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Research Article

Solvable and Nilpotent Radicals of the Fuzzy Lie Algebras

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Abstract In this paper, we apply the concepts of fuzzy sets to Lie algebras in order to introduce and to study the notions of solvable and nilpotent fuzzy radicals. We present conditions to prove the existence and uniqueness of such radicals.

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1 Introduction

Lie algebras were discovered by Sophus Lie [4]. There are many applications of Lie algebras in several branches of physics. The notion of fuzzy sets was introduced by Zadeh [8]. Fuzzy set theory has been developed in many directions by many scholars and has evoked a great interest among mathematicians working in different fields of mathematics. Many mathematicians have been involved in extending the concepts and results of abstract algebra. The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were first introduced by Yehia in [7]. In this paper, we introduce the notion of solvable and nilpotent fuzzy radical of a fuzzy algebra of Lie algebras and investigate some of their properties. The results presented in this paper are strongly connected with the results proved in [1,2,3].

2 Fuzzy sets

In this section, we present the basic concepts on fuzzy sets which will be used throughout this paper. A new notion is introduced and results are proved for guiding the construction of the main theorems of this work.

Definition 1. A mapping of a non-empty set $\mathfrak{X}$ into the closed unit interval $[0,1]$ is called a *fuzzy set* of $\mathfrak{X}$. Let μ be any fuzzy set of $\mathfrak{X}$, then the set $\{\mu(x) \mid x \in \mathfrak{X}\}$ is called the *image* of μ and is denoted by $\mu(\mathfrak{X})$. The set $\{x \mid x \in \mathfrak{X},\ \mu(x) > 0\}$ is called the *support* of μ and is denoted by μ^*. In particular, μ is called a *finite fuzzy set* if μ^* is a finite set, and an *infinite fuzzy set* otherwise. For all real $t \in [0,1]$ the subset

$$[\mu]_t = \{x \in \mathfrak{X} \mid \mu(x) \geq t\}$$

is called a *t-level set* of μ.

Definition 2. Let $\mathfrak{X}$ be a non-empty set and $\{\nu_i\}_{i\in I}$ an arbitrary family of fuzzy sets of $\mathfrak{X}$. One defines the fuzzy set of $\mathfrak{X}$ $\bigcup_{i\in I}\nu_i$, called *union*, as

$$\Big(\bigcup_{i\in I}\nu_i\Big)(x) = \bigvee_{i\in I}\nu_i(x),$$

for all $x \in \mathfrak{X}$.

Remark 3. Let us note that if $\{\nu_i\}_{i\in I}$ is a family of fuzzy sets of $\mathfrak{X}$, then $\bigcup_{i\in I}[\nu_i]_t \subseteq \big[\bigcup_{i\in I}\nu_i\big]_t$, for all $t \in]0,1]$.

Definition 4. Let $\mathfrak{X}$ be a non-empty set. One says that a family of fuzzy sets of $\mathfrak{X}$ $\{\nu_i\}_{i\in I}$ satisfies the *second sup property* if for all $x \in \mathfrak{X}$ there is an index $i_0 = i_0(x) \in I$ such that $(\bigcup_{i\in I}\nu_i)(x) = \nu_{i_0}(x)$.

Thus, a family of fuzzy sets of $\mathfrak{X}$ $\{\nu_i\}_{i\in I}$ satisfies the second sup property if, and only if, $\big(\bigcup_{i\in I}\nu_i\big)(x) \in \{\nu_i(x) \mid i \in I\}$, for all $x \in \mathfrak{X}$.

Proposition 5. *Let $\mathfrak{X}$ be a non-empty set and $\{\nu_i\}_{i\in I}$ an arbitrary family of fuzzy sets of $\mathfrak{X}$. Then $\big[\bigcup_{i\in I}\nu_i\big]_t = \bigcup_{i\in I}[\nu_i]_t$ for all $t \in]0,1]$ if, and only if, the family $\{\nu_i\}_{i\in I}$ satisfies the second sup property.*

Proof. Let us take $x \in \mathfrak{X}$ and let $\big(\bigcup_{i\in I}\nu_i\big)(x)=\alpha$. If $\alpha=0$, then the result is evident. Now if $\alpha>0$, then $x\in \big[\bigcup_{i\in I}\nu_i\big]_\alpha$ which implies that $x\in\bigcup_{i\in I}[\nu_i]_\alpha$. It follows that there is an index $i_0\in I$ such that $x\in[\nu_{i_0}]_\alpha$. Thus, we have $\big(\bigcup_{i\in I}\nu_i\big)(x)\geq \nu_{i_0}(x)\geq\alpha$. This implies that $\big(\bigcup_{i\in I}\nu_i\big)(x)=\nu_{i_0}(x)$. So $\{\nu_i\}_{i\in I}$ satisfies the second sup property.

