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Quantization of the q-analog Virasoro-like algebras

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Abstract

We use the general method of quantization by Drinfel'd twist element to quantize explicitly the Lie bialgebra structures on the q-analog Virasoro-like algebras studied in *Comm. Algebra*, **37** (2009), 1264–1274.

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1 Introduction

The study of Lie bialgebras [1, 2] is now well established as an infinitesimalization of the notion of a quantum group or Hopf algebra. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ provided with a Lie cobracket which is related to the Lie bracket by a certain compatibility condition. According to quantum groups theory, a quantum group is essentially a formal deformation of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, the semiclassical structure associated with such a deformation is a Lie bialgebra structure on $\mathfrak{g}$. Constructing quantizations of Lie bialgebras is an important method to produce new quantum groups. Using the method twisting the coproduct by a Drinfel'd twist element but keeping the product unchanged, Grunspan [3] presented the quantization of a class of infinite dimensional Lie algebras containing Virasoro algebras studied in [4] (see also [5, 6]). Using the same technique, Hu and Wang [7] quantized some Lie algebras presented in [8]. In a recent paper [9], the Lie bialgebra structures of q-analog Virasoro-like algebras $\mathfrak{L}$ with the basis $\{L_\alpha, d_1, d_2 \mid \alpha \in \mathbb{Z}^2\backslash\{(0,0)\}\}$ and brackets

$$\big[L_\alpha, L_\beta\big] = \big(q^{\alpha_2\beta_1} - q^{\alpha_1\beta_2}\big)L_{\alpha+\beta}, \quad \big[d_i, L_\alpha\big] = \alpha_i L_\alpha,\ i = 1, 2, \tag{1.1}$$

for $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2\backslash\{(0,0)\}$, were considered, where $0 \neq q \in \mathbb{C}$ is a fixed non-root of unity. Here we treat $L_{0,0}$ as zero. Obviously, the Lie algebra $\mathfrak{L}$ is $\mathbb{Z}^2$-graded (however its structural constant $q^{\alpha_2\beta_1} - q^{\alpha_1\beta_2}$ is not linearly dependent on the gradings α, β; in this case, the Lie algebra $\mathfrak{L}$ is called *non-linear*). This Lie algebra is closely related to the Virasoro and Virasoro-like algebras and the Lie algebras of Cartan type S and H (cf. [15, 16]), which is probably why this type of Lie algebras has attracted some attentions in the literature (cf. [10, 11, 12, 13, 14, 17, 18]).

In this paper, we will use the techniques developed in [3, 7] to construct the quantization of this type of bialgebra. However, since in our case the Lie algebra is non-linear, some of our arguments may render rather technical.

We fix a field $\mathbb{F}$ of characteristic zero. Let $\mathcal{A}$ be a unitary $\mathcal{R}$-algebra ($\mathcal{R}$ is a ring). For $z \in \mathcal{A}$, $n \in \mathbb{Z}$, we set

$$z^{\langle n\rangle} = z(z+1)\cdots(z+n-1), \quad z^{[n]} = z(z-1)\cdots(z-n+1)$$

and set $z^{\langle 0\rangle} = 1$ and $z^{[0]} = 1$. If $a \in \mathcal{R}$ is any scalar, set $z_a^{\langle n\rangle} = (z+a)^{\langle n\rangle}$ and $z_a^{[n]} = (z+a)^{[n]}$, that is

$$z_a^{\langle n\rangle} = (z+a)(z+a+1)\cdots(z+a+n-1), \tag{1.2}$$

$$z_a^{[n]} = (z+a)(z+a-1)\cdots(z+a-n+1). \tag{1.3}$$

Obviously $z^{\langle n\rangle} = z_0^{\langle n\rangle}$, $z^{[n]} = z_0^{[n]}$.

The following lemma can be found in [3].

Lemma 1.1. *Let z be any element of a unitary $\mathbb{F}$-algebras $\mathcal{A}$. For $a, d \in \mathbb{F}$, and $m, n, r \in \mathbb{Z}$, one has*

$$z_a^{\langle m+n\rangle} = z_a^{\langle m\rangle} z_{a+m}^{\langle n\rangle}, \quad z_a^{[m+n]} = z_a^{[m]} z_{a-m}^{[n]}, \quad z_a^{[m]} = z_{a-m+1}^{\langle m\rangle}, \tag{1.4}$$

$$\begin{aligned} &\sum_{m+n=r} \frac{(-1)^n}{m!n!} z_a^{[m]} z_d^{\langle n\rangle} = \binom{a-d}{r}, \\ &\sum_{m+n=r} \frac{(-1)^n}{m!n!} z_a^{[m]} z_{d-m}^{[n]} = \binom{a-d+r-1}{r}, \end{aligned} \tag{1.5}$$

where in general $\binom{a}{b}$ is the binomial coefficient.

