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Interface Regularity of the Solutions to Maxwell Systems on Riemannian Manifolds

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Abstract. In this paper we study the interface regularity of the solutions to the differential systems defined by differential forms (for example, stationary Maxwell systems) on $N(\geq 3)$-dimensional Riemannian manifolds. Our results are natural extensions of the results of *Interface regularity of the solutions for the rotation free and the divergence free systems* and *Interface vanishing for solutions to Maxwell and Stokes systems*.

1. Introduction

We will start by describing previous results. First, let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a $C^{2,1}$-Lipschitz boundary. Let $\mathcal{M}$ be a hypersurface in $\mathbf{R}^3$. Assume that $\mathcal{M}$ divides Ω into two domains $\Omega_\pm$, i.e., $\Omega = \Omega_+ \cup (\Omega \cap \mathcal{M}) \cup \Omega_-$ (disjoint). Let $\Gamma = \Gamma_\pm = \partial\Omega_\pm \cap \mathcal{M}$, and let ν be the outer unit normal vector field on Γ_-. Γ is called an interface. If $\mathcal{M}$ is of $C^{k,1}$, then ν has a $C^{k-1,1} \cap W^{k,\infty}$-extension to Ω, which is denoted by the same symbol ν. Let B be an $\mathbf{R}^3$-vector valued function on Ω. For $x \in \Gamma$, set

$$B_\pm(x) := \lim_{\Omega_\pm \ni \xi \to x} B(\xi)\,, \quad [B]_-^+ = B_+ - B_- \ \text{ on } \ \Gamma\,.$$

Let J be an $\mathbf{R}^3$-vector valued function on $\Omega_\pm$, and let ψ be a function on $\Omega_\pm$. The regularity properties of the solutions to the systems

$$(1)\ \begin{cases} \operatorname{rot} B = J\,, \\ \operatorname{div} B = 0\,, \end{cases} \text{ in } \ \Omega_\pm\,, \quad (2)\ \begin{cases} \operatorname{rot} B = 0\,, \\ \operatorname{div} B = \psi\,, \end{cases} \text{ in } \ \Omega_\pm\,,$$

have been obtained by T. Kobayashi, T. Suzuki, and K. Watanabe [8] for (1) (Maxwell system), and M. Kanou, T. Sato, and K. Watanabe [5] for (2):

THEOREM 1.1 ([8]). *Let $\mathcal{M} \subset \mathbf{R}^3$ be a $C^{2,1}$-surface, and* $\operatorname{rot} J \in L^2(\Omega_\pm)^3$. *If $B \in H^1(\Omega)^3$ is a solution to* (1), *then $\nu \cdot B \in H^2_{loc}(\Omega)$.*

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THEOREM 1.2 ([5]). *Let $\mathcal{M} \subset \mathbf{R}^3$ be a $C^{2,1}$-surface, and $\psi \in H^1(\Omega_\pm)$. If $B \in H^1(\Omega)^3$ is a solution to* (2), *then $\nu \times B \in H^2_{loc}(\Omega)^3$.*

Here, we will describe a historical background. In [3], Geselowitz studied the problem for Magnetoencephalography (MEG), which arises in mathematical medicine. We will give a rigorous explanation for MEG. Ω_+ is a "head", Ω_- is the outside of the head, and Γ is the surface of the head. Let B be a magnetic field, and let J be an electric current. The problem is: whether we can determine the electric current J by measuring the magnetic field B in Ω_- (the outside). For (1), J has a discontinuity across the interface Γ. (For example, $J \not\equiv 0$ on Γ, $J \equiv 0$ in Ω_-.) In [9], T. Suzuki, K. Watanabe, and M. Shimogawara investigated some properties of the solutions to (1) by using the Newton potential. They also studied the inverse problem, under the assumption that J is a dipole.

When J has a discontinuity across Γ, we do not expect that B itself gains a one-rank higher regularity in Ω. However, we have a higher regularity property in Ω of the normal component of B (cf. [6, 8]). Precisely, in [7], T. Kobayashi, T. Suzuki, and K. Watanabe obtained the same result as Theorem 1.1 by assuming that $\mathcal{M}$ is a C^2-surface. In [8], they improved this result and obtained Theorem 1.1 as stated above. In order to prove Theorem 1.1, they used the Green and the Gauss formulas instead of the Newton potential. In [5], M. Kanou, T. Sato, and K. Watanabe obtained Theorem 1.2 as stated above.

In [6], we studied an extension to the Euclidean space $\Omega \subset \mathbf{R}^N$ ($N \geq 3$). We used differential forms, 1 or 2-forms and write them as

$$B = \sum_{i=1}^{N} B^i dx_i \text{ (1-form)}, \quad J = \sum_{1 \leq i < j \leq N} J^{ij} dx_i \wedge dx_j \text{ (2-form)}.$$

For 1-forms $A = \sum_{i=1}^N A^i dx_i$ and $B = \sum_{i=1}^N B^i dx_i$, an inner product is defined by $(A, B) := \sum_{i=1}^N A^i B^i$.

We write $\partial B^i / \partial x_j$ as B^i_j. The differential operators $d_0, d_1, \delta_0, \delta_1$ on forms are defined by

$$d_0 f := \sum_{i=1}^{N} f_i dx_i, \quad d_1 B := \sum_{1 \leq i < j \leq N} (B^j_i - B^i_j) dx_i \wedge dx_j,$$

$$\delta_0 B := -\sum_{i=1}^{N} B^i_i, \quad \delta_1 J := -\sum_{i=1}^{N} \left(\sum_{l=1}^{N} J^{li}_l \right) dx_i,$$

where f is a function, B is a 1-form, and J is a 2-form. Let $H^m(D; \mathbf{R}^K)$ be a ($\mathbf{R}^K$-vector valued) Sobolev space of rank m on D which is a domain in $\mathbf{R}^N$. Denote the outer unit normal vector field on Γ_- by ν. Assume that ν has an extension to Ω, and that ν is identified with a 1-form $\nu = \sum_{i=1}^N \nu^i dx_i$.

Consider the following systems:

$$\text{(3)} \quad \begin{cases} d_1 B = J\,, \\ \delta_0 B = 0\,, \end{cases} \qquad \text{in } \ \Omega_\pm\,, \quad \delta_1 J \in L^2(\Omega_\pm; \mathbf{R}^N)\,,$$

$$\text{(4)} \quad \begin{cases} d_1 B = 0\,, \\ \delta_0 B = \psi\,, \end{cases} \qquad \text{in } \ \Omega_\pm\,, \quad \psi \in H^1(\Omega_\pm)\,.$$

B^ν and B^τ are defined by

$$B^\nu := (\nu, B)\nu\,, \quad B^\tau := B - B^\nu\,.$$

THEOREM 1.3 ([6]). *Let B and J satisfy* (3). *If $B \in H^1(\Omega; \mathbf{R}^N)$ and $[B]^+_- = 0$ on Γ, then $(\nu, B) \in H^2_{loc}(\Omega)$ holds.*

THEOREM 1.4 ([6]). *Let B and ψ satisfy* (4). *If $B \in H^1(\Omega; \mathbf{R}^N)$ and $[B]^+_- = 0$ on Γ, then $B^\tau \in H^2_{loc}(\Omega)$ holds.*

The main purpose of this paper is to generalize Theorems 1.3 and 1.4 to Riemannian manifolds. That is, we will prove Theorems 2.1 and 2.2. We consider this problem from mathematical interest.