Now, let us suppose that $\{\nu_i\}_{i\in I}$ satisfies the second sup property and let us take $t\in]0,1]$ and $x\in\big[\bigcup_{i\in I}\nu_i\big]_t$. Since there is an index $i_0\in I$ such that $(\bigcup_{i\in I}\nu_i)(x)=\nu_{i_0}(x)$, then $x\in[\nu_{i_0}]_t$. It follows that $x\in\bigcup_{i\in I}[\nu_i]_t$ which implies that $\big[\bigcup_{i\in I}\nu_i\big]_t\subseteq\bigcup_{i\in I}[\nu_i]_t$. So $[\bigcup_{i\in I}\nu_i]_t=\bigcup_{i\in I}[\nu_i]_t$, by Remark 3. □

Example 6. Let $\mathfrak{X}$ be a non-empty set and $\mathfrak{Y}$ an arbitrary subset of $\mathfrak{X}$. Let us consider $I\subset[0,1]$ an infinite subset of the rational numbers such that $\sup I=\sqrt{2}/2$ and let us take $\{\nu_i\}_{i\in I}$ the family of the fuzzy sets of $\mathfrak{X}$, by setting $\nu_i=i\chi_{\mathfrak{Y}}$, for all $i\in I$. In this case, we have $[\bigcup_{i\in I}\nu_i]_{\sqrt{2}/2}=\mathfrak{Y}$ and $\bigcup_{i\in I}[\nu_i]_{\sqrt{2}/2}=\emptyset$.

Definition 7. A set $S\subset[0,1]$ is said to be an *upper well-ordered set* if for all non-empty subsets $C\subset S$, then $\sup C\in C$.

Definition 8. Let $\mathfrak{X}$ be a non-empty set and S an upper well-ordered set. One defines the set

$$\mathfrak{F}(\mathfrak{X},S)=\big\{\nu\mid \nu \text{ is an arbitrary fuzzy set of } \mathfrak{X} \text{ such that } \nu(\mathfrak{X})\subseteq S\big\}.$$

Proposition 9. *Let $\mathfrak{X}$ be a non-empty set and S an upper well-ordered set. Then, every family of fuzzy sets $\{\nu_i\}_{i\in I}$ of $\mathfrak{F}(\mathfrak{X},S)$ satisfies the second sup property.*

In particular, for every family of fuzzy sets $\{\nu_i\}_{i\in I}$ of $\mathfrak{F}(\mathfrak{X},S)$ one has $\bigcup_{i\in I}\nu_i\in\mathfrak{F}(\mathfrak{X},S)$.

Let $\mathfrak{X}$ be a non-empty set. All fuzzy sets of $\mathfrak{X}$ μ can be written in terms of its t-level sets $[\mu]_t$ as [9]

$$\mu(x)=\bigvee\big\{\alpha\mid x\in[\mu]_\alpha\big\}.$$

Now, let us consider the converse problem: a family $(\mathfrak{A}_\alpha)_\alpha$ of subsets of $\mathfrak{X}$ is given; is there a fuzzy set $\mu:\mathfrak{X}\to[0,1]$ such that $[\mu]_\alpha=\mathfrak{A}_\alpha$, for every α?

The necessary and sufficient conditions are given in the following.

Theorem 10. *Let $(\mathfrak{A}_\alpha)_\alpha$ be a family of subsets of $\mathfrak{X}$. The necessary and sufficient conditions for the existence of a fuzzy set $\mu:\mathfrak{X}\to[0,1]$, such that $[\mu]_\alpha=\mathfrak{A}_\alpha$ $(0\leq\alpha\leq1)$, are as follows:*

(i) $\alpha\leq\beta$ *implies that* $\mathfrak{A}_\alpha\supseteq\mathfrak{A}_\beta$,
(ii) $\alpha_1\leq\alpha_2\leq\cdots$ *and* $\alpha_n\to\alpha$ *imply that* $\bigcap_{n=1}^{\infty}\mathfrak{A}_{\alpha_n}=\mathfrak{A}_\alpha$.

The proof is given by Negoita and Ralescu [6] and will be omitted here.

3 Fuzzy algebras and ideals

In this section, we present the basic concepts of fuzzy Lie algebras, and fuzzy Lie subalgebras, fuzzy ideals and solvable (resp., nilpotent) fuzzy ideals, of a fuzzy Lie algebra. Relationships between these concepts with operations of sum and product of the fuzzy sets are studied.

Definition 11. Let $\mathfrak{L}$ be a Lie algebra over a field $\mathfrak{F}$. A fuzzy set μ of $\mathfrak{L}$ is called a *fuzzy Lie algebra* of $\mathfrak{L}$ if

(i) $\mu(ax+by)\geq\mu(x)\wedge\mu(y)$,
(ii) $\mu(xy)\geq\mu(x)\wedge\mu(y)$,
(iii) $\mu(0)=1$,

for all $a,b\in\mathfrak{F}$ and $x,y\in\mathfrak{L}$.

A fuzzy set ν of $\mathfrak{L}$ is called a *fuzzy subalgebra* of μ if ν is a fuzzy Lie algebra of $\mathfrak{L}$ satisfying $\nu(x)\leq\mu(x)$ for all $x\in\mathfrak{L}$.

Clearly, if μ is a fuzzy algebra of $\mathfrak{L}$, then μ^* is a subalgebra of $\mathfrak{L}$. Also, μ is a fuzzy Lie algebra of $\mathfrak{L}$ if, and only if, the t-level sets $[\mu]_t$ are subalgebras of $\mathfrak{L}$, for all $t\in]0,1]$. Moreover, ν is a fuzzy subalgebra of μ, if, and only if, the t-level sets $[\nu]_t$ are subalgebras of $[\mu]_t$, for all $t\in]0,1]$.