Denote by $(U(\mathfrak{L}), \mu, \tau, \Delta_0, S_0, \epsilon_0)$ the natural Hopf algebra structure on $U(\mathfrak{L})$ (the universal enveloping algebra of the Lie algebra $\mathfrak{L}$), that is, the coproduct Δ_0, the antipode S_0 and the counit ϵ_0 are respectively defined by

$$\begin{aligned} &\Delta_0\big(L_\beta\big) = L_\beta \otimes 1 + 1 \otimes L_\beta, \quad \Delta_0\big(d_i\big) = d_i \otimes 1 + 1 \otimes d_i, \\ &S_0\big(L_\beta\big) = -L_\beta, \quad S_0\big(d_i\big) = -d_i, \\ &\epsilon_0\big(L_\beta\big) = 0, \quad \epsilon_0\big(d_i\big) = 0 \quad \text{for } \beta \in \mathbb{Z}^2 \setminus \{(0,0)\},\ i = 1,2. \end{aligned}$$

The following definition and well-known result can be found in [2].

Definition 1.2. Let $(\mathcal{H}, \mu, \tau, \Delta_0, S_0, \epsilon_0)$ be a Hopf algebra over a commutative ring. An element $\mathscr{F} \in \mathcal{H} \otimes \mathcal{H}$ is called Drinfel'd twist element, if it is invertible such that

$$(\mathscr{F} \otimes 1)\big(\Delta_0 \otimes Id\big)(\mathscr{F}) = (1 \otimes \mathscr{F})\big(Id \otimes \Delta_0\big)(\mathscr{F}), \tag{1.6}$$

$$(\epsilon_0 \otimes Id)(\mathscr{F}) = 1 \otimes 1 = \big(Id \otimes \epsilon_0\big)(\mathscr{F}). \tag{1.7}$$

Lemma 1.3. *Let $(\mathcal{H}, \mu, \tau, \Delta_0, S_0, \epsilon_0)$ be a Hopf algebra over a commutative ring, and let $\mathscr{F}$ be a Drinfel'd twist element of $\mathcal{H} \otimes \mathcal{H}$, then*

(1) $\mathscr{U} = \mu(Id \otimes S_0)(\mathscr{F})$ is an invertible element of $\mathcal{H}$ with $\mathscr{U}^{-1} = \mu(S_0 \otimes Id)(\mathscr{F}^{-1})$;

(2) the algebra $(\mathcal{H}, \mu, \tau, \Delta, S, \epsilon)$ is a new Hopf algebra if we keep the counit undeformed (i.e., $\epsilon = \epsilon_0$) and define $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $S : \mathcal{H} \to \mathcal{H}$ by

$$\Delta(h) = \mathscr{F} \Delta_0(h) \mathscr{F}^{-1}, \quad S(h) = u S_0(h) u^{-1}.$$

Let $(U(\mathfrak{g}),\mu,\tau,\Delta_0,S_0,\epsilon_0)$ be the natural Hopf algebra structure, where $\mathfrak{g}$ is a triangular Lie bialgebra, and denote by $U(\mathfrak{g})[[t]]$ an associative $\mathbb{F}$-algebra of formal power series with coefficients in $U(\mathfrak{g})$. Naturally, $U(\mathfrak{g})[[t]]$ is equipped with an induced Hopf algebra structure arising from that on $U(\mathfrak{g})$.

Definition 1.4. For a triangular Lie bialgebra $\mathfrak{g}$ over $\mathbb{F}$, the Hopf algebra $(U(\mathfrak{g})[[t]],\mu,\tau,\Delta_r, S_r,\epsilon_0)$ is called a quantization of $(U(\mathfrak{g}),\mu,\tau,\Delta_0,S_0,\epsilon_0)$ by a Drinfel'd twist element $\mathscr{F}$, if $U(\mathfrak{g})[[t]]/tU(\mathfrak{g})[[t]]\cong U(\mathfrak{g})$ and $\mathscr{F}$ is determined by its r-matrix r.

We will fix the following notations, for $x_1,x_2\in\mathbb{Z}$,

$$\begin{aligned}&T=x_1d_1+x_2d_2\in\operatorname{span}\{d_1,d_2\},\\&E=L_\alpha \text{ for } \alpha\in\mathbb{Z}^2\setminus(0,0)\quad\text{satisfying } [T,E]=E.\end{aligned}\tag{1.8}$$

The following result is obtained in [9].

Lemma 1.5. *There is a triangular Lie bialgebra structure on the Lie algebras $\mathfrak{L}$ given by the r-matrix $T\otimes E-E\otimes T$, where T and E are defined in* (1.8).

The main result of this paper is the following theorem.

