{5.17}$$

$$\phi^1_{u^2u^2} = \frac{1}{u^2}\phi^1_{u^2}, \tag{5.18}$$

which implies that

$$\phi^1 = A(x^1, x^2, u^1)(u^2)^2 + B(x^1, x^2, u^1) \tag{5.19}$$

for some functions A and B of x^1, x^2 and u^1 only.

Similarly, we find that

$$\begin{aligned} \text{R.H.S. of (5.3)} = & \frac{2}{u^2}(\phi^2_i u^2_i - \phi^1_i u^1_i) \\ & + \frac{2}{u^2}(\phi^2_{u^1} - \phi^1_{u^2}) < \nabla_0 u^1, \nabla_0 u^2 > \\ & + \left(\frac{\phi^2}{(u^2)^2} - \frac{2}{u^2}\phi^1_{u^1} + 2\frac{\xi^1_1}{u^2} \right) |\nabla_0 u^1|^2 \\ & + \left(-\frac{\phi^2}{(u^2)^2} + \frac{2\phi^2_{u^2}}{u^2} - \frac{2\xi^1_1}{u^2} \right) |\nabla_0 u^2|^2 \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \text{L.H.S. of (5.3)} = & \sum_i \left(\phi^2_{ii} + 2\phi^2_{iu^\beta} u^\beta_i \right) \\ & + \left(2\phi^2_{u^1u^2} + \frac{2}{u^2}\phi^2_{u^1} \right) < \nabla_0 u^1, \nabla_0 u^2 > \\ & + \left(\phi^2_{u^1u^1} - \frac{1}{u^2}(\phi^2_{u^2} - 2\xi^1_1) \right) |\nabla_0 u^1|^2 \\ & + \left(\phi^2_{u^2u^2} + \frac{1}{u^2}(\phi^2_{u^2} - 2\xi^1_1) \right) |\nabla_0 u^2|^2. \end{aligned} \tag{5.21}$$

Then comparing the corresponding terms as before, we have

$$\Delta_0 \phi^2 = 0, \tag{5.22}$$

$$\phi^2_{iu^\beta} u^\beta_i = \frac{1}{u^2}\left(\phi^2_i u^2_i - \phi^1_i u^1_i \right). \tag{5.23}$$

$$\phi^2_{u^1u^1} - \frac{1}{u^2}\phi^2_{u^2} = \frac{\phi^2}{(u^2)^2} - \frac{2}{u^2}\phi^1_{u^1}, \tag{5.24}$$

$$\phi^2_{u^1u^2} + \frac{1}{u^2}\phi^1_{u^2} = 0, \tag{5.25}$$

$$\phi^2_{u^2u^2} - \frac{1}{u^2}\phi^2_{u^2} + \frac{1}{(u^2)^2}\phi^2 = 0, \tag{5.26}$$

which is an Euler equation and gives

$$\phi^2 = C(x^1, x^2, u^1)u^2 + D(x^1, x^2, u^1)u^2 \log u^2, \tag{5.27}$$

where C and D are functions of (x^1, x^2) and u^1 only. Now substituting (5.19) and (5.27) into (5.25) and comparing the coefficients, we have

$$D_{u^1} = 0 \quad \text{and} \quad C_{u^1} + 2A = 0. \tag{5.28}$$

Then substituting (5.28), (5.19) and (5.27) into (5.24) yields

$$D = 0 \quad \text{and} \quad B_{u^1} = C. \tag{5.29}$$

Therefore (5.28) and (5.29) imply:

$$\phi^1 = -\frac{1}{2}B_{u^1u^1}(x^1, x^2, u^1)(u^2)^2 + B(x^1, x^2, u^1), \tag{5.30}$$