The remainder of this paper is organized as follows: In §2, we state the terminologies of Riemannian geometry, and some of the main theorems. In §3, we provide the Green and the Stokes formulas of L^2-type. In §4, we present proofs of the theorems.

2. Riemannian Geometry and Main Theorems

We follow the terminology of [2]. Let (M, g) be a compact orientable C^∞-Riemannian manifold of dimension N. Let $TM = \cup_{x\in M} T_x M$ and $\wedge^p T^* M = \cup_{x \in M} \wedge^p T^*_x M$.

Let $\mathcal{X}(M)$ be the set of smooth vector fields on M, and $\Lambda^p(M)$ be the set of p-forms on M. The exterior differential operator $d_p : \Lambda^p(M) \to \Lambda^{p+1}(M)$ $(p = 0, 1, 2)$ is defined by, for $X, Y, Z \in \mathcal{X}(M)$, as

$$\begin{aligned}
(d_0\omega)(X) :=& X\omega\,, \\
(d_1\omega)(X, Y) :=& X\omega(Y) - Y\omega(X) - \omega([X, Y])\,, \\
(d_2\omega)(X, Y, Z) :=& X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\
& - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)\,,
\end{aligned}$$

for $\omega \in \Lambda^p(M)$.

∇ is called a connection if ∇ satisfies the following three properties:
$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, $(X, Y) \mapsto \nabla_X Y$ and for $f, f_1, f_2 \in C^\infty(M)$ and $X, Y, Z \in \mathcal{X}(M)$,

$$\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z\,,$$

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z\,,$$

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y\,.$$

Furthermore, the Levi-Civita connection ∇ satisfies

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\,, \tag{2.1}$$

$$\nabla_X Y - \nabla_Y X = [X,Y]\,. \tag{2.2}$$

Throughout this paper, ∇ denotes the Levi-Civita connection. The covariant derivative $\nabla\omega \in \Lambda^{p+1}(M)$ of a p-form ω ($p = 0,1,2$) is defined, for $\omega \in \Lambda^p(M)$ ($p = 0,1,2$) and $X, Y, Z \in \mathcal{X}(M)$, by

$$(\nabla\omega)(X) := (\nabla_X\omega) := X\omega\,,$$

$$(\nabla\omega)(X,Y) := (\nabla_Y\omega)(X) := Y\omega(X) - \omega(\nabla_Y X)\,,$$

$$(\nabla\omega)(X,Y,Z) := (\nabla_Z\omega)(X,Y) := Z\omega(X,Y) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y)\,.$$

Let $\{e_k\}_{k=1,\dots,N}$ be a locally defined orthonormal frame field of TM, i.e., $g(e_i, e_j) = \delta_{ij}$. And let $\{\xi_j\}_{1\le j\le N}$ be the dual coframe field satisfying $e_i(\xi_j) = \delta_{ij}$. We denote ∇_{e_i} by ∇_i. The co-derivative δ_p ($p = 0,1$) is defined as follows:

$$(\delta_0\omega) = -\sum_{i=1}^{N}(\nabla_i\omega)(e_i), \quad (\delta_1\omega)(X) = -\sum_{i=1}^{N}(\nabla_i\omega)(e_i, X)\,.$$

We set $g_{kl} := g(\partial/\partial x_k, \partial/\partial x_l)$, and $(g^{kl}) = (g_{kl})^{-1}$. The operator $\sharp : \Lambda^1(M) \to \mathcal{X}(M)$ is defined by $\omega(X) = g(\omega^\sharp, X)$, for all $X \in \mathcal{X}(M)$. In a local coordinate system, for $\omega = \sum_{i=1}^{N}\omega^i dx_i$, we have

$$\omega^\sharp = \sum_{i=1}^{N}\sum_{j=1}^{N} g^{ij}\omega^j \frac{\partial}{\partial x_i}\,.$$

Furthermore, $\flat$ denotes the dual operator of $\sharp$.

We define an inner product $(\cdot,\cdot)_p$ on $\bigwedge^p T_x^*M$ ($p = 1,2$) as follows: for $\omega = \sum_{i=1}^{N}\omega^i dx_i$ and $\eta = \sum_{i=1}^{N}\eta^i dx_i \in T_x^*M$,

$$(\omega,\eta)_1 := g(\omega^\sharp, \eta^\sharp) = \sum_{i,j=1}^{N} g^{ij}\omega^i\eta^j\,;$$

for $\omega = \sum_{1\le i<j\le N}\omega^{ij}dx_i\wedge dx_j$ and $\eta = \sum_{1\le i<j\le N}\eta^{ij}dx_i\wedge dx_j \in \bigwedge^2 T_x^*M$,

$$(\omega,\eta)_2 := \sum_{k,l,m,n=1}^{N}\omega^{kl}\eta^{mn}(g^{km}g^{ln} - g^{lm}g^{kn})\,.$$

Moreover, for $\omega = \sum_{i=1}^{N} \tilde{\omega}^i \xi_i$ and $\eta = \sum_{i=1}^{N} \tilde{\eta}^i \xi_i$ we have

$$(\omega, \eta)_1 = g(\omega^\sharp, \eta^\sharp) = \sum_i \tilde{\omega}^i \tilde{\eta}^i ,$$

and for $\omega = \sum_{1 \le i < j \le N} \tilde{\omega}^{ij} \xi_i \wedge \xi_j$ and $\eta = \sum_{1 \le i < j \le N} \tilde{\eta}^{ij} \xi_i \wedge \xi_j$ we have

$$(\omega, \eta)_2 = \sum_{1 \le k < l \le N} \tilde{\omega}^{kl} \tilde{\eta}^{kl} .$$

Let $D \subset M$ be a connected open submanifold. We define an inner product on $\Lambda^p(D)$ $(p = 1, 2)$ as follows: for $\omega, \eta \in \Lambda^p(D)$,

$$\langle \omega, \eta \rangle_{p,D} = \int_D (\omega(x), \eta(x))_p dv_g(x) ,$$

where dv_g is the volume element. As $\omega(x) \in \bigwedge^p T_x^* D$ for $\omega \in \Lambda^p(D)$, we use $(\omega, \eta)_p := (\omega(x), \eta(x))_p$ for $\omega, \eta \in \Lambda^p(D)$ throughout this paper.

Let $L^2(\Lambda^p(D))$ be the completion of $\Lambda^p(D)$, with respect to the norm $\| \cdot \|_{p,D}$ corresponding to the inner product $\langle \cdot, \cdot \rangle_{p,D}$.