Theorem 1.6. *Let $\mathfrak{L}$ be the q-analog Virasoro-like algebras with $[T,E]=E$ (cf. (1.8)), then there exists a noncommutative and noncocommutative Hopf algebra structure $(U(\mathfrak{L})[[t]],\mu,\tau,\Delta,S,\epsilon)$ on $U(\mathfrak{L})[[t]]$, such that $U(\mathfrak{L})[[t]]/tU(\mathfrak{L})[[t]]=U(\mathfrak{L})$, which preserves the product and the counit of $U(\mathfrak{L})[[t]]$, but the coproduct and antipode are defined by*

$$\Delta(L_\beta)=L_\beta\otimes(1-Et)^c+\sum_{k=0}^{\infty}(-1)^k a_k T^{\langle k\rangle}\otimes(1-Et)^{-k}L_{\beta+k\alpha}t^k,\tag{1.9}$$

$$\Delta(d_i)=d_i\otimes 1+1\otimes d_i+\alpha_i T\otimes(1-Et)^{-1}Et,\tag{1.10}$$

$$S(L_\beta)=-(1-Et)^{-c}\sum_{k=0}^{\infty}a_kL_{\beta+k\alpha}T_1^{\langle k\rangle}t^k,\tag{1.11}$$

$$S(d_i)=\alpha_iT(1-Et)^{-1}(Et-E^2t^2)-d_i,\tag{1.12}$$

where

$$c=x_1\beta_1+x_2\beta_2,\quad a_k=\frac{1}{k!}\prod_{p=1}^{k}\left(q^{\alpha_2(\beta_1+(p-1)\alpha_1)}-q^{\alpha_1(\beta_2+(p-1)\alpha_2)}\right),\quad c_0=1,\quad i=1,2.$$

In fact, we can introduce the operator $\mathscr{D}_{(n)}$ $(n\in\mathbb{N})$ on $U(\mathfrak{L})$ defined by $\mathscr{D}_{(n)}:=\frac{1}{n!}(\operatorname{ad}E)^n$; it is easy to check that

$$\mathscr{D}_{(n)}(L_\beta)=a_nL_{\beta+n\alpha}.\tag{1.13}$$

Thus, (1.9) and (1.11) in Theorem 1.6 can be rewritten as

$$\Delta(L_\beta)=L_\beta\otimes(1-Et)^c+\sum_{p=0}^{\infty}(-1)^pT^{\langle p\rangle}\otimes(1-Et)^{-p}\mathscr{D}_{(p)}(L_\beta)t^p,\tag{1.14}$$

$$S(L_\beta)=-(1-Et)^{-c}\sum_{p=0}^{\infty}\mathscr{D}_{(p)}(L_\beta)T_1^{\langle p\rangle}t^p.\tag{1.15}$$

2 Proof of the main results

From above, in order to quantize the Lie bialgebra structures on q-analog Virasoro-like algebras, the key is to construct the Drinfel'd twisting, thus we have to do some necessary computation.

Lemma 2.1. *Let $\mathfrak{L}$ be the q-analog Virasoro-like algebras. The following equations hold in $U(\mathfrak{L})$:*

$$L_\beta T_a^{[m]} = T_{a-c}^{[m]} L_\beta, \quad L_\beta T_a^{\langle m\rangle} = T_{a-c}^{\langle m\rangle} L_\beta, \tag{2.1}$$

$$E^n T_a^{[m]} = T_{a-n}^{[m]} E^n, \quad E^n T_a^{\langle m\rangle} = T_{a-n}^{\langle m\rangle} E^n, \tag{2.2}$$

$$d_n^k T_a^{[m]} = T_a^{[m]} d_n^k, \quad d_n^k T_a^{\langle m\rangle} = T_a^{\langle m\rangle} d_n^k, \tag{2.3}$$

$$L_\beta L_\gamma^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \prod_{p=1}^{i} \big(q^{\gamma_2(\beta_1+(p-1)\gamma_1)} - q^{\gamma_1(\beta_2+(p-1)\gamma_2)}\big) L_\gamma^{m-i} L_{\beta+i\gamma}, \tag{2.4}$$

$$d_n L_\gamma^m = m\gamma_n L_\gamma^m + L_\gamma^m d_n, \tag{2.5}$$

where $T = x_1 d_1 + x_2 d_2 \in \operatorname{span}\{d_1, d_2\}$, $E = L_\alpha$ satisfying $[T, E] = E$ (cf. (1.8)), $\beta, \gamma \in \mathbb{Z}^2 \setminus \{(0,0)\}$, $c = x_1\beta_1 + x_2\beta_2$, $a \in \mathbb{C}$ and $n = 1, 2$.