Definition 12. A fuzzy set ν of $\mathfrak{L}$ is called a *fuzzy ideal* of $\mathfrak{L}$ if

(i) $\nu(ax+by)\geq\nu(x)\wedge\nu(y)$,
(ii) $\nu(xy)\geq\nu(x)\vee\nu(y)$,
(iii) $\nu(0)=1$,

for all $a,b\in\mathfrak{F}$ and $x,y\in\mathfrak{L}$.

A fuzzy set ν of $\mathfrak{L}$ is called a *fuzzy ideal* of μ if ν is a fuzzy Lie ideal of $\mathfrak{L}$ satisfying $\nu(x) \leq \mu(x)$ for all $x \in \mathfrak{L}$.

Clearly, if ν is a fuzzy ideal of $\mathfrak{L}$, then ν^* is an ideal of $\mathfrak{L}$. Also, ν is a fuzzy ideal of $\mathfrak{L}$ if, and only if, the t-level sets $[\nu]_t$ are ideals of $\mathfrak{L}$, for all $t \in]0,1]$. Moreover, any fuzzy ideal of $\mathfrak{L}$ is a fuzzy algebra of $\mathfrak{L}$ and any fuzzy ideal of μ is a fuzzy subalgebra of μ.

Definition 13. For any fuzzy Lie algebra μ of $\mathfrak{L}$, the fuzzy set of $\mathfrak{L}$, denoted and defined by

$$o(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

is a fuzzy algebra (resp., fuzzy ideal) of μ, called the *null fuzzy algebra* of μ (resp., *null fuzzy ideal* of μ). A fuzzy set ν of $\mathfrak{L}$ is the null fuzzy algebra of μ if, and only if, $[\nu]_t = \{0\}$, for all $t \in]0,1]$.

A fuzzy Lie algebra μ of $\mathfrak{L}$ is called *abelian* if $\mu^2 = o$ and *non-abelian* otherwise.

Let us observe that if $S \subset [0,1]$ is an upper well-ordered set, then $o \in \mathfrak{F}(\mathfrak{L}, S)$ if, and only if, $0, 1 \in S$. Thus, throughout this paper we will always assume that our upper-ordered set has the real numbers 0 and 1.

Definition 14. Let $\mathfrak{L}$ be a Lie algebra over a field $\mathfrak{F}$. One defines the following:

(i) the fuzzy set $\sum_{i=1}^{n} \nu_i$ of $\mathfrak{L}$ (*sum*), if $\nu_1, \dots, \nu_n$ are fuzzy sets of $\mathfrak{L}$, as

$$\left(\sum_{i=1}^{n} \nu_i\right)(x) = \bigvee \left\{ \bigwedge_{i=1}^{n} \nu_i(x_i) \mid x = \sum_{i=1}^{n} x_i \right\},$$

for all $x \in \mathfrak{L}$,

(ii) the fuzzy set $\nu_1\nu_2$ of $\mathfrak{L}$ (*product*), if ν_1 and ν_2 are fuzzy sets of $\mathfrak{L}$, as

$$(\nu_1\nu_2)(x) = \bigvee \left\{ \bigwedge_{i=1}^{m} \{\nu_1(c_i) \wedge \nu_2(d_i)\} \mid x = \sum_{i=1}^{m} c_i d_i \right\},$$

for all $x \in \mathfrak{L}$.

Remark 15. Let us note that if ν_1, ν_2 are fuzzy sets of $\mathfrak{L}$, then $[\nu_1]_t + [\nu_2]_t \subseteq [\nu_1 + \nu_2]_t$ and $[\nu_1]_t[\nu_2]_t \subseteq [\nu_1\nu_2]_t$, for all $t \in]0,1]$.

Lemma 16. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. If ν_1 and ν_2 are fuzzy subalgebras of μ and $t \in]0,1]$, then*

(i) *for all $x \in [\nu_1 + \nu_2]_t$, there are elements $c \in [\nu_1]_t$ and $d \in [\nu_2]_t$ such that $x = c + d$ and $(\nu_1 + \nu_2)(x) = \nu_1(c) \wedge \nu_2(d)$,*

(ii) *for all $x \in [\nu_1\nu_2]_t$, there are elements $c_i \in [\nu_1]_t$ and $d_i \in [\nu_2]_t$ $(i = 1, \dots, n)$ such that $x = \sum_{i=1}^{n} c_i d_i$ and $(\nu_1\nu_2)(x) = \bigwedge_{i=1}^{n} \{\nu_1(c_i) \wedge \nu_2(d_i)\}$.*

In particular, the following holds:

(iii) $[\nu_1 + \nu_2]_t = [\nu_1]_t + [\nu_2]_t$,

(iv) $[\nu_1\nu_2]_t = [\nu_1]_t[\nu_2]_t$, *for all* $t \in]0,1]$.