First, we define the Sobolev space of order $m \in \mathbf{N} \cup \{0\}$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (\mathbf{N} \cup \{0\})^N$, we denote $e^\alpha := e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_N^{\alpha_N}$. We define

$$\begin{aligned} H^m(\Lambda^0(D)) =& H^m(D) \\ :=& \{f \in L^2(D);\ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N = m, e^\alpha f \in L^2(D)\} , \end{aligned}$$

and we define the Sobolev space of order $m + \sigma$ where $m \in \mathbf{N} \cup \{0\}$ and $0 < \sigma < 1$, as follows: $f \in H^{m+\sigma}(D)$ if and only if f is in $H^m(D)$ and satisfies, for $|\alpha| = m$,

$$\int_D \int_D \frac{|e^\alpha f(x) - e^\alpha f(y)|^2}{d(x, y)^{N+2\sigma}} dv_g(x) dv_g(y) < \infty ,$$

where $d(x, y)$ is the distance between x and y. For $\Lambda^1(D)$ and $\Lambda^2(D)$, we define the Sobolev spaces as

$$\begin{aligned} H^{m(+\sigma)}(\Lambda^1(D)) &= \{\omega \in L^2(\Lambda^1(D));\ 1 \le \forall i \le N, \omega(e_i) \in H^{m(+\sigma)}(D)\} , \\ H^{m(+\sigma)}(\Lambda^2(D)) &= \{\omega \in L^2(\Lambda^2(D));\ 1 \le \forall i \le N, 1 \le \forall j \le N , \\ &\qquad \omega(e_i, e_j) \in H^{m(+\sigma)}(D)\} . \end{aligned}$$

For $s < 0$, we define $H^s(D)$ as the dual space of
$H_0^{-s}(D) = \{\varphi \in H^{-s}(D); \varphi$ has a compact support$\}$. For $p = 0, 1, 2$, we set

$$\begin{aligned} \mathcal{H}_{1,p} &:= H(d_p; D) := \{\omega \in L^2(\Lambda^p(D)); d_p\omega \in L^2(\Lambda^{p+1}(D))\} , \\ \mathcal{H}_{2,p} &:= H(\delta_p; D) := \{\omega \in L^2(\Lambda^{p+1}(D)); \delta_p\omega \in L^2(\Lambda^p(D))\} , \end{aligned}$$

and define norms $\|\cdot\|_{\mathcal{H}_{1,p}}$ and $\|\cdot\|_{\mathcal{H}_{2,p}}$ by

$$\|\omega\|_{\mathcal{H}_{1,p}} := \|\omega\|_{p,D} + \|d_p\omega\|_{p+1,D}\,,$$

$$\|\omega\|_{\mathcal{H}_{2,p}} := \|\omega\|_{p+1,D} + \|\delta_p\omega\|_{p,D}\,.$$

Let $\Omega \subset M$ be a smooth connected open submanifold, and let H be a smooth submanifold with codimension 1. We assume that H divides Ω into two submanifolds $\Omega_\pm$. Let $\Gamma = \Gamma_\pm = \partial\Omega_\pm \cap H$, and let $\nu \in \mathcal{X}(M)$ be an outer unit normal vector field on Γ_-.

DEFINITION. For $B \in H^1(\Lambda^1(\Omega))$, we set

$$B^\nu := (\nu^\flat, B)_1\nu^\flat\,, \quad B^\tau = B - B^\nu\,.$$

We consider the following system:

$$\begin{cases} d_1 B = J, \\ \delta_0 B = \psi\,, \end{cases} \quad \text{in } \Omega_\pm\,, \quad J \in H(\delta_1, \Omega_\pm)\,, \quad d_2 J = 0\,, \;\; \psi \in H^1(\Omega_\pm)\,. \tag{2.3}$$

We set, for a function f and $x \in \Gamma$,

$$f_\pm(x) := \lim_{\Omega_\pm \ni \xi \to x} f(\xi)\,, \quad [f]_-^+ = f_+ - f_- \text{ on } \Gamma\,.$$

For any $1 \le i \le N$, $[B(e_i)]_-^+ = 0$ if and only if $[B]_-^+ = 0$.

We say that B satisfies (2.3) in a submanifold $D \subset M$, in the sense of distributions, if for $C \in \Lambda_0^2(D) := \{C \in \Lambda^2(D);\, C$ has a compact support$\}$ and $\varphi \in \Lambda_0^0(D) = \{\varphi \in \Lambda^0(D);\, \varphi$ has a compact support$\}$,

$$\langle B, \delta_1 C\rangle_{1,D} = \langle J, C\rangle_{2,D} \quad \text{and} \quad \langle B, d_0\varphi\rangle_{1,D} = \langle\psi, \varphi\rangle_{0,D}$$

hold.

THEOREM 2.1. *Assume that $[\psi]_-^+ = 0$ on Γ, and let B satisfy* (2.3) *in $\Omega_\pm$ in the sense of distributions. If $B \in H^1(\Lambda^1(\Omega))$, then B satisfies* (2.3) *in Ω in the sense of distributions. Moreover, $(\nu^\flat, B)_1 \in H^2_{loc}(\Omega)$ holds, that is, $(\nu^\flat, B)_1 \in H^2(\Omega')$ for any N-dimension compact submanifold $\Omega' \subset \Omega$.*

THEOREM 2.2. *Assume that $[J]_-^+ = 0$ on Γ, and let B satisfy* (2.3) *in $\Omega_\pm$ in the sense of distributions. If $B \in H^1(\Lambda^1(\Omega))$, then B satisfies* (2.3) *in Ω in the sense of distributions. Moreover, $B^\tau \in H^2_{loc}(\Lambda^1(\Omega))$ holds, that is, $B^\tau \in H^2(\Lambda^1(\Omega'))$ for any N-dimension compact submanifold $\Omega' \subset \Omega$.*

REMARK 2.1. We see that if $\phi \in H^1(\Omega_\pm)$ satisfies $[\phi]_-^+ = 0$ on Γ, then $\phi \in H^1(\Omega)$.

3. Preliminaries

In order to prove the Green and the Stokes formulas of L^2-type for $\mathcal{H}_{1,p}$ and $\mathcal{H}_{2,p}$, we will now prepare some lemmas.

We define the divergence of $X \in \mathcal{X}(\Omega)$, $\operatorname{div}(X)$, by

$$\operatorname{div}(X) = \sum_{l=1}^{N} g(e_l, \nabla_l X) .$$

LEMMA 3.1. *Let $D \subset \Omega$ be an open submanifold. For $X \in \mathcal{X}(\Omega)$, we have*

$$\int_D \operatorname{div}(X) dv_g = \int_{\partial D} g(X, \nu) dS_g ,$$

where ν is the outer unit normal vector field on ∂D and dS_g is the surface element.

PROOF. See [1]. □

The relationship between the Levi-Civita connection ∇ and the exterior differentiation d_1 is given by

$$(d_1\omega)(X, Y) = -(\nabla\omega)(X, Y) + (\nabla\omega)(Y, X) ,$$

where $\omega \in \Lambda^1(\Omega)$, $X, Y \in \mathcal{X}(\Omega)$.