Proof. Since $[T, L_\beta] = cL_\beta$, we have $L_\beta T = (T - c)L_\beta$. It is easy to see that (2.1) is true for $m = 1$. We can suppose that the first equation of (2.1) is true for m, then for $m+1$, we have

$$\begin{aligned} L_\beta T_a^{[m+1]} &= L_\beta T_a^{[m]}(T + a - m) = T_{a-c}^{[m]} L_\beta (T + a - m) \\ &= T_{a-c}^{[m]}(T + a - c - m)L_\beta = T_{a-c}^{[m+1]} L_\beta. \end{aligned}$$

Thus we get (2.1) by induction on m. The second equation in (2.1), (2.2) and (2.3) can be verified in a similar way. Since

$$\big(\operatorname{ad} L_\gamma\big)^i L_\beta = \prod_{p=1}^{i} \big(q^{\gamma_2(\beta_1+(p-1)\gamma_1)} - q^{\gamma_1(\beta_2+(p-1)\gamma_2)}\big) L_{\beta+i\gamma}, \tag{2.6}$$

for any $L_\beta, L_\gamma \in \mathfrak{L}$, then for (2.4), we have

$$\begin{aligned} L_\beta L_\gamma^m &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_\gamma^{m-i} \big(\operatorname{ad} L_\gamma\big)^i \big(L_\beta\big) \\ &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \prod_{p=1}^{i} \big(q^{\gamma_2(\beta_1+(p-1)\gamma_1)} - q^{\gamma_1(\beta_2+(p-1)\gamma_2)}\big) L_\gamma^{m-i} L_{\beta+i\gamma}. \end{aligned}$$

Similarly, we can obtain (2.5). □

For $a \in \mathbb{F}$, we set

$$\begin{aligned} &\mathscr{F}_a := \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes E^i t^i, \quad F_a := \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{\langle i\rangle} \otimes E^i t^i, \\ &\mathscr{U}_a := \mu \cdot \big(S_0 \otimes Id\big)\big(F_a\big), \quad \mathscr{V}_a := \mu \cdot \big(Id \otimes S_0\big)\big(\mathcal{F}_a\big), \end{aligned}$$

where t denotes a formal variable. Denote $\mathscr{F} = \mathscr{F}_0$, $F = F_0$, $\mathscr{U} = \mathscr{U}_0$, $\mathscr{V} = \mathscr{V}_0$. Since $S_0(T_a^{\langle i\rangle}) = (-1)^i T_{-a}^{[i]}$, $S_0(E^i) = (-1)^i E^i$, we have

$$\mathscr{U}_a = \mu\big(S_0 \otimes Id\big)\left(\sum_{i=0}^{\infty} \frac{1}{i!} T_a^{\langle i\rangle} \otimes E^i t^i\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a}^{[i]} E^i t^i,$$

$$\mathscr{V}_a = \mu\big(Id \otimes S_0\big)\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes E^i t^i\right) = \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} E^i t^i.$$

Lemma 2.2. *For $a, d \in \mathbb{C}$, one has*

$$\mathscr{F}_a F_d = 1 \otimes (1 - Et)^{(a-d)}, \quad \mathscr{V}_a \mathscr{U}_d = (1 - Et)^{-(a+d)}. \tag{2.7}$$

Therefore the elements $\mathscr{F}_a$, F_a, $\mathscr{U}_a$, $\mathscr{V}_a$ are invertible elements with $\mathscr{F}_a^{-1} = F_a$, $\mathscr{U}_a^{-1} = \mathscr{V}_{-a}$.

Proof. Using the formula (1.5), we have

$$\begin{aligned}
\mathscr{F}_a F_d &= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{i+j=m} \frac{(-1)^j}{i!j!} T_a^{[i]} T_d^{\langle j\rangle}\right) \otimes E^m t^m \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{a-d}{m} \otimes E^m t^m \\
&= 1 \otimes (1 - Et)^{a-d}.
\end{aligned}$$

For the second equation, using (2.2) and (1.5), we have

$$\begin{aligned}
\mathscr{V}_a \mathscr{U}_d &= \sum_{m=0}^{\infty} \left(\sum_{i+j=m} \frac{(-1)^j}{i!j!} T_a^{[i]} T_{-d-i}^{[j]}\right) E^{i+j} t^{i+j} \\
&= \sum_{m=0}^{\infty} \binom{a+d+m-1}{m} E^m t^m \\
&= (1 - Et)^{-(a+d)}.
\end{aligned}$$

□

Lemma 2.3. *For any positive integer m and any $a \in \mathbb{F}$, one has*

$$\Delta_0\big(T^{[m]}\big) = \sum_{i=0}^{m} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]}. \tag{2.8}$$

In particular, one has

$$\Delta_0\big(T^{[m]}\big) = \sum_{i=0}^{m} \binom{m}{i} T^{[i]} \otimes T^{[m-i]}.$$