Proof. (i) and (iii) Since $\mathfrak{L}$ is finite dimensional, then both ν_1 and ν_2 take finite values in $[0,1]$, by [5]. This implies that there are two finite sets of real numbers $\{r_0 = 0 < r_1 < \cdots < r_{m-1} < 1 = r_m\}$ and $\{s_0 = 0 < s_1 < \cdots < s_{n-1} < 1 = s_n\}$ such that $[\nu_1]_r = [\nu_1]_{r_i}$, for all $r \in]r_{i-1}, r_i]$ $(i = 1, \dots, m)$, and $[\nu_2]_s = [\nu_2]_{s_j}$, for all $s \in]s_{j-1}, s_j]$ $(j = 1, \dots, n)$. Thus, for $t \in]0,1]$ and for $x \in [\nu_1 + \nu_2]_t$, let $(\nu_1 + \nu_2)(x) = \alpha$. Then, there are integers $1 \leq i \leq m$ and $1 \leq j \leq n$ such that $r_{i-1} < \alpha \leq r_i$ and $s_{j-1} < \alpha \leq s_j$ which implies $(r_{i-1} \vee s_{j-1}) < \alpha \leq (r_i \wedge s_j)$. It follows that there are elements $c, d \in \mathfrak{L}$ such that $x = c + d$ and $\nu_1(c) \wedge \nu_2(d) > r_{i-1} \vee s_{j-1}$. Hence, $\nu_1(c) > r_{i-1}$ and $\nu_2(d) > s_{j-1}$, which implies that $\nu_1(c) \geq r_i$ and $\nu_2(d) \geq s_j$, that is, $\nu_1(c) \wedge \nu_2(d) \geq r_i \wedge s_j \geq \alpha$. This implies that $(\nu_1 + \nu_2)(x) = \nu_1(c) \wedge \nu_2(d)$. Moreover, since $c \in [\nu_1]_t$ and $d \in [\nu_2]_t$, then $[\nu_1 + \nu_2]_t \subset [\nu_1]_t + [\nu_2]_t$. From Remark 15 we obtain $[\nu_1 + \nu_2]_t = [\nu_1]_t + [\nu_2]_t$.

Similarly, we prove the cases (ii) and (iv). $\square$

Definition 17. Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. For any fuzzy subalgebra ν of μ we define inductively the *derived series* of ν as the descending chain of fuzzy subalgebras of μ $\nu^{(1)} \geq \nu^{(2)} \geq \nu^{(3)} \geq \cdots$, by setting $\nu^{(1)} = \nu$ and $\nu^{(n+1)} = (\nu^{(n)})^2$ for every $n \geq 1$, and the *lower central series* of ν as the descending chain of fuzzy algebras of μ $\nu^1 \geq \nu^2 \geq \nu^3 \geq \cdots$, by defying $\nu^1 = \nu$ and $\nu^n = \nu\nu^{n-1}$ for every $n \geq 2$.

The fuzzy subalgebra ν is said *solvable* (resp., *nilpotent*) if there exists an integer $k = k(\nu) \geq 1$ such that $\nu^{(k)} = o$ (resp., $\nu^k = o$). The smallest strict positive integer k such that $\nu^{(k)} = o$ (resp., $\nu^k = o$) is called the *index of solvability* (resp., *index of nilpotency*) of ν.

Clearly, Definition 17 generalizes the concept of solvability (resp., nilpotency) as defined in the class of the Lie algebras [4].

Corollary 18. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. If ν is a fuzzy subalgebra of μ, then*

$$\left[\nu^{(n)}\right]_t = \left[\nu_t\right]^{(n)} \left(\textit{resp.},\ \left[\nu^n\right]_t = \left[\nu_t\right]^n\right),$$

for all integer $n \geq 1$ and $t \in]0,1]$.

Thus, ν is solvable (resp., nilpotent) if, and only if, there exists an integer $k = k(\nu) \geq 1$ such that ${[\nu]_t}^{(k)} = \{0\}$ (resp., ${[\nu]_t}^k = \{0\}$), for all $t \in]0,1]$.

Proposition 19. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. If ν_1, ν_2 are fuzzy ideals of μ, then the sum $\nu_1 + \nu_2$ and the product $\nu_1\nu_2$ are fuzzy ideals of μ. Moreover,*

(i) $\left([\nu_1]_t + [\nu_2]_t\right)^{(2l+1)} \subseteq [\nu_1]_t^{(l)} + [\nu_2]_t^{(l)}$, *for all $t \in]0,1]$ and integer $l \geq 1$,*
(ii) *each non-associative product of l $(= l_{\nu_1} + l_{\nu_2})$ terms $[\nu_{i_1}]_t \cdots [\nu_{i_l}]_t$, $i_j = 1$ or 2 $(1 \leq j \leq l)$, where l_{ν_1} terms are formed by the t-level sets $[\nu_1]_t$ and l_{ν_2} terms are formed by the t-level sets $[\nu_2]_t$, is a subset of both the sets $[\nu_1]_t^{l_{\nu_1}}$ and $[\nu_2]_t^{l_{\nu_2}}$.*

Proof. The first part of the proposition can be easily shown. Next, by the Jacobi identity and using the Principle of Mathematical Induction, we can also demonstrate that

(a) $[\nu_1]_t^{(l)}[\nu_2]_t^{(l)} \subseteq [\nu_i]_t^{(l)}$ $(i = 1,2)$, for all integer $l \geq 1$,
(b) $[\nu_i]_t^{(l)}[\nu_j]_t^{(l+1)} \subseteq [\nu_j]_t^{(l+1)}$ $(i,j = 1,2;\ i \neq j)$, for all integer $l \geq 1$.