$$\phi^2 = B_{u^1}(x^1, x^2, u^1)u^2. \tag{5.31}$$

Substituting these into (5.17), we conclude that

$$B_{u^1u^1u^1} = 0,$$

so that

$$B = E(x^1, x^2)(u^1)^2 + F(x^1, x^2)u^1 + G(x^1, x^2) \tag{5.32}$$

for some functions E, F and G of (x^1, x^2) only. By (5.30), (5.31) and (5.32), we see that (5.16) becomes identity and (5.15) implies that

$$\nabla_0 E = \nabla_0 F = \nabla_0 G = 0,$$

and hence E, F and G are real constants, says $E = a$, $F = b$ and $G = c$. The remaining equations (5.23), (5.14) and (5.22) are then satisfied trivially. Therefore we arrive at our final answer

$$\phi^1 = a\left((u^1)^2 - (u^2)^2\right) + bu^1 + c,$$

$$\phi^2 = 2au^1u^2 + bu^2.$$

Together with (5.6) and (5.10), the proof of the theorem is completed.
q.e.d.

It is easy to see that the infinitesimal generators are combinations of infinitesimal generators obtained by the local conformal transformations on the domain and the isometries of the target. Therefore, we have

Theorem 5.2. *The Lie point symmetry group of the system of harmonic map equations from a domain in* $\mathbb{R}^2$ *into* $\mathbb{H}^2$ *is the product of the local group of local conformal transformations of the domain and the isometry group of* $\mathbb{H}^2$.

From now on, we will use Theorem 5.1 to find all the group-invariant harmonic maps up to conformal equivalence. The procedure is to integrate the vector field given by Theorem 5.1 in order to obtain the orbit of a particular solution under the corresponding one-parameter subgroup. Then by eliminating the parameter, one get enough invariants to obtain group-invariant solutions. The interested readers can consult Chapter 3 of the book [15].

Theorem 5.3. *A group-invariant harmonic map from a domain in* $\mathbb{R}^2$ *into* $\mathbb{H}^2$ *is equivalent, up to a conformal transformation of the domain and an isometry on* $\mathbb{H}^2$, *to one of the following forms:*

1. *Harmonic maps invariant under the group generated by a translation on* $\mathbb{R}^2$ *and a translation on the upper-half plane model of* $\mathbb{H}^2$:
$$\begin{cases} f(x,y) = & x + h_1(y), \\ g(x,y) = & h_2(y), \end{cases}$$
where h_1, h_2 *are functions of* y *only with* $h_2 > 0$.

2. *Harmonic maps invariant under the group generated by a translation on* $\mathbb{R}^2$ *and a dilation on the upper-half plane model of* $\mathbb{H}^2$:
$$\begin{cases} f(x,y) = & e^x h_1(y), \\ g(x,y) = & e^x h_2(y), \end{cases}$$
where h_1, h_2 *are functions of* y *only with* $h_2 > 0$.

3. *Harmonic maps invariant under the group generated by a translation on* $\mathbb{R}^2$ *and a rotation on the unit disc model of* $\mathbb{H}^2$:
$$\zeta(x,y) + \sqrt{-1}\eta(x,y) = e^{\sqrt{-1}x}\left(h_1(y) + \sqrt{-1}h_2(y)\right),$$
where h_1, h_2 *are functions of* y *only and satisfy* $h_1^2 + h_2^2 < 1$.

Proof. Using the complex form of the admissible vector field in Theorem 5.1, the orbit of a harmonic map $(z, u(z))$ under a one-parameter subgroup is given by the solution of the following initial value problem of ordinary differential system

$$
\begin{cases} \frac{d\tilde{z}}{d\epsilon} = F(\tilde{z}), & \tilde{z}(0) = z, \\ \frac{d\tilde{u}}{d\epsilon} = a\tilde{u}^2 + b\tilde{u} + c, & \tilde{u}(0) = u. \end{cases} \tag{5.33}
$$