Proof. In order to get the result, we want to use induction. Since $\Delta_0(T) = T \otimes 1 + 1 \otimes T$, it is easy to see that the result is true for $m = 1$; suppose that it is true for m, then it is enough to consider the condition for $m+1$,

$$
\begin{aligned}
\Delta_0\big(T^{[m+1]}\big) &= \Delta_0\big(T^{[m]}\big)\Delta_0(T-m) \\
&= \left(\sum_{i=0}^{m} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]}\right) \\
&\quad \times \big((T-a-m)\otimes 1 + 1 \otimes (T+a-m) + m(1\otimes 1)\big) \\
&= 1 \otimes T_a^{[m+1]} + T_{-a}^{[m+1]} \otimes 1 + m\left(\sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]}\right) \\
&\quad + (T-a) \otimes T_a^{[m]} + T_{-a}^{[m]} \otimes (T+a) + \sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i+1]} \otimes T_a^{[m-i]} \\
&\quad + \sum_{i=1}^{m-1} (i-m) \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} + \sum_{i=1}^{m-1} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i+1]} \\
&\quad + \sum_{i=1}^{m-1} (-i) \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]} \\
&= 1 \otimes T_a^{[m+1]} + T_{-a}^{[m+1]} \otimes 1 + \sum_{i=1}^{m} \left(\binom{m}{i-1} + \binom{m}{i}\right) T_{-a}^{[i]} \otimes T_a^{[m+1-i]} \\
&= \sum_{i=0}^{m+1} \binom{m+1}{i} T_{-a}^{[i]} \otimes T_a^{[m+1-i]}.
\end{aligned}
$$

Therefore, the result is proved by induction. $\square$

Proposition 2.4. *$\mathscr{F} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T^{[i]} \otimes E^i t^i$ is a Drinfel'd twist element of $(U(\mathfrak{L})[[t]], \mu, \tau, \Delta_0, S_0, \epsilon_0)$, that is $\mathscr{F}$ satisfies (1.6) and (1.7).*

Proof. The proof of (1.7) is easy, we just need to check (1.6). Since

$$
\begin{aligned}
(\mathscr{F} \otimes 1)\big(\Delta_0 \otimes Id\big)(\mathscr{F}) &= \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T^{[i]} \otimes E^i t^i \otimes 1\right) \\
&\quad \cdot \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{j} \binom{j}{k} T_{-i}^{[k]} \otimes T_i^{[j-k]} \otimes E^j t^j\right) \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!j!} \sum_{k=0}^{j} \binom{j}{k} T^{[i]} T_{-i}^{[k]} \otimes E^i T_i^{[j-k]} \otimes E^j t^{i+j} \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!j!} \sum_{k=0}^{j} \binom{j}{k} T^{[i+k]} \otimes T^{[j-k]} E^i \otimes E^j t^{i+j},
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
(1\otimes\mathscr{F})\big(Id\otimes\Delta_0\big)(\mathscr{F}) &= \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}1\otimes T^{[r]}\otimes E^r t^r\right)\\
&\quad\cdot\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}T^{[s]}\otimes\sum_{q=0}^{s}\binom{s}{q}E^q\otimes E^{s-q}t^s\right)\\
&=\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}}{r!s!}\sum_{q=0}^{s}\binom{s}{q}T^{[s]}\otimes T^{[r]}E^q\otimes E^{r+s-q}t^{r+s},
\end{aligned}
$$

thus, to verify (1.6), it suffices to show for a fixed m that

$$
\begin{aligned}
&\sum_{i+j=m}\frac{1}{i!j!}\sum_{k=0}^{j}\binom{j}{k}T^{[i+k]}\otimes T^{[j-k]}E^i\otimes E^j\\
&\quad=\sum_{r+s=m}\frac{1}{r!s!}\sum_{q=0}^{s}\binom{s}{q}T^{[s]}\otimes T^{[r]}E^q\otimes E^{r+s-q}.
\end{aligned}
$$

Now, fix r, s, q such that $r+s=m$, $0\le q\le s$, set $i=q$, $i+k=s$, then we have $j=m-q$, $j-k=r$. We see that the coefficients of $T^{[s]}\otimes T^{[r]}E^q\otimes E^{m-q}$ in both sides are equal. So the result follows. □