Now

$$\left(\left[\nu_1\right]_t + \left[\nu_2\right]_t\right)^{(3)} \subseteq \left[\nu_1\right]_t^{(1)} + \left[\nu_2\right]_t^{(1)}.$$

Then using again the Principle of Mathematical Induction, (a), and (b) we have, for an integer $l \geq 1$,

$$\begin{aligned}\left(\left[\nu_1\right]_t + \left[\nu_2\right]_t\right)^{(2l+3)} &= \left(\left(\left[\nu_1\right]_t + \left[\nu_2\right]_t\right)^{(2l+2)}\right)^2 = \left(\left(\left(\left[\nu_1\right]_t + \left[\nu_2\right]_t\right)^{(2l+1)}\right)^2\right)^2 \\ &\subseteq \left(\left(\left[\nu_1\right]_t^{(l)} + \left[\nu_2\right]_t^{(l)}\right)^2\right)^2 \subseteq \left(\left[\nu_1\right]_t^{(l+1)} + \left[\nu_1\right]_t^{(l)} + \left[\nu_2\right]_t^{(l+1)}\right)^2 \subseteq \left[\nu_1\right]_t^{(l+1)} + \left[\nu_2\right]_t^{(l+1)}.\end{aligned}$$

So $\left([\nu_1]_t + [\nu_2]_t\right)^{(2l+1)} \subseteq [\nu_1]_t^{(l)} + [\nu_2]_t^{(l)}$, for all integer $l \geq 1$.

Next, by the Jacobi identity and the Principle of Mathematical Induction again, we can demonstrate that

(c) $[\nu_i]_t^{l_1}[\nu_i]_t^{l_2} \subseteq [\nu_i]_t^{l_1+l_2}$ $(i = 1,2)$, for any integers $l_1, l_2 \geq 1$,

Now (ii) is evident, for $l = 2$. Hence, let us consider an integer l $(= l_{\nu_1} + l_{\nu_2}) \geq 2$ and a non-associative product of l terms $[\nu_{i_1}]_t \cdots [\nu_{i_l}]_t$, where $i_j = 1$ or 2 $(1 \leq j \leq l)$, l_{ν_1} terms are formed by the t-level sets $[\nu_1]_t$ and l_{ν_2} terms are formed by the t-level sets $[\nu_2]_t$. It follows that we can write the previous product as a product of two non-associative products

$$\left[\nu_{i_1}\right]_t \cdots \left[\nu_{i_l}\right]_t = \left(\left[\nu_{p_1}\right]_t \cdots \left[\nu_{p_r}\right]_t\right)\left(\left[\nu_{q_1}\right]_t \cdots \left[\nu_{q_s}\right]_t\right),$$

$p_i, q_j = 1$ or 2 $(1 \leq i \leq r; 1 \leq j \leq s)$, where $[\nu_{p_1}]_t \cdots [\nu_{p_r}]_t$ and $[\nu_{q_1}]_t \cdots [\nu_{q_s}]_t$ are products of r $(= r_{\nu_1} + r_{\nu_2})$ and s $(= s_{\nu_1} + s_{\nu_2})$ terms, respectively, with r_{ν_1} and s_{ν_1} terms formed by the t-level sets $[\nu_1]_t$ and, r_{ν_2} and s_{ν_2} terms formed by the t-level sets $[\nu_2]_t$. It follows that $l = r + s$, $l_{\nu_1} = r_{\nu_1} + s_{\nu_1}$, and $l_{\nu_2} = r_{\nu_2} + s_{\nu_2}$. From the Principle of Mathematical Induction, we have that $[\nu_{p_1}]_t \cdots [\nu_{p_r}]_t$ is a subset of both t-level sets $[\nu_1]_t^{r_{\nu_1}}$ and $[\nu_2]_t^{r_{\nu_2}}$ and $[\nu_{q_1}]_t \cdots [\nu_{q_s}]_t$ is a subset of both t-level sets $[\nu_1]_t^{s_{\nu_1}}$ and $[\nu_2]_t^{s_{\nu_2}}$. This implies that $[\nu_{i_1}]_t \cdots [\nu_{i_l}]_t$ is a subset of both the sets $[\nu_1]_t^{r_{\nu_1}} [\nu_1]_t^{s_{\nu_1}}$ and $[\nu_2]_t^{r_{\nu_2}} [\nu_2]_t^{s_{\nu_2}}$. From the condition (c), we conclude that $[\nu_{i_1}]_t \cdots [\nu_{i_l}]_t$ is a subset of both the sets $[\nu_1]_t^{l_{\nu_1}}$ and $[\nu_1]_t^{l_{\nu_2}}$.

The proposition is proved. □

Proposition 20. *Let μ be a fuzzy Lie algebra of $\mathfrak{L}$. If ν_1 and ν_2 are solvable (resp., nilpotent) fuzzy ideals of μ, then $\nu_1 + \nu_2$ and the $\nu_1\nu_2$ are also solvable (resp., nilpotent) fuzzy ideals of μ.*

Proof. Let $k_{\nu_1} = k_{\nu_1}(t) \geq 1$ and $k_{\nu_2} = k_{\nu_2}(t) \geq 1$ be the indices of solvability of ν_1 and ν_2, respectively. Let us take $k = k(t) = \max\{k_{\nu_1}, k_{\nu_2}\}$. It follows that $\left[(\nu_1+\nu_2)^{(2k+1)}\right]_t = \{0\}$, for all $t \in]0,1]$, by Proposition 19(i), Lemma 16(iii) and Corollary 18. Thus $\nu_1 + \nu_2$ is a solvable fuzzy ideal of μ.