If $F \equiv 0$, then $\tilde{z} = z$ and $\tilde{u} = \gamma(\epsilon)u$ for some one-parameter subgroup of the Möbius group. Hence there is no invariant harmonic map in this case, and we may assume that F does not vanish since we are only interested in local solutions at this moment. We define a holomorphic, hence locally conformal, change of the domain by

$$
s + it = \int^z \frac{d\zeta}{F(\zeta)},
$$

where the choice of the base point is not important. Then in terms of this new variable,

$$
2\mathrm{Re}\ (F(z)\partial_z) = \partial_s.
$$

Since we are interested in classifying the local group-invariant harmonic maps up to conformal change of the domain and isometry on $\mathbb{H}^2$, we may as well assume that $F(z) = 1$ and the system (5.33) becomes

$$
\begin{cases} \frac{d\tilde{x}}{d\epsilon} = 1, & \tilde{x}(0) = x, \\ \frac{d\tilde{y}}{d\epsilon} = 0, & \tilde{y}(0) = y, \\ \frac{d\tilde{u}}{d\epsilon} = a\tilde{u}^2 + b\tilde{u} + c, & \tilde{u}(0) = u, \end{cases} \tag{5.34}
$$

where $z = x + iy$. The first two equations of (5.34) give

$$
\begin{cases} \tilde{x} = & x + \epsilon, \\ \tilde{y} = & y. \end{cases} \tag{5.35}
$$

Hence, y is an invariant of the corresponding one-parameter subgroup generated by X.

Let $\delta = b^2 - 4ac$. We consider the following cases. The first case is $\delta = 0$. Suppose also that $a = 0$. Then $b = 0$ and the third equation of (5.34) gives

$$
\tilde{u} = u + c\epsilon.
$$

Therefore, in this case, the orbit of a harmonic map under the group action is

$$\begin{cases} \tilde{x} = & x + \epsilon, \\ \tilde{y} = & y, \\ \tilde{u} = & u + c\epsilon. \end{cases}$$

By eliminating the ϵ, we obtain three invariants of the one-parameter group

$$y, \quad f - cx, \quad g.$$

Therefore, the group-invariant solution is of the form

$$\begin{cases} f - cx & = h_1(y), \\ g & = h_2(y), \end{cases}$$

for some functions h_1 and h_2 of one variable. This is of course equivalent to the first case.

Suppose $\delta = 0$, but $a \neq 0$. Then

$$\frac{-1}{\tilde{u} + \frac{b}{2a}} = a\epsilon + \frac{-1}{u + \frac{b}{2a}}.$$

Hence the invariants in this case are

$$y, \quad \mathrm{Re}\left(\frac{-1}{u + \frac{b}{2a}} - ax\right), \quad \mathrm{Im}\left(\frac{-1}{u + \frac{b}{2a}} - ax\right).$$

Let $U = \frac{-1}{u + b/(2a)}$. Then U is harmonic and in fact equivalent to u in our sense. Therefore, in this case, the group invariant harmonic map is also equivalent to

$$\begin{cases} \mathrm{Re}\, U & = ax + h_1(y), \\ \mathrm{Im}\, U & = h_2(y), \end{cases}$$

for some functions h_1 and h_2 of one variable. This is also equivalent to the first case. Hence, we have proved that for $\delta = 0$, all group invariant harmonic maps are equivalent to the first case.

Secondly, we assume that $\delta > 0$. Suppose also that $a = 0$. Then $b \neq 0$ and we get

$$\tilde{u} + \frac{c}{b} = e^{b\epsilon}\left(u + \frac{c}{b}\right).$$

As before, we can find three invariants in this case

$$y, \quad e^{-bx}\left(f + \frac{c}{b}\right), \quad e^{-bx} g.$$

Therefore, the corresponding group-invariant harmonic map is of the form

$$\begin{cases} e^{-bx}\left(f + \frac{c}{b}\right) & = h_1(y), \\ e^{-bx}g & = h_2(y), \end{cases}$$

for some functions h_1 and h_2 of one variable. This is of course equivalent to the second case.