The following lemma shows relationships between the Riemannian metric and the Levi-Civita connection.

LEMMA 3.2. (a) $g(\nabla_j e_i, e_i) = 0$. (b) $g(\nabla_i e_i, e_j) = -g(\nabla_i e_j, e_i)$.

PROOF. (a) From (2.1) and $g(e_i, e_j) \equiv \delta_{ij}$, we obtain

$$0 = e_j g(e_i, e_i) = g(\nabla_j e_i, e_i) + g(e_i, \nabla_j e_i) = 2g(\nabla_j e_i, e_i) .$$

(b) We have

$$0 = e_i g(e_i, e_j) = g(\nabla_i e_i, e_j) + g(e_i, \nabla_i e_j) ,$$

which shows that $g(\nabla_i e_i, e_j) = -g(\nabla_i e_j, e_i)$. □

We will use a trace operator and an extension operator, defined as follows.

LEMMA 3.3. *Let $D \subset M$ be an open set with C^∞-boundary ∂D (codimension 1). There exists a bounded operator (trace operator) $\gamma : H^1(D) \to H^{1/2}(\partial D)$, such that $\gamma\Phi = \Phi|_{\partial D}$ on ∂D for $\Phi \in C^\infty(D)$. Moreover, there exists a bounded operator (extension operator) $E : H^{1/2}(\partial D) \to H^1(D)$, such that $\gamma E\varphi = \varphi$ for $\varphi \in C^\infty(\partial D)$.*

For a proof, see §8 in [10].

Though, the Green and the Stokes formulas of L^2-type are well-known, we can not find the proofs. So, we give proofs of Propositions 3.1 and 3.2. Let $D \subset \Omega$ be an open submanifold and let ν be the outer unit normal vector field on ∂D.

PROPOSITION 3.1. *For any $\omega \in H(\delta_0; D)$, we have $\gamma(\omega, \nu^\flat)_1 \in H^{-1/2}(\partial D)$. Furthermore, for $f \in H^1(D)$,*

$$\langle \delta_0 \omega, f \rangle_{0,D} = \langle \omega, d_0 f \rangle_{1,D} - \int_{\partial D} \gamma(\omega, \nu^\flat)_1 \gamma f dS_g \tag{3.1}$$

holds.

PROOF. When no confusion can arise, we omit γ. First, we prove the formula (3.1) for $\omega \in \Lambda^1(D)$ and $f \in C^\infty(D)$. Note that

$$\mathrm{div}(\omega^\sharp) = -\delta_0(\omega).$$

We have

$$\begin{aligned}
\delta_0(f\omega) &= -\sum_{i=1}^{N} (\nabla_i f\omega)(e_i) \\
&= -\sum_{i=1}^{N} \{e_i(f\omega(e_i)) - f\omega(\nabla_i e_i)\} \\
&= -\sum_{i=1}^{N} \{e_i(f)\omega(e_i) + f e_i(\omega(e_i)) - f\omega(\nabla_i e_i)\} \\
&= -(d_0 f, \omega)_1 + f\delta_0(\omega).
\end{aligned}$$

Integrating both sides and using Lemma 3.1, we obtain

$$\begin{aligned}
\langle \delta_0 \omega, f \rangle_{0,D} &= \langle \omega, d_0 f \rangle_{1,D} - \int_{\partial D} f g(\omega^\sharp, \nu) dS_g \\
&= \langle \omega, d_0 f \rangle_{1,D} - \int_{\partial D} \gamma(\omega, \nu^\flat)_1 \gamma f dS_g.
\end{aligned}$$

Remark that (3.1) holds for $\omega \in \Lambda^1(D)$ and $f \in H^1(D)$.

Next, we need to prove $\gamma(\omega, \nu^\flat)_1 \in H^{-1/2}(\partial D)$ for $\omega \in H(\delta_0; D)$. It is sufficient to prove that $\int_{\partial D} \gamma(\omega, \nu^\flat)_1 \tilde{f} dS_g$ is well-defined for any $\tilde{f} \in H^{1/2}(\partial D)$. From Lemma 3.3, we see $f := E\tilde{f} \in H^1(D)$. We know that $\Lambda^1(D)$ is dense in $H(\delta_0; D)(= \mathcal{H}_{1,0})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_{1,0}}$. Therefore, we can find some sequence $\{\omega_k\}_k \subset \Lambda^1(D)$, such that

$$\|\omega_k - \omega\|_{\mathcal{H}_{1,0}} \to 0 \quad (k \to \infty).$$

For ω_k, we see that

$$\int_{\partial D} \gamma(\omega_k, \nu^\flat)_1 \tilde{f} dS_g = \langle \omega_k, d_0 f \rangle_{1,D} - \langle \delta_0 \omega_k, f \rangle_{0,D}.$$

The right hand side converges to $\langle \omega, d_0 f \rangle_{1,D} - \langle \delta_0 \omega, f \rangle_{0,D}$. Hence, the left hand side $\gamma(\omega, \nu^\flat)_1$ is determined as an element of $H^{-1/2}(\partial D)$. □

PROPOSITION 3.2. *For any $\omega \in H(d_1; D)$, we have $\gamma(\nu^\flat \wedge \omega) \in H^{-1/2}(\Lambda^2(\partial D))$. Furthermore, for $\eta \in H^1(\Lambda^2(D))$,*

$$\langle d_1\omega, \eta\rangle_{2,D} = \langle \omega, \delta_1\eta\rangle_{1,D} + \int_{\partial D} (\nu^\flat \wedge \omega, \eta)_2 dS_g \tag{3.2}$$

holds.

PROOF. By a similar argument to the proof of Proposition 3.1, we see that it suffices to prove (3.2) for $\omega \in \Lambda^1(D)$ and $\eta \in \Lambda^2(D)$. Let $\omega = \omega^k\xi_k$ and $\eta = \eta^{ij}\xi_i \wedge \xi_j$ for $i < j$. We calculate the inner product between $d_1\omega$ and η. Note that

$$\xi_i(\cdot) = g(\cdot, e_i)\,.$$

Fixing $i < j$, we calculate $\langle d_1\omega, \eta\rangle_{2,D}$ and $\langle \omega, \delta_1\eta\rangle_{1,D}$ for the cases [I] $k = i$; [II] $k = j$; and [III] $k \neq i,\ k \neq j$, separately.