Now, let $k_{\nu_1} = k_{\nu_1}(t) \geq 1$ and $k_{\nu_2} = k_{\nu_2}(t) \geq 1$ be the indices of nilpotency of ν_1 and ν_2, respectively. Let us take $k = k(t) = k_{\nu_1} + k_{\nu_2}$. Then, for all $t \in]0,1]$, we have $\left[(\nu_1+\nu_2)^k\right]_t = \{0\}$, Proposition 19(ii), Lemma 16(iii) and Corollary 18. So $\nu_1 + \nu_2$ is a nilpotent fuzzy ideal of μ. □

Corollary 21. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. If S is an upper well-ordered set and ν_1 and ν_2 are solvable (resp., nilpotent) fuzzy ideals of μ in $\mathfrak{F}(\mathfrak{L}, S)$, then $\nu_1 + \nu_2$ and $\nu_1\nu_2$ are solvable (resp., nilpotent) fuzzy ideals of μ in $\mathfrak{F}(\mathfrak{L}, S)$.*

4 The solvable and nilpotent fuzzy radical

In this section, we present the main results of this paper. We prove that every fuzzy Lie algebra has a unique maximal solvable (resp., nilpotent) fuzzy ideal, called the solvable (resp., nilpotent) fuzzy radical.

Let us begin by introducing the following definition.

Definition 22. Let μ be a fuzzy Lie algebra of $\mathfrak{L}$ and S an upper well-ordered set. One says that a fuzzy subalgebra ν of μ is a *maximal element* of μ in $\mathfrak{F}(\mathfrak{L}, S)$ if $\nu \in \mathfrak{F}(\mathfrak{L}, S)$ and for every fuzzy algebra ν^* of μ in $\mathfrak{F}(\mathfrak{L}, S)$ such that $\nu \subset \nu^*$, then $\nu = \nu^*$. In this case, one says that the fuzzy algebra μ has a *maximum* in $\mathfrak{F}(\mathfrak{L}, S)$.

Theorem 23. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and S an upper well-ordered set. Then each fuzzy algebra μ of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ has a maximal solvable (resp., nilpotent) fuzzy ideal in $\mathfrak{F}(\mathfrak{L}, S)$.*

Proof. Let μ be a fuzzy algebra of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ and let us consider the set

$$\Xi = \left\{\nu \mid \nu \text{ is a solvable fuzzy ideal of } \mu \text{ in } \mathfrak{F}(\mathfrak{L}, S)\right\}.$$

Obviously, the set Ξ is non-empty and partially ordered by $\leq$. Let us take a subset $\{\nu_i\}_{i\in I}$ of Ξ totally ordered by $\leq$. Let us show that $\nu = \bigcup_{i\in I}\nu_i$ is an upper bound of $\{\nu_i\}_{i\in I}$ in Ξ. In fact, for every $i \in I$, we have $\nu_i(x) \leq \mu(x)$ for all $x \in \mathfrak{L}$. Hence

$$\nu(x) = \left(\bigcup_{i\in I}\nu_i\right)(x) = \bigvee_{i\in I}\nu_i(x) \leq \bigvee_{i\in I}\mu(x) \leq \mu(x),$$

for all $x \in \mathfrak{L}$. Now let us consider arbitrary elements $i, j \in I$. As the set $\{\nu_i\}_{i\in I}$ is totally ordered by $\leq$, then either $\nu_i \leq \nu_j$ or $\nu_j \leq \nu_i$ implies either

$$\nu_j(x) \wedge \nu_j(y) \geq \nu_i(x) \wedge \nu_j(y) \text{ or } \nu_i(x) \wedge \nu_i(y) \geq \nu_i(x) \wedge \nu_j(y),$$

for all $x, y \in \mathfrak{L}$. Thus, for all $x, y \in \mathfrak{L}$ we have

$$\nu(x+y) = \left(\bigcup_{i\in I}\nu_i\right)(x+y) = \bigvee_{i\in I}\nu_i(x+y) \geq \bigvee_{i\in I}\left(\nu_i(x) \wedge \nu_i(y)\right) \geq \left(\bigvee_{i\in I}\nu_i(x)\right)\bigwedge\left(\bigvee_{j\in I}\nu_j(y)\right) = \nu(x) \wedge \nu(y),$$

Next, for all $a \in \mathfrak{F}$ and $x \in \mathfrak{L}$ we have

$$\nu(ax) = \left(\bigcup_{i\in I}\nu_i\right)(ax) = \bigvee_{i\in I}\nu_i(ax) \geq \bigvee_{i\in I}\nu_i(x) = \nu(x).$$

Also, for all $x, y \in \mathfrak{L}$ we have

$$\nu(xy) = \left(\bigcup_{i\in I}\nu_i\right)(xy) = \bigvee_{i\in I}\nu_i(xy) \geq \bigvee_{i\in I}\big(\nu_i(x)\vee\nu_i(y)\big) = \Big(\bigvee_{i\in I}\nu_i(x)\Big)\bigvee\Big(\bigvee_{j\in I}\nu_j(y)\Big) = \nu(x)\vee\nu(y),$$

Finally, we have

$$\nu(0) = \left(\bigcup_{i\in I}\nu_i\right)(0) = \bigvee_{i\in I}\nu_i(0) = 1.$$

So ν is a fuzzy ideal of μ.