Lemma 2.5. *One has for any $a\in\mathbb{F}$ and $L_\beta\in\mathfrak{L}$*

$$
\begin{aligned}
\big(L_\beta\otimes 1\big)F_a &= F_{a-c}\big(L_\beta\otimes 1\big), &(2.9)\\
\big(1\otimes L_\beta\big)F_a &= \sum_{l=0}^{\infty}(-1)^l a_l F_{a+l}\big(T_a^{\langle l\rangle}\otimes L_{\beta+l\alpha}t^l\big), &(2.10)\\
L_\beta\mathscr{U}_a &= \mathscr{U}_{a+c}\sum_{l=0}^{\infty}a_l L_{\beta+l\alpha}T_{1-a}^{\langle l\rangle}t^l, &(2.11)\\
\big(d_i\otimes 1\big)F_a &= F_a\big(d_i\otimes 1\big), &(2.12)\\
\big(1\otimes d_i\big)F_a &= F_{a+1}\big(T_a^{\langle 1\rangle}\otimes\alpha_i Et\big)+F_a\big(1\otimes d_i\big), &(2.13)\\
d_i\mathscr{U}_a &= -\alpha_i T_{-a}^{[1]}\mathscr{U}_{a+1}Et+\mathscr{U}_a d_i, &(2.14)\\
E\mathscr{U}_a &= \mathscr{U}_{a+1}E, &(2.15)\\
\mathscr{V}_a T_{-a}^{[1]} &= T_{-a}^{[1]}\mathscr{V}_a - T_a^{[1]}\mathscr{V}_{a-1}Et, &(2.16)
\end{aligned}
$$

where

$$
a_l=\frac{1}{l!}\prod_{p=1}^{k}\big(q^{\alpha_2(\beta_1+(p-1)\alpha_1)}-q^{\alpha_1(\beta_2+(p-1)\alpha_2)}\big),\quad c=x_1\beta_1+x_2\beta_2,\ i=1,2.
$$

Proof. By the second equation of (2.1) we have

$$
\big(L_\beta\otimes 1\big)F_a=\sum_{i=0}^{\infty}\frac{1}{i!}L_\beta T_a^{\langle i\rangle}\otimes E^i t^i
$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_{a-c}^{\langle i\rangle} L_\beta \otimes E^i t^i \\
&= F_{a-c}\big(L_\beta \otimes 1\big);
\end{aligned}
$$

this prove (2.12). For (2.10), using (2.4), we have

$$
\begin{aligned}
\big(1 \otimes L_\beta\big) F_a &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{\langle i\rangle} \otimes L_\beta E^i t^i \\
&= \sum_{i=0}^{\infty} \sum_{l=0}^{i} (-1)^l \frac{1}{(i-l)!} a_l T_a^{\langle i\rangle} \otimes E^{i-l} L_{\beta+l\alpha} t^i \\
&= \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{1}{i!} a_l T_a^{\langle i+1\rangle} \otimes E^i L_{\beta+l\alpha} t^{i+l} \\
&= \sum_{l=0}^{\infty} (-1)^l a_l \sum_{i=0}^{\infty} \frac{1}{i!} T_{a+l}^{\langle i\rangle} \otimes E^i t^i T_a^{\langle l\rangle} \otimes L_{\beta+l\alpha} t^l \\
&= \sum_{l=0}^{\infty} (-1)^l a_l F_{a+l}\big(T_a^{\langle l\rangle} \otimes L_{\beta+l\alpha} t^l\big);
\end{aligned}
$$

this proves (2.10). The following two equations give the proofs of (2.11) and (2.12):

$$
\begin{aligned}
L_\beta \mathscr{U}_a &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-c}^{[r]} L_\beta E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-c}^{[r]} \sum_{l=0}^{r} (-1)^l \frac{r!}{(r-l)!} a_l E^{r-l} L_{\beta+l\alpha} t^r \\
&= \sum_{r,l=0}^{\infty} \frac{(-1)^r}{r!} a_l T_{-a-c}^{[r+l]} E^r L_{\beta+l\alpha} t^{r+l} \\
&= \sum_{r,l=0}^{\infty} \frac{(-1)^r}{r!} a_l T_{-a-c}^{[r]} T_{-a-c-r}^{[l]} E^r L_{\beta+l\alpha} t^{r+l} \\
&= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \left(\frac{(-1)^r}{r!} a_l T_{-a-c}^{[r]} E^r t^r \right) T_{-a-c}^{[l]} L_{\beta+l\alpha} t^l \\
&= \mathscr{U}_{a+c} \sum_{l=0}^{\infty} a_l T_{-a-c}^{[l]} L_{\beta+l\alpha} t^l \\
&= \mathscr{U}_{a+c} \sum_{l=0}^{\infty} a_l L_{\beta+l\alpha} T_{1-a}^{\langle l\rangle} t^l,
\end{aligned}
$$

$$
\begin{aligned}
\big(d_i \otimes 1\big) F_a &= \big(d_i \otimes 1\big) \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} d_i T_a^{\langle r\rangle} \otimes E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} d_i \otimes E^r t^r
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r t^r\right) \big(d_i \otimes 1\big)\\
&= F_a\big(d_i \otimes 1\big).
\end{aligned}
$$