If $\delta > 0$ and $a \neq 0$, then

$$\frac{\tilde{u} + \lambda_1}{\tilde{u} + \lambda_2} = \frac{u + \lambda_1}{u + \lambda_2} e^{\sqrt{\delta}\epsilon},$$

where $\lambda_1 = (b - \sqrt{\delta})/(2a)$ and $\lambda_2 = (b + \sqrt{\delta})/(2a)$. Hence the invariants are

$$y, \quad \mathrm{Re}\ \frac{u + \lambda_1}{u + \lambda_2} e^{-\sqrt{\delta}x}, \quad \mathrm{Im}\ \frac{u + \lambda_1}{u + \lambda_2} e^{-\sqrt{\delta}x}.$$

Therefore, by letting $V = (u + \lambda_1)/(u + \lambda_2)$, the group invariant harmonic map is also equivalent to

$$\begin{cases} \mathrm{Re}\ V & = e^{\sqrt{\delta}x} h_1(y), \\ \mathrm{Im}\ V & = e^{\sqrt{\delta}x} h_2(y), \end{cases}$$

for some functions h_1 and h_2 of one variable. This is also equivalent to the second case. Hence, we have proved that for $\delta > 0$, all group invariant harmonic maps are equivalent to the second case.

Finally, if $\delta < 0$, then $a \neq 0$. Similarly, the group invariant solution is equivalent to

$$\begin{cases} \mathrm{Re}\ V & = e^{\sqrt{\delta}x} h_1(y), \\ \mathrm{Im}\ V & = e^{\sqrt{\delta}x} h_2(y). \end{cases}$$

However, V does not map into the upper-half space since λ_k are complex-valued. In fact, V is a mapping into the unit disc. So we see that the group invariant solution is equivalent to a harmonic map into the unit disc model of $\mathbb{H}^2$ of the form

$$\zeta + i\eta = e^{i\sqrt{|\delta|}x}\left(h_1(y) + ih_2(y)\right)$$

for some functions h_1 and h_2 of one variable. This is equivalent to the third case. The proof of theorem is completed. q.e.d.

The remaining task for us now is to determine the functions h_1 and h_2 in each case.

Case 1. For the first case, using the harmonic map equation for the upper-half space model, we see that h_1 and h_2 satisfy

$$\begin{cases} h_2 h_1'' &= 2h_1' h_2', \\ h_2 h_2'' &= (h_2')^2 - (h_1')^2 - 1. \end{cases}$$

It is easy to obtain the first integral of this system

$$\begin{cases} h_1' &= c_1 (h_2)^2, \\ (h_2')^2 &= c_2 (h_2)^2 - c_1^2 (h_2)^4 + 1 \end{cases} \tag{5.36}$$

for some constants c_1 and c_2. Therefore, the general solution can be expressed in terms of elliptic functions explicitly. Instead of giving the general solution which is straightforward by consulting table of elliptic integrals, we would like to point out that the second equation in (5.36) implies that if $c_1 \neq 0$, then the corresponding harmonic map is not surjective. So if one is interested in finding examples of harmonic diffeomorphisms, one can assume $c_1 = 0$. Then

$$\begin{cases} h_1 &= c_3, \\ (h_2')^2 &= c_2 (h_2)^2 + 1 \end{cases}$$

for some constant c_3. The second equation has the general solution

$$h_2 = \frac{1}{\sqrt{c_2}} \sinh(\pm\sqrt{c_2} y + c_4)$$

for some constant c_4, provided $c_2 > 0$. Therefore the harmonic map is equivalent to

$$\begin{cases} f(x,y) &= x, \\ g(x,y) &= \frac{1}{\sqrt{c_2}} \sinh(\sqrt{c_2} y), \end{cases}$$

which is the example given earlier by Li-Tam [L-T 2]. If $c_2 \leq 0$, then one can check that the harmonic maps are equivalent to the identity

$$\begin{cases} f(x,y) &= x, \\ g(x,y) &= y \end{cases}$$

for $c_2 = 0$, and to

$$\begin{cases} f(x,y) &= x, \\ g(x,y) &= \frac{1}{\sqrt{|c_2|}} \sin(\sqrt{|c_2|} y), \end{cases}$$

which is not surjective for $c_2 < 0$.