[I]. $k = i$. By Lemma 3.2 (a), we see that

$$\begin{aligned} d_1\omega(e_i, e_j) &= -e_j\omega^i - \omega^i\xi_i(\nabla_i e_j) + \omega^i\xi_i(\nabla_j e_i) \\ &= -e_j\omega^i - \omega^i g(\nabla_i e_j, e_i) + \omega^i g(\nabla_j e_i, e_i) \\ &= -e_j\omega^i - \omega^i g(\nabla_i e_j, e_i)\,, \end{aligned}$$

and

$$\begin{aligned} \delta_1(\eta)(e_i) &= -\sum_{l=1}^N (\nabla_l\eta)(e_l, e_i) = -\sum_{l=1}^N \{e_l\eta(e_l, e_i) - \eta(\nabla_l e_l, e_i) - \eta(e_l, \nabla_l e_i)\} \\ &= -\sum_{l=1}^N e_l\big[\eta^{ij}\{\xi_i(e_l)\xi_j(e_i) - \xi_i(e_i)\xi_j(e_l)\}\big] \\ &\qquad + \sum_{l=1}^N \eta^{ij}\{\xi_i(\nabla_l e_l)\xi_j(e_i) - \xi_i(e_i)\xi_j(\nabla_l e_l)\} \\ &\qquad + \sum_{l=1}^N \eta^{ij}\{\xi_i(e_l)\xi_j(\nabla_l e_i) - \xi_i(\nabla_l e_i)\xi_j(e_l)\} \\ &= e_j\eta^{ij} - \eta^{ij}\sum_{l=1}^N \xi_j(\nabla_l e_l) + \eta^{ij}\{\xi_j(\nabla_i e_i) - \xi_i(\nabla_j e_i)\} \\ &= e_j\eta^{ij} + \eta^{ij}\left\{-\sum_{l=1}^N g(\nabla_l e_l, e_j) + g(\nabla_i e_i, e_j) - g(\nabla_j e_i, e_i)\right\} \\ &= e_j\eta^{ij} + \eta^{ij}\left\{-\sum_{l=1}^N g(\nabla_l e_l, e_j) + g(\nabla_i e_i, e_j)\right\}. \end{aligned}$$

From Lemma 3.2 (b), we obtain

$$\begin{aligned}(d_1\omega, \eta)_2 - (\omega, \delta_1\eta)_1 &= -e_j(\omega^i)\eta^{ij} - \omega^i e_j(\eta^{ij}) + \omega^i\eta^{ij}\sum_{l=1}^{N} g(\nabla_l e_l, e_j)\\ &= -e_j(\omega^i\eta^{ij}) - \omega^i\eta^{ij}\sum_{l=1}^{N} g(e_l, \nabla_l e_j)\\ &= -\mathrm{div}(\omega^i\eta^{ij}e_j),\end{aligned}$$

where we used $\mathrm{div}(fX) = Xf + f\mathrm{div}(X)$, as $X = e_j$ and $f = \omega^i\eta^{ij}$. Therefore, from Lemma 3.1, we have

$$\langle d_1\omega, \eta\rangle_{2,D} - \langle\omega, \delta_1\eta\rangle_{1,D} = -\int_{\partial D} g(\omega^i\eta^{ij}e_j, \nu)dS_g\,.$$

[II]. $k = j$. We see that

$$\begin{aligned}d_1\omega(e_i, e_j) &= e_i\omega(e_j) - \omega(\nabla_i e_j) - e_j\omega(e_i) + \omega(\nabla_j e_i)\\ &= e_i\omega^j - \omega^j\xi_j(\nabla_i e_j) + \omega^j\xi_j(\nabla_j e_i)\\ &= e_i\omega^j - \omega^j g(\nabla_i e_j, e_j) + \omega^j g(\nabla_j e_i, e_j)\\ &= e_i\omega^j + \omega^j g(\nabla_j e_i, e_j),\end{aligned}$$

and

$$\begin{aligned}\delta_1(\eta)(e_j) &= -\sum_{l=1}^{N}(\nabla_l\eta)(e_l, e_j) = -\sum_{l=1}^{N}\big\{e_l\eta(e_l, e_j) - \eta(\nabla_l e_l, e_j) - \eta(e_l, \nabla_l e_j)\big\}\\ &= -e_i\eta^{ij} + \eta^{ij}\left\{\sum_{l=1}^{N} g(\nabla_l e_l, e_i) + g(\nabla_j e_i, e_j)\right\}.\end{aligned}$$

This implies that

$$\begin{aligned}&(d_1\omega, \eta)_2 - (\omega, \delta_1\eta)_1\\ &\quad= (e_i\omega^j + \omega^j g(\nabla_j e_i, e_j))\eta^{ij} - \omega^j\left(-e_i\eta^{ij} + \eta^{ij}\left\{\sum_{l=1}^{N} g(\nabla_l e_l, e_i) + g(\nabla_j e_i, e_j)\right\}\right)\\ &\quad= \eta^{ij}e_i\omega^j + \omega^j e_i\eta^{ij} - \omega^j\eta^{ij}\sum_{l=1}^{N} g(\nabla_l e_l, e_i)\\ &\quad= \mathrm{div}(\omega^j\eta^{ij}e_i)\,.\end{aligned}$$

Hence, integrating both sides yields

$$\langle d_1\omega, \eta\rangle_{2,D} - \langle\omega, \delta_1\eta\rangle_{1,D} = \int_{\partial D} g(\omega^j\eta^{ij}e_i, \nu)dS_g\,.$$

[III]. $k \neq i, j$. We see that

$$\begin{aligned} d_1\omega(e_i, e_j) &= (\nabla_i \omega)(e_j) - (\nabla_j \omega)(e_i) \\ &= e_i\omega(e_j) - \omega(\nabla_i e_j) - e_j\omega(e_i) + \omega(\nabla_j e_i) \\ &= -\omega^k \xi_k(\nabla_i e_j) + \omega^k \xi_k(\nabla_j e_i) \\ &= -\omega^k \{g(\nabla_i e_j, e_k) - g(\nabla_j e_i, e_k)\}, \end{aligned}$$

and

$$\begin{aligned} \delta_1(\eta)(e_k) &= -\sum_{l=1}^N (\nabla_l \eta)(e_l, e_k) = -\sum_{l=1}^N \{e_l \eta(e_l, e_k) - \eta(\nabla_l e_l, e_k) - \eta(e_l, \nabla_l e_k)\} \\ &= -\sum_{l=1}^N e_l\big[\eta^{ij}\{\xi_i(e_l)\xi_j(e_k) - \xi_i(e_k)\xi_j(e_l)\}\big] \\ &\quad + \sum_{l=1}^N \eta^{ij}\{\xi_i(\nabla_l e_l)\xi_j(e_k) - \xi_i(e_k)\xi_j(\nabla_l e_l)\} \\ &\quad + \sum_{l=1}^N \eta^{ij}\{\xi_i(e_l)\xi_j(\nabla_l e_k) - \xi_i(\nabla_l e_k)\xi_j(e_l)\} \\ &= \eta^{ij}\{\xi_j(\nabla_i e_k) - \xi_i(\nabla_j e_k)\} \\ &= \eta^{ij}\{g(\nabla_i e_k, e_j) - g(\nabla_j e_k, e_i)\}. \end{aligned}$$