Using (1.4) and (2.5), we have

$$
\begin{aligned}
\big(1 \otimes d_i\big) F_a &= \big(1 \otimes d_i\big) \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r t^r\\
&= \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes d_i E^r t^r\\
&= \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes \big(r\alpha_i E^r + E^r d_i\big) t^r\\
&= \sum_{r=0}^{\infty} \frac{1}{(r-1)!} T_a^{\langle r\rangle} \otimes \alpha_i E^r t^r + \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r d_i t^r\\
&= \sum_{r=0}^{\infty} \frac{1}{(r-1)!} T_a^{\langle 1\rangle} T_{a+1}^{\langle r-1\rangle} \otimes \alpha_i E^r t^r + F_a\big(1 \otimes d_i\big)\\
&= F_{a+1}\big(T_a^{\langle 1\rangle} \otimes \alpha_i E t\big) + F_a\big(1 \otimes d_i\big),
\end{aligned}
$$

which gives (2.12). The equations (2.14) and (2.15) follow from the following computations:

$$
\begin{aligned}
d_i \mathscr{U}_a &= d_i \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} E^r t^r\\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} d_i E^r t^r\\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} \big(r\alpha_i E^r + E^r d_i\big) t^r\\
&= \sum_{r=0}^{\infty} \alpha_i T_{-a}^{[1]} \frac{(-1)^r}{(r-1)!} T_{-a-1}^{[r-1]} E^r t^r + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} E^r t^r d_i\\
&= -\alpha_i T_{-a}^{[1]} \mathscr{U}_{a+1} E t + \mathscr{U}_a d_i,\\
E \mathscr{U}_a &= E \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a}^{[i]} E^i t^i\\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a-1}^{[i]} E^{i+1} t^i = \mathscr{U}_{a+1} E.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathscr{V}_a T_{-a}^{[1]} &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} E^i t^i T_{-a}^{[1]}\\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} (T - a - i) E^i t^i
\end{aligned}
$$

$$
\begin{aligned}
&= T_{-a}^{[1]}\mathscr{V}_a - \sum_{i=0}^{\infty}\frac{1}{(i-1)!}(T+a)T_{a-1}^{[i-1]}E^i t^i \\
&= T_{-a}^{[1]}\mathscr{V}_a - T_a^{[1]}\mathscr{V}_{a-1}Et,
\end{aligned}
$$

which proves the last equation of the lemma. □

Now we can prove our main theorem in this paper.

Proof of Theorem 1.6. For arbitrary elements, $L_\beta \in \mathfrak{L}$, $i = 1, 2$. First, using (2.7), (2.12) and (2.10), we have

$$
\begin{aligned}
\Delta\big(L_\beta\big) &= \mathscr{F}\Delta_0\big(L_\beta\big)\mathscr{F}^{-1} \\
&= \mathscr{F}\big(L_\beta \otimes 1\big)\mathscr{F}^{-1} + \mathscr{F}\big(1 \otimes L_\beta\big)\mathscr{F}^{-1} \\
&= \mathscr{F}F_{-c}\big(L_\beta \otimes 1\big) + \mathscr{F}\sum_{l=0}^{\infty}(-1)^l a_l F_l\big(T^{\langle l\rangle} \otimes L_{\beta+l\alpha}t^l\big) \\
&= \big(1 \otimes (1-Et)^c\big)\big(L_\beta \otimes 1\big) \\
&\quad + \sum_{l=0}^{\infty}(-1)^l a_l\big(1 \otimes (1-Et)^{-l}\big) \otimes \big(T^{\langle l\rangle} \otimes L_{\beta+l\alpha}t^l\big) \\
&= L_\beta \otimes (1-Et)^c + \sum_{l=0}^{\infty}(-1)^l a_l T^{\langle l\rangle} \otimes (1-Et)^{-l}L_{\beta+l\alpha}t^l.
\end{aligned}
$$

Using (2.7), (2.12) and (2.13), we have

$$
\begin{aligned}
\Delta\big(d_i\big) &= \mathscr{F}\Delta\big(d_i\big)\mathscr{F}^{-1} \\
&= \mathscr{F}\big(d_i \otimes 1 + 1 \otimes d_i\big)F \\
&= \mathscr{F}\big(d_i \otimes 1\big)F + \mathscr{F}\big(1 \otimes d_i\big)F \\
&= \mathscr{F}F\big(d_i \otimes 1\big) + \mathscr{F}\big(F_1\big(T^{\langle 1\rangle} \otimes \alpha_i Et\big) + F\big(1 \otimes d_i\big)\big) \\
&= d_i \otimes 1 + 1 \otimes d_i + 1 \otimes (1-Et)^{-1}\big(T^{\langle 1\rangle} \otimes \alpha_i Et\big) \\
&= d_i \otimes 1 + 1 \otimes d_i + \alpha_i T^{\langle 1\rangle} \otimes (1-Et)^{-1}Et.
\end{aligned}
$$