Now, let us consider $t \in]0,1]$. Since $\mathfrak{L}$ is finite dimensional, then $\mathfrak{L}$ has a unique maximal solvable ideal. It follows that each t-level set $[\nu_i]_t$, for $i \in I$, is also a solvable ideal of $\mathfrak{L}$ which implies that $\bigcup_{i\in I}[\nu_i]_t$ is a solvable ideal of $\mathfrak{L}$, because the family $\{\nu_i\}_{i\in I}$ of Ξ is totally ordered by $\leq$. By Propositions 5 and 9, we have that $\big[\bigcup_{i\in I}\nu_i\big]_t$ is a solvable ideal of $\mathfrak{L}$. Therefore, ν is a solvable fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L},S)$ and so an upper bound of $\{\nu_i\}_{i\in I}$ in Ξ. From Zorn's lemma, Ξ possesses at least one maximal element.

Similarly, we prove the nilpotent case. □

Theorem 24. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and S an upper well-ordered set. Then, every solvable (resp., nilpotent) fuzzy ideal ν of μ in $\mathfrak{F}(\mathfrak{L},S)$ is contained in a unique maximal solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L},S)$, called solvable (resp., nilpotent) fuzzy radical of μ in $\mathfrak{F}(\mathfrak{L},S)$ and denoted by $\mathfrak{R}(\mu,S)$ (resp., $\mathfrak{N}(\mu,S)$).*

Proof. Let $\mathfrak{S}$ be a maximal solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L},S)$. If ν is a solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L},S)$, then $\mathfrak{S}+\nu$ is a solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L},S)$, by Corollary 21, and $\mathfrak{S}(x) = \mathfrak{S}(x)\wedge\nu(0) \leq \bigvee\{\mathfrak{S}(c)\wedge\nu(d) \mid x = c+d\} = (\mathfrak{S}+\nu)(x)$, for all $x \in \mathfrak{L}$. Thus $\mathfrak{S} \leq \mathfrak{S}+\nu$. Since $\mathfrak{S}$ is maximal, then $\mathfrak{S}+\nu \leq \mathfrak{S}$. So $\mathfrak{S}+\nu = \mathfrak{S}$. Hence $\nu(x) = \mathfrak{S}(0)\wedge\nu(x) \leq \bigvee\{\mathfrak{S}(c)\wedge\nu(d) \mid x = c+d\} = (\mathfrak{S}+\nu)(x)$, for all $x \in \mathfrak{L}$. So $\nu \leq \mathfrak{S}$. Let $\mathfrak{R}(\mu,S) = \mathfrak{S}$ (resp., $\mathfrak{N}(\mu,S) = \mathfrak{S}$) be. □

5 Simple and semisimple fuzzy Lie ideals

In this section, we introduce the notions of simple and semisimple fuzzy ideals and establish relations with the solvable fuzzy radical.

Definition 25. Let $\mathfrak{L}$ be a Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie ideal of $\mathfrak{L}$. One says that μ is a *simple fuzzy ideal* if

(i) μ is a fuzzy ideal non-abelian;
(ii) for all fuzzy ideals ν of μ, one has either $[\nu]_t = [\mu]_t$ or $[\nu]_t = \{0\}$, for all $t \in]0,1]$.

Definition 26. Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$, S an upper well-ordered set and μ a fuzzy ideal of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L},S)$. One says that μ is a *semisimple fuzzy ideal* in $\mathfrak{F}(\mathfrak{L},S)$ if

(i) μ is a fuzzy ideal non-abelian;
(ii) its solvable fuzzy radical in $\mathfrak{F}(\mathfrak{L},S)$ is o, that is, $\mathfrak{R}(\mu,S) = o$.

Theorem 27. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie ideal of $\mathfrak{L}$ non-abelian. If μ is a simple fuzzy ideal, then μ^* is a non solvable ideal of $\mathfrak{L}$.*

Moreover, μ is a simple fuzzy ideal if, and only if, μ^* is a minimal ideal of $\mathfrak{L}$.

Proof. First, let us observe that μ^* is not a zero ideal, since μ^2 is non-null. Since $\mathfrak{L}$ is finite dimensional, there is a finite set of real numbers $\{r_0 = 0 < r_1 < \cdots < r_{m-1} < 1 = r_m\}$ such that $[\mu]_t = [\mu]_{r_i}$ for all $t \in]r_{i-1}, r_i]$ $(i = 1,\dots,m)$. This implies that $[\mu]_t = [\mu]_{r_1} = \mu^*$, for all $t \in]0,r_1]$. If μ^* is a solvable ideal of $\mathfrak{L}$, then $\mu^* \supsetneq (\mu^*)^{(2)} \supsetneq \cdots$ and there is an integer $k \geq 1$ such that $(\mu^*)^{(k)} = \{0\}$. Thus, by Theorem 10, we can construct the fuzzy set ν of $\mathfrak{L}$ defined by t-level sets; $[\nu]_0 = \mathfrak{L}$, $[\nu]_t = (\mu^*)^{(2)}$ for all $t \in]0,r_1]$, and $[\nu]_t = \{0\}$ for all $t \in]r_1,1]$. It follows that ν is a solvable fuzzy ideal of $\mathfrak{L}$ such that $[\nu]_t \neq [\mu]_t$ for some $t \in]0,r_1]$. This an absurd.