By performing calculations similar to those performed in [I] and [II], we obtain

$$(d_1\omega, \eta)_2 = (\omega, \delta_1\eta)_1 .$$

Moreover, in general we find that

$$\langle d_1\omega, \eta\rangle_{2,D} - \langle \omega, \delta_1\eta\rangle_{1,D} = \sum_{i<j} \int_{\partial D} (g(\omega^j \eta^{ij} e_i, \nu) - g(\omega^i \eta^{ij} e_j, \nu)) dS_g .$$

Let $\nu = \sum_{i=1}^N \nu^i e_i$. Since $\nu^\flat = \sum_{i=1}^N \nu^i \xi_i$, we have

$$\begin{aligned} \sum_{i<j}(g(\omega^j \eta^{ij} e_i, \nu) - g(\omega^i \eta^{ij} e_j, \nu)) &= \sum_{i<j}(\omega^j \eta^{ij} \nu^i - \omega^i \eta^{ij} \nu^j) \\ &= \sum_{i<j}(\omega^j \nu^i - \omega^i \nu^j)\eta^{ij} = (\nu^\flat \wedge \omega, \eta)_2. \end{aligned}$$

Therefore, we obtain

$$\langle d_1\omega, \eta\rangle_{2,D} = \langle \omega, \delta_1\eta\rangle_{1,D} + \int_{\partial D} (\nu^\flat \wedge \omega, \eta)_2 dS_g \,.$$

□

4. Proofs of Theorems

LEMMA 4.1. *If $f \in H^1(\Omega)$, then we have* (a) $[f]_-^+ = 0$ *on Γ as an element of $H^{1/2}(\Gamma)$, and* (b) $[\nu^\flat \wedge d_0 f]_-^+ = 0$ *on Γ as an element of $H^{-1/2}(\Lambda^2(\Gamma))$.*

PROOF. Let $\{f_n\}_n \subset C^\infty(\Omega)$ be a sequence approximating f in $H^1(\Omega)$. Let $\gamma_\pm : H^1(\Omega_\pm) \to H^{1/2}(\Gamma_\pm)$ be the trace operators by Lemma 3.3. Then, we see that $\gamma_\pm f \in H^{1/2}(\Gamma)$. We now prove that $\|\gamma_+ f - \gamma_- f\|_{H^{1/2}(\Gamma)} = 0$:

$$\begin{aligned}\|\gamma_+ f - \gamma_- f\|_{H^{1/2}(\Gamma)} &= \|\gamma_+ f - \gamma_+ f_n + \gamma_- f_n - \gamma_- f\|_{H^{1/2}(\Gamma)} \\ &\le c(\|f - f_n\|_{H^1(\Omega_+)} + \|f_n - f\|_{H^1(\Omega_-)}) \\ &\to 0\,,\end{aligned}$$

for some constant $c > 0$ as $n \to \infty$. For any $C \in \Lambda_0^2(\Omega)$, we have

$$\begin{aligned}0 &= \langle d_1 d_0 f_n, C\rangle_{2,\Omega} \\ &= \int_{\Omega_+} (d_1 d_0 f_n, C)_2 dv_g + \int_{\Omega_-} (d_1 d_0 f_n, C)_2 dv_g \\ &= \langle d_0 f_n, \delta_1 C\rangle_{1,\Omega} + \int_\Gamma [(\nu^\flat \wedge d_0 f_n, C)_2]_-^+ dS_g \\ &= \langle f_n, \delta_0\delta_1 C\rangle_{0,\Omega} + \int_\Gamma [f_n(\nu^\flat, \delta_1 C)_1]_-^+ dS_g + \int_\Gamma [(\nu^\flat \wedge d_0 f_n, C)_2]_-^+ dS_g \\ &= \int_\Gamma [(\nu^\flat \wedge d_0 f_n, C)_2]_-^+ dS_g \,.\end{aligned}$$

By letting $n \to \infty$, we obtain $d_1 d_0 f = 0$. Hence, we can deduce that $d_0 f \in \mathcal{H}_{2,0}$, and $[\nu^\flat \wedge d_0 f]_-^+ = 0$ on Γ as an element of $H^{-1/2}(\Lambda^2(\Gamma))$. □

DEFINITION. We define differential operators (ν, d_0) by

$$(\nu, d_0) f := (d_0 f)(\nu) = \nu(f)\,, \quad \text{for } f \in H^1(D)\,,$$

$$(\nu, d_0) B := \sum_{i=1}^N (\nu, d_0)(B(e_i))\xi_i = \sum_{i=1}^N d_0(B(e_i))(\nu)\xi_i\,, \quad \text{for } B \in H^1(\Lambda^1(D))\,.$$

From the definitions of the operators $\sharp$, $\flat$, and (ν, d_0), it follows that

$$\begin{aligned}(d_0(\nu^\flat, B)_1, \nu^\flat)_1 &= g((d_0(\nu^\flat, B)_1)^\sharp, \nu) = (d_0(\nu^\flat, B)_1)(\nu) \\ &= (\nu, d_0)(\nu^\flat, B)_1 \,. \end{aligned} \tag{4.1}$$

DEFINITION. For $B \in H^1(\Lambda^1(D))$, we set

$$\delta_{0\nu} B := -(\nu^\flat, (\nu, d_0)B)_1 , \quad \delta_{0\tau} := \delta_0 - \delta_{0\nu} .$$

PROPOSITION 4.1. *Let $B \in H^1(\Lambda^1(\Omega))$. Then, we can decompose $\delta_0 B$ as follows:*

$$\delta_0 B = ((\nu, d_0)\nu^\flat, B^\tau)_1 - (d_0(\nu^\flat, B)_1, \nu^\flat)_1 + \delta_{0\tau} B^\tau + (\nu^\flat, B)_1 \delta_0(\nu^\flat) \tag{4.2}$$

in $L^2_{loc}(\Omega)$ (which is the set of $L^2(\Omega')$-functions for any N-dimension compact submanifold $\Omega' \subset \Omega$). Moreover, assume that $\delta_0 B = \psi \in H^1(\Omega)$. Then we have

$$[(d_0(\nu^\flat, B)_1, \nu^\flat)_1]^+_- = 0 \tag{4.3}$$

as an element of $H^{-1/2}(\Gamma)$.