Case 2. The second case is in fact handled in Section 4 already by an isometric change of the Poincaré upper-half plane to a horizontal infinite strip with width equal to π. The only difference is that we are interested in finding the harmonic diffeomorphism with suitable boundary data in Section 4. As a matter of fact, this is equivalent to showing that one can obtain the harmonic diffeomorphisms with the required boundary data by selecting the suitable integral constants. Moreover, one can also check that all the other group-invariant solutions except those obtained in Section 4 are not surjective as in the first case.

Case 3. Finally, for the third case, we use geodesic polar coordinates (ρ, θ) on the unit disc model of the hyperbolic 2-space. Then the harmonic maps should have the form

$$\rho = h_1(y), \quad \theta = x + h_2(y)$$

for some functions h_1 and h_2 of y only. Also the equations of h_1 and h_2 become

$$\begin{cases} h_1'' - \sinh h_1 \cosh h_1 \left(1 + (h_2')^2\right) & = 0, \\ h_2'' + 2\frac{\cosh h_1}{\sinh h_1} h_1' h_2' & = 0. \end{cases}$$

Similarly, the first integral can be found

$$\begin{cases} (h_1')^2 & = \sinh^2 h_1 - c_1^2 \sinh^{-2} h_1 + c_2, \\ h_2' & = c_1 \sinh^{-2} h_1. \end{cases}$$

Again, general solutions can be expressed in terms of elliptic functions. Also, one sees that $c_1 = 0$ and $c_2 = \lambda^2 \geq 0$ in order to have a surjective solution. In this case,

$$(h_1')^2 = \sinh^2(h_1) + \lambda^2 \quad \text{and} \quad h_2 = c_3.$$

Letting $\xi = \cosh h_1$, the equation for h_1 becomes

$$(\xi')^2 = (\xi^2 - 1)(\xi^2 - 1 + \lambda^2).$$

Therefore, the solutions are equivalent to:

1. If $\lambda = 0$, then $(x, y) \in (-\infty, +\infty) \times (-\infty, 0)$ and the map is

$$(x, y) \mapsto e^{y+ix}.$$

2. If $0 < \lambda < 1$, then $(x, y) \in (-\infty, +\infty) \times (0, K(k))$ and the map is

$$(x, y) \mapsto \frac{1 + \operatorname{cn}(y, k) - \operatorname{sn}(y, k)}{1 + \operatorname{cn}(y, k) + \operatorname{sn}(y, k)} e^{ix},$$

where $k = \sqrt{1 - \lambda^2}$ and sn, cn and $K(k)$ are Jacobi elliptic functions and the complete elliptic integral corresponding to k respectively [3, p.51].

3. If $\lambda = 1$, then $(x, y) \in (-\infty, +\infty) \times (0, 2\pi)$ and the map is

$$(x, y) \mapsto \frac{1 + \sin y - \cos y}{1 + \sin y + \cos y} e^{ix}.$$

4. If $\lambda > 1$, then $(x, y) \in (-\infty, +\infty) \times (0, K(k'))$ and the map is

$$(x, y) \mapsto \frac{1 + \operatorname{sn}(\lambda y, k') - \operatorname{cn}(\lambda y, k')}{1 + \operatorname{sn}(\lambda y, k') + \operatorname{cn}(\lambda y, k')} e^{ix},$$

where $k' = \sqrt{1 - \lambda^{-2}}$ and sn, cn and $K(k')$ are Jacobi elliptic functions and the complete elliptic integral corresponding to k' respectively [3, p.45].

One can then see that all the solutions are not diffeomorphic in this case.

Corollary 5.4. *The only quasi-conformal and group invariant harmonic diffeomorphisms from $\mathbb{H}^2$ onto itself are those obtained in Theorem 4.3.*

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