Now let us consider $\mathfrak{I}$ an ideal of $\mathfrak{L}$ with $\mathfrak{I} \subseteq \mu^*$. By Theorem 10, let us consider the fuzzy set ν of $\mathfrak{L}$ defined by t-level sets; $[\nu]_0 = \mathfrak{L}$, $[\nu]_t = \mathfrak{I}$ for all $t \in]0,r_1]$, and $[\nu]_t = \{0\}$ for all $t \in]r_1,1]$. Clearly, ν is a fuzzy ideal of μ and so we have either $[\nu]_t = [\mu]_t$ or $[\nu]_t = \{0\}$, for all $t \in]0,1]$. It follows that $\mathfrak{I} = \{0\}$ or $\mathfrak{I} = \mu^*$. So μ^* is a minimal ideal of $\mathfrak{L}$. Reciprocally, let us consider a fuzzy ideal ν of μ. Then the t-level set $[\nu]_t$ is an ideal of the algebra $\mathfrak{L}$ and $[\nu]_t \subseteq [\mu]_t \subseteq \mu^*$, for all $t \in]0,1]$. Since μ^* is a minimal ideal of $\mathfrak{L}$, then either $[\nu]_t = \mu^*$ or $[\nu]_t = \{0\}$, for all $t \in]0,1]$. This implies that $[\nu]_t = [\mu]_t$ or $[\nu]_t = \{0\}$, for all $t \in]0,1]$. So μ is a simple fuzzy ideal. □

Corollary 28. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie ideal of $\mathfrak{L}$. If μ is a fuzzy simple ideal, then $[\mu]_t = \mu^*$ or $[\mu]_t = \{0\}$, for all $t \in]0,1]$.*

Theorem 29. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and S an upper well-ordered set. Then any simple fuzzy ideal μ of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ is semisimple in $\mathfrak{F}(\mathfrak{L}, S)$.*

Proof. Let us consider ν a solvable fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$. Then $[\nu]_t \subseteq \mu^*$ is a solvable ideal of $\mathfrak{L}$, for all $t \in]0,1]$. Since μ is simple, then $[\nu]_t = \{0\}$ for all $t \in]0,1]$, by Theorem 27, which implies $\nu = o$. Consequently, $\mathfrak{R}(\mu, S) = o$. □

Theorem 30. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$, S an upper well-ordered set, and μ a fuzzy ideal of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ non-abelian. If μ is semisimple in $\mathfrak{F}(\mathfrak{L}, S)$, then μ^* is a non-solvable ideal of $\mathfrak{L}$.*

Moreover, μ is semisimple in $\mathfrak{F}(\mathfrak{L}, S)$ if, and only if, μ^* does not contain non-trivial solvable ideals of $\mathfrak{L}$.

Proof. First, from a similar construction, as shown in the demonstration of Theorem 27, if μ^* is a solvable ideal of $\mathfrak{L}$, then we can construct the fuzzy set ν of $\mathfrak{L}$ defined by t-level sets; $[\nu]_0 = \mathfrak{L}$, $[\nu]_t = (\mu^*)^{(2)}$ for all $t \in]0, r_1]$, and $[\nu]_t = \{0\}$ for all $t \in]r_1, 1]$, by Theorem 10. Its follows that ν is a solvable fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$. This implies that $\nu = o$ and so μ is abelian. This is absurd.

Now, let us consider $\mathfrak{J}$ as a solvable ideal of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ with $\mathfrak{J} \subseteq \mu^*$. By Theorem 10 again, let us consider the fuzzy set ν of μ defined by $[\nu]_0 = \mathfrak{L}$, $[\nu]_t = \mathfrak{J}$ for all $t \in]0, r_1]$, and $[\nu]_t = \{0\}$ for all $t \in]r_1, 1]$. Clearly, ν is a solvable fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$ which implies that $\nu = 0$. Thus, we have $\mathfrak{J} = 0$. So μ^* does not contain non-trivial solvable ideals of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$. Reciprocally, let us consider a solvable fuzzy ideal ν of μ in $\mathfrak{F}(\mathfrak{L}, S)$. Then, for all $t \in]0,1]$ the t-level set $[\nu]_t$ is a solvable ideal of $\mathfrak{L}$ such that $[\nu]_t \subseteq [\mu]_t \subseteq \mu^*$. This implies that $[\nu]_t = \{0\}$, for all $t \in [0,1]$. This implies that $\nu = o$. So $\mathfrak{R}(\mu, S) = o$. □

References

[1] M. Akram, *Fuzzy Lie ideals of Lie algebras with interval-valued membership functions*, Quasigroups Related Systems, 16 (2008), 1–12.
[2] M. Akram, *Generalized fuzzy Lie subalgebras*, J. Gen. Lie Theory Appl., 2 (2008), 261–268.
[3] M. Akram, *Redefined fuzzy Lie subalgebras*, Quasigroups Related Systems, 16 (2008), 119–132.
[4] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, vol. 9 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1972.
[5] R. Lowen, *Convex fuzzy sets*, Fuzzy Sets and Systems, 3 (1980), 291–310.
[6] C. V. Negoiţă and D. A. Ralescu, *Applications of fuzzy sets to systems analysis*, John Wiley & Sons, New York, 1975.
[7] S. E.-B. Yehia, *Fuzzy ideals and fuzzy subalgebras of Lie algebras*, Fuzzy Sets and Systems, 80 (1996), 237–244.
[8] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338–353.
[9] L. A. Zadeh, *Similarity relations and fuzzy orderings*, Information Sci., 3 (1971), 177–200.