PROOF. First, we prove (4.2). We start by showing that

$$(\nu^\flat, (\nu, d_0)\nu^\flat)_1 = 0 .$$

Since $(\nu^\flat, \nu^\flat)_1 = 1$, we have

$$0 = (\nu, d_0) \sum_{i=1}^N \nu^\flat(e_i)^2 = \sum_{i=1}^N d_0(\nu^\flat(e_i)^2)(\nu) = 2 \sum_{i=1}^N \nu(\nu^\flat(e_i)) \cdot \nu^\flat(e_i) .$$

This implies that

$$\begin{aligned}(\nu^\flat, (\nu, d_0)\nu^\flat)_1 &= \left(\nu^\flat, \sum_{i=1}^N d_0(\nu^\flat(e_i))(\nu)\xi_i\right)_1 \\ &= \sum_{i=1}^N (\nu^\flat, \xi_i)_1 d_0(\nu^\flat(e_i))(\nu) \\ &= \sum_{i=1}^N \nu^\flat(e_i) d_0(\nu^\flat(e_i))(\nu) \\ &= \sum_{i=1}^N \nu^\flat(e_i)\nu(\nu^\flat(e_i)) = 0 .\end{aligned}$$

We can then compute $\delta_{0\nu} B^\tau$, $\delta_{0\tau} B^\nu$, and $\delta_{0\nu} B^\nu$ as follows:

$$\begin{aligned}\delta_{0\nu} B^\tau &= -(\nu^\flat, (\nu, d_0)B^\tau)_1 = -(\nu, d_0)(\nu^\flat, B^\tau)_1 + ((\nu, d_0)\nu^\flat, B^\tau)_1 \\ &= ((\nu, d_0)\nu^\flat, B^\tau)_1 , \\ \delta_{0\tau} B^\nu &= \delta_0 B^\nu - \delta_{0\nu} B^\nu = \delta_0 B^\nu + (\nu^\flat, (\nu, d_0)B^\nu)_1 \\ &= \delta_0((\nu^\flat, B)_1 \nu^\flat) + (\nu^\flat, (\nu, d_0)((\nu^\flat, B)_1 \nu^\flat))_1 \\ &= -(d_0(\nu^\flat, B)_1, \nu^\flat)_1 + (\nu^\flat, B)_1 \delta_0(\nu^\flat) + (\nu, d_0)((\nu^\flat, (\nu^\flat, B)_1 \nu^\flat)_1)\end{aligned}$$

$$
\begin{aligned}
&\quad - ((\nu, d_0)\nu^\flat, (\nu^\flat, B)_1\nu^\flat)_1 \\
&= -(d_0(\nu^\flat, B)_1, \nu^\flat)_1 + (\nu^\flat, B)_1\delta_0(\nu^\flat) + (\nu, d_0)((\nu^\flat, B)_1) \\
&\quad - (\nu^\flat, B)_1((\nu, d_0)\nu^\flat, \nu^\flat)_1 \\
&= (\nu^\flat, B)_1\delta_0(\nu^\flat)\,, \\
\delta_{0\nu}B^\nu &= -(\nu^\flat, (\nu, d_0)((\nu^\flat, B)_1\nu^\flat))_1 \\
&= -(\nu, d_0)(\nu^\flat, (\nu^\flat, B)_1\nu^\flat)_1 + ((\nu, d_0)\nu^\flat, (\nu^\flat, B)_1\nu^\flat) \\
&= -(\nu, d_0)(\nu^\flat, B)_1 = -(d_0(\nu^\flat, B)_1, \nu^\flat)_1.
\end{aligned}
$$

It follows that

$$
\delta_0 B = ((\nu, d_0)\nu^\flat, B^\tau)_1 - (d_0(\nu^\flat, B)_1, \nu^\flat)_1 + \delta_{0\tau}B^\tau + (\nu^\flat, B)_1\delta_0(\nu^\flat)\,.
$$

Now, we prove (4.3). From (4.2) and $[\psi]_-^+ = 0$ on Γ, we have

$$
\begin{aligned}
0 &= [\psi]_-^+ = [\delta_0 B]_-^+ \\
&= [((\nu, d_0)\nu^\flat, B^\tau)_1]_-^+ - [(d_0(\nu^\flat, B)_1, \nu^\flat)_1]_-^+ + [\delta_{0\tau}B^\tau]_-^+ + [(\nu^\flat, B)_1\delta_0(\nu^\flat)]_-^+ \\
&= [(d_0(\nu^\flat, B)_1, \nu^\flat)_1]_-^+ + [\delta_{0\tau}B^\tau]_-^+\,.
\end{aligned}
$$

Therefore, it suffices to prove that $[\delta_{0\tau}B^\tau]_-^+ = 0$ as an element of $H^{-1/2}(\Gamma)$.

We remark that $\nabla_i e_i$ is represented as

$$
\nabla_i e_i = \sum_{j=1}^N a_{ij}e_j\,,
$$

for some $a_{ij} \in C^\infty(M)$. For $C = \sum_{j=1}^N C^j\xi_j \in \Lambda^1(M)$ and $\nu^i := \nu(e_i)$, we define

$$
\delta_0^{(i)}C^l := -e_i(C^l) + \sum_{j=1}^N a_{ij}C^j\,, \quad \delta_{0\nu}^{(i)}C^l := -\nu^i(\nu, d_0)C^l\,,
$$

and

$$
\delta_{0\tau}^{(i)}C^l := \delta_0^{(i)}C^l - \delta_{0\nu}^{(i)}C^l\,.
$$

Then, we see that

$$
\delta_{0\tau}B^\tau = \sum_{i=1}^N \delta_{0\tau}^{(i)}B^{\tau i}\,.
$$

Therefore, for each $1 \le i \le N$, it suffices to prove that $[\delta_{0\tau}^{(i)}B^{\tau i}]_-^+ = 0$ on Γ. For $f \in \Lambda^0(M)$, we have

$$\nu^\flat \wedge d_0 f = \sum_{k=1}^N \nu^k \xi_k \wedge \sum_{l=1}^N e_l f \xi_l = \sum_{1\le k<l\le N} \{\nu^k e_l f - \nu^l e_k f\}\xi_k \wedge \xi_l \,.$$

If $f \in H^1(\Omega)$, then we have

$$[\nu^k e_l f - \nu^l e_k f]_-^+ = 0\,, \quad k < l \quad \text{on } \Gamma\,, \tag{4.4}$$

from Lemma 4.1 (b). Therefore, we have

$$\begin{aligned}
\Big[\delta_{0\tau}^{(i)} B^{\tau i}\Big]_-^+ &= \Big[- e_i(B^{\tau i}) + \nu^i(\nu, d_0)B^{\tau i}\Big]_-^+ \\
&= \Big[- \sum_{k=1}^N (\nu^k)^2 e_i(B^{\tau i}) + \sum_{k=1}^N \nu^i \nu^k e_k(B^{\tau i})\Big]_-^+ \\
&= \Big[- \sum_{k=1}^N \nu^k\{\nu^k e_i(B^{\tau i}) - \nu^i e_k(B^{\tau i})\}\Big]_-^+ = 0\,.
\end{aligned}$$

□

PROOF OF THEOREM 2.1. Let $\Delta = -\delta_0 d_0$ and $B \in H^1(\Lambda^1(\Omega))$. Since B satisfies (2.3), $B \in H^2_{loc}(\Lambda^1(\Omega_\pm))$. Set

$$f = \begin{cases} (\nu^\flat, B)_1\,, & \text{in } \Omega_+\,, \\ (\nu^\flat, B)_1\,, & \text{in } \Omega_-\,. \end{cases}$$

For any $\varphi \in \Lambda_0^0(\Omega)$, we will prove that

$$\langle f, \Delta\varphi\rangle_{0,\Omega} = \langle \Delta f, \varphi\rangle_{0,\Omega}\,.$$