Using (2.7) and (2.11), we have

$$
\begin{aligned}
S\big(L_\beta\big) &= \mathscr{U}^{-1}S_0\big(L_\beta\big)\mathscr{U} \\
&= -\mathscr{V}L_\beta\mathscr{U} \\
&= -\mathscr{V}\mathscr{U}_b\left(\sum_{l=0}^{\infty}a_l L_{\beta+l\alpha}T_1^{\langle l\rangle}t^l\right) \\
&= -(1-Et)^{-b}\left(\sum_{l=0}^{\infty}a_l L_{\beta+l\alpha}T_1^{\langle l\rangle}t^l\right).
\end{aligned}
$$

Using (2.7), (2.14), (2.15) and (2.16), we have

$$
\begin{aligned}
S\big(d_i\big) &= \mathscr{U}^{-1}S_0\big(d_i\big)\mathscr{U} \\
&= -\mathscr{V}d_i\mathscr{U} \\
&= -\mathscr{V}\big(-\alpha_i T^{[1]}\mathscr{U}_1 Et + \mathscr{U}d_i\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_i \big(T\mathscr{V} - T\mathscr{V}Et\big)\mathscr{U}_1 Et - d_i \\
&= \alpha_i T\mathscr{V}\mathscr{U}_1 Et - \alpha_i T\mathscr{V}\mathscr{U}_2 E^2 t^2 - d_i \\
&= \alpha_i T(1-Et)^{-1}Et - \alpha_i T(1-Et)^{-1}E^2t^2 - d_i \\
&= \alpha_i T(1-Et)^{-1}\big(Et - E^2t^2\big) - d_i.
\end{aligned}
$$

This completes the proof of the theorem. □

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References

[1] V. Drinfel'd, Constant quasiclassical solutions of the Yang-Baxter quantum equation, *Dokl. Akad. Nauk SSSR*, **273** (1983), 531–535.

[2] V. Drinfel'd, Quantum groups, *Proceedings of the International Congress of Mathematicians* (Berkeley, CA, 1986), 798–820, American Mathematical Society, Providence, RI, USA, 1987.

[3] C. Grunspan, Quantizations of the Witt algebra and of simple Lie algebras in characteristic p, *J. Algebra*, **280** (2004), 145–161.

[4] W. Michaelis, A Class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, *Adv. Math.*, **107** (1994), 365–392.

[5] E. J. Taft, Witt and Virasoro algebras as Lie bialgebras, *J. Pure Appl. Algebra*, **87** (1993), 301–312.

[6] S. H. Ng and Earl J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Algebra*, **151** (2000), 67–88.

[7] N. Hu and X. Wang, Quantizations of generalized-Witt algebra and of Jacobson-Witt algebra in modular case, *J. Algebra*, **312** (2007), 902–929.

[8] G. Song and Y. Su, Lie bialgebras of generalized Witt type, *Sci. China Ser. A*, **49** (2006), 533–544.

[9] Y. Cheng and Y. Shi, Lie bialgebra structures on the q-analog Virasoro-like algebras, *Comm. Algebra*, **37** (2009), 1264–1274.

[10] W. Lin and S. Tan, Harish-Chandra modules for the q-analog Virasoro-like algebra, *J. Algebra*, **297** (2006), 254–272.

[11] R. Shen and Y. Su, Classification of irreducible weight modules with a finite-dimensional weight space over twisted Heisenberg-Virasoro algebra, *Acta Math. Sin. (Engl. Ser.)*, **23** (2007), 189–192.

[12] Y. Su, Quasifinite representations of a Lie algebra of Block type, *J. Algebra*, **276** (2004), 117–128.

[13] Y. Su and X. Xu, Structure of divergence-free Lie algebras, *J. Algebra*, **243** (2001), 557–595.

[14] Y. Su and J. Zhou, Structure of the Lie algebras related to those of block, *Comm. Algebra*, **30** (2002), 3205–3226.

[15] X. Xu, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, *J. Algebra*, **224** (2000), 23–58.

[16] X. Xu, Generalizations of Block algebras, *Manuscripta Math.*, **100** (1999), 489–518.

[17] Y. Wu, G. Song and Y. Su, Lie bialgebras of generalized Virasoro-like type, *Acta Math. Sin. (Engl. Ser.)*, **22** (2006), 1915–1922.

[18] X. Yue and Y. Su, Highest weight representations of a family of Lie algebras of Block type, *Acta Math. Sin. (Engl. Ser.)*, **24** (2008), 687–696.

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