From Proposition 3.1, we obtain

$$\begin{aligned}
\langle d_0 f, d_0\varphi\rangle_{1,\Omega} &= \left(\int_{\Omega_+} + \int_{\Omega_-}\right)(d_0 f, d_0\varphi)_1 dv_g \\
&= \left(\int_{\Omega_+} + \int_{\Omega_-}\right) f\delta_0(d_0\varphi)dv_g + \int_\Gamma [f(d_0\varphi, \nu^\flat)_1]_-^+ dS_g \\
&= \langle f, \Delta\varphi\rangle_{0,\Omega}\,.
\end{aligned}$$

On the other hand, from Proposition 3.1 and (4.3), we have

$$\begin{aligned}
\langle d_0 f, d_0\varphi\rangle_{1,\Omega} &= -\left(\int_{\Omega_+} + \int_{\Omega_-}\right)(\delta_0 d_0 f, \varphi)_0 dv_g - \int_\Gamma [(d_0 f, \nu^\flat)_1]_-^+ \varphi dS_g \\
&= \int_\Omega (\Delta f)\varphi dv_g = \langle \Delta f, \varphi\rangle_{0,\Omega}\,.
\end{aligned}$$

Hence, by the elliptic regularity theorem (cf. [4]), we have that $(\nu^\flat, B)_1 \in H^2_{loc}(\Omega)$. $\square$

PROOF OF THEOREM 2.2. Let $\Delta = -(\delta_1 d_1 + d_0\delta_0)$ and $B \in H^1(\Lambda^1(\Omega))$. By Proposition 3.1 and Lemma 4.1 (b), we obtain for any $C \in \Lambda^1_0(\Omega) := \{C \in \Lambda^1(\Omega); C$ has a compact support$\}$

$$\begin{aligned}\langle \delta_0 B^\tau, \delta_0 C\rangle_{0,\Omega} &= \left(\int_{\Omega_+} + \int_{\Omega_-}\right)\delta_0(B^\tau)\delta_0(C)dv_g \\ &= \langle B^\tau, d_0\delta_0 C\rangle_{1,\Omega} - \int_\Gamma [(B^\tau, \nu^\flat)_1\delta_0(C)]^+_- dS_g \\ &= \langle B^\tau, d_0\delta_0 C\rangle_{1,\Omega}\,.\end{aligned}$$

In addition, from Proposition 3.1, we have

$$\langle \delta_0 B^\tau, \delta_0 C\rangle_{0,\Omega} = \langle d_0\delta_0 B^\tau, C\rangle_{1,\Omega} - \int_\Gamma [\delta_0(B^\tau)(C, \nu^\flat)_1]^+_- dS_g\,.$$

We will prove that $[\delta_0 B^\tau]^+_- = 0$. From the definition of δ_0, we see that

$$\delta_0(fC) = -(d_0 f, C)_1 + f\delta_0 C\,.$$

Therefore, we have

$$\begin{aligned}\delta_0 B^\tau &= \delta_0(B - (\nu^\flat, B)\nu^\flat) \\ &= -\sum_{i=1}^N \{e_i B(e_i) - B(\nabla_i e_i)\} + (d_0(\nu^\flat, B)_1, \nu^\flat)_1 - (\nu^\flat, B)_1\delta_0(\nu^\flat),\end{aligned}$$

which implies that

$$[\delta_0 B^\tau]^+_- = \left[-\sum_{i=1}^N e_i B(e_i) + (d_0(\nu^\flat, B)_1, \nu^\flat)_1\right]^+_-\,.$$

From (4.1) and (4.4), we obtain

$$\begin{aligned}[\delta_0 B^\tau]^+_- &= \left[-\sum_{i=1}^N e_i B(e_i) + (\nu, d_0)(\nu^\flat, B)_1\right]^+_- \\ &= \left[-\sum_{i=1}^N e_i B(e_i) + (\nu^\flat, (\nu, d_0)B)_1 + ((\nu, d_0)\nu^\flat, B)_1\right]^+_- \\ &= \left[-\sum_{i=1}^N e_i B(e_i) + \sum_{i=1}^N \nu(B(e_i))\nu(\xi_i) + ((\nu, d_0)\nu^\flat, B)_1\right]^+_- \\ &= \left[-\sum_{i=1}^N e_i B(e_i) + \sum_{i=1}^N\sum_{l=1}^N \nu(\xi_l)e_l(B(e_i))\nu(\xi_i)\right]^+_-\end{aligned}$$

$$= \Big[-\sum_{i=1}^{N}\sum_{l=1}^{N}(\nu(\xi_l))^2 e_i B(e_i) + \sum_{i=1}^{N}\sum_{l=1}^{N}\nu(\xi_l)e_l(B(e_i))\nu(\xi_i)\Big]_-^+$$
$$= -\Big[\sum_{i=1}^{N}\sum_{l=1}^{N}\nu(\xi_l)\{\nu(\xi_l)e_i B(e_i) - \nu(\xi_i)e_l B(e_i)\}\Big]_-^+$$
$$= 0\,.$$

Hence, we have

$$\langle B^\tau, d_0\delta_0 C\rangle_{1,\Omega} = \langle d_0\delta_0 B^\tau, C\rangle_{1,\Omega}\,.$$

Similarly, we can easily see that

$$\langle d_1 B^\tau, d_1 C\rangle_{2,\Omega} = \langle B^\tau, \delta_1 d_1 C\rangle_{1,\Omega}\,.$$

From Proposition 3.2, we have

$$\langle d_1 B^\tau, d_1 C\rangle_{2,\Omega} = \langle \delta_1 d_1 B^\tau, C\rangle_{1,\Omega} + \int_\Gamma [(d_1 B^\tau, \nu^\flat \wedge C)_2]_-^+ dS_g\,. \tag{4.5}$$

We will now prove that $[d_1 B^\tau]_-^+ = 0$ on Γ. From $d_{p+q}(\omega\wedge\eta) = (d_p\omega)\wedge\eta + (-1)^p\omega\wedge d_q\eta$ and Lemma 4.1 (b), we obtain

$$\begin{aligned}[d_1 B^\tau]_-^+ &= [d_1 B - d_1(\nu^\flat(\nu^\flat, B)_1)]_-^+ = [J - d_1(\nu^\flat(\nu^\flat, B)_1)]_-^+\\ &= -[(\nu^\flat, B)_1 d_1\nu + (-1)^1\nu^\flat\wedge d_0(\nu^\flat, B)_1]_-^+\\ &= [\nu^\flat\wedge d_0(\nu^\flat, B)_1]_-^+\\ &= 0\,.\end{aligned}$$

Therefore, we have

$$\langle \delta_1 d_1 B^\tau, C\rangle_{1,\Omega} = \langle B^\tau, \delta_1 d_1 C\rangle_{1,\Omega}\,.$$

Hence, we obtain

$$\langle \Delta B^\tau, C\rangle_{1,\Omega} = \langle B^\tau, \Delta C\rangle_{1,\Omega}\,.$$

As a result, we have $B^\tau \in H^2_{loc}(\Lambda^1(\Omega))$. □

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