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On Expressions of Theta Series by η-products

Akihiko OKAMOTO

Waseda University

Abstract. In this paper, we give a certain identity between an η-product of weight 1 and theta series associated with a pair of binary quadratic forms. We also have explicit description of Siegel's theorem by an η-product. For quadratic forms Q_1 and Q_2 which are in the same genus, we express the difference $\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)$ by an η-product.

1. Introduction

The Dedekind η-function is defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\,,$$

where τ lies in the complex upper half plane $\mathcal{H} = \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$ and $q = \exp(2\pi i\tau)$. An η-quotient is defined to be a product of the form

$$f(\tau) = \prod_{\substack{i|N \\ i>0}} \eta(i\tau)^{e_i}\,,$$

where $e_i \in \mathbf{Z}$. This is a modular form of weight $k = \frac{1}{2}\sum_{0<i|N} e_i$ with a multiplier system (cf. [3]). When all of the $e_i's$ are non-negative, we say that $f(\tau)$ is an η-product. Fourier coefficients of η-products are related to many well-known number-theoretic functions, including partition functions and quadratic form representation numbers (cf. [4], [10]). Here we study connections between η-products of weight 1 and theta series associated with a pair of binary quadratic forms.

Let p be a prime number such that $p \equiv -1 \pmod{24}$. Consider the following pair of primitive binary quadratic forms with discriminant $-p$:

$$Q_1 : 6x^2 + xy + \frac{p+1}{24}y^2\,, \quad Q_2 : 6x^2 + 5xy + \frac{p+25}{24}y^2\,.$$

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Serre [12] has given the following identity.

$$\frac{1}{2}\big(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)\big) = \eta(\tau)\eta(p\tau)\,,$$

where $\vartheta_{Q_1}(\tau)$ and $\vartheta_{Q_2}(\tau)$ are the theta series associated with Q_1 and Q_2 respectively. We will extend this relation and our result is the following:

THEOREM 1. *Let N be a square-free positive integer such that $N \equiv -1 \pmod{24}$. Let Q_1 and Q_2 be two primitive binary quadratic forms which are given by*

$$Q_1 :\ 6x^2 + xy + \frac{N+1}{24}y^2\,, \qquad Q_2 :\ 6x^2 + 5xy + \frac{N+25}{24}y^2$$

respectively. Then we have the equality

$$\frac{1}{2}\big(\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)\big) = \eta(\tau)\eta(N\tau).$$

2. Preliminaries

In this section, we recall some known results about η-products and theta series associated with a quadratic form.

PROPOSITION 1 ([3], p. 174). *Suppose that $f(\tau) = \prod_{0<i|N} \eta(i\tau)^{e_i}$ is an η-product which satisfies the following two properties*

$$(1)\ \sum_{0<i|N} ie_i \equiv 0 \pmod{24};$$

$$(2)\ \sum_{0<i|N} \frac{N}{i}e_i \equiv 0 \pmod{24}.$$

Then an η-product $f(\tau)$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, *where* $k = \frac{1}{2}\sum_{0<i|N} e_i$, $\chi(d) = (\frac{(-1)^k s}{d})$ (Jacobi symbol), and $s = \prod_{0<i|N} i^{e_i}$.

Hence $f(\tau)$ is in the vector space $\mathcal{M}_k(\Gamma_0(N), \chi)$ of modular forms on $\Gamma_0(N)$ with weight k and character χ, holomorphic in $\mathcal{H}$ and at the cusps of $\Gamma_0(N)$. These cusps can be represented by rational numbers a/c, where $c \mid N$, $c > 0$ and gcd(a, c) = 1 (cf.[2] p.103). The order of $f(\tau)$ at the cusp a/c is

$$\nu_{a/c} = \frac{h_c}{24}\sum_{0<i|N} \frac{gcd(i,c)^2}{i}e_i\,, \tag{1}$$

where $h_c = \frac{N}{\gcd(c^2,N)}$ is the width of the cusp a/c (cf.[6], proposition 3.2.8).

Next, we review the theta series associated with a quadratic form. Let A be an even integral symmetric r x r matrix, i.e. $a_{ij} = a_{ji}$ is an integer and a_{ii} is an even integer. Let $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}A^t\mathbf{x} = \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}x_i x_j$ ($\mathbf{x} = (x_1, \ldots, x_r)$) be a positive definite quadratic form, that is $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. The theta series associated with a quadratic form Q is defined by

$$\vartheta_Q(\tau) = \sum_{x \in \mathbf{Z}^r} q^{Q(x)}\,.$$

Assume $r = 2k$ is even. The following result is given by Schoeneberg.

PROPOSITION 2 ([9], Theorem 20). *We have* $\vartheta_Q(\tau) \in \mathcal{M}_k(\Gamma_0(N), \chi)$, *where* N *is the least positive integer such that* NA^{-1} *is even integral and* $\chi(d) = (\frac{(-1)^k \mathrm{detA}}{d})$ (*Jacobi symbol*).

In Theorem 1 we can write $Q_1(x, y) = \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 12 & 1 \\ 1 & \frac{N+1}{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Since $N\begin{pmatrix} 12 & 1 \\ 1 & \frac{N+1}{12} \end{pmatrix}^{-1}$ is even integral, we have $\vartheta_{Q_1}(\tau) \in \mathcal{M}_1(\Gamma_0(N), \chi_{-N})$. Similarly, we have $\vartheta_{Q_2}(\tau) \in \mathcal{M}_1(\Gamma_0(N), \chi_{-N})$.

3. Proof of Theorem 1

In order to show Theorem 1 , we calculate the orders of $\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)$ and $\eta(\tau)\eta(N\tau)$ at the cusps of $\Gamma_0(N)$. Since N is a square-free integer, a complete set of representatives for the cusps of $\Gamma_0(N)$ is

$$\mathcal{C}_N = \left\{ \frac{1}{c} \,\middle|\, c \mid N \right\}.$$

Let $1/c \in \mathcal{C}_N$. First, we consider the η-product $\eta(\tau)\eta(N\tau)$. From (1), the order of $\eta(\tau)\eta(N\tau)$ at $1/c$ is

$$\nu_{1/c} = \frac{N + c^2}{24c}\,.$$

We have $\nu_{1/c} = \frac{N+c^2}{24c} \in \mathbf{N}$, because $c|N$, $24|N + 1$ and $24|c^2 - 1$. Hence the product $\eta(\tau)\eta(N\tau)$ vanishes at all cusps of $\Gamma_0(N)$, and then we obtain $\eta(\tau)\eta(N\tau) \in \mathcal{S}_1(\Gamma_0(N), \chi_{-N})$. Next, we consider the theta series. We put

$$A_1 = \begin{pmatrix} 12 & 1 \\ 1 & \frac{N+1}{12} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 12 & 5 \\ 5 & \frac{N+25}{12} \end{pmatrix}$$

and put $\vartheta(\tau; A_1, N) = \vartheta_{Q_1}(\tau)$, $\vartheta(\tau; A_2, N) = \vartheta_{Q_2}(\tau)$. For a cusp $1/c$, we take $\gamma = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in SL_2(\mathbf{Z})$. Then we have $\gamma\infty = 1/c$. The following equalities are obvious by definition of the theta series.

$$\vartheta(\tau; A_1, N) = \sum_{\substack{\mathbf{g}\equiv 0 \pmod N \\ \mathbf{g}\in \mathbf{Z}^2/N\mathbf{Z}^2}} \vartheta(c\tau; \mathbf{g}, cA_1, cN) \quad (\text{for all c} \in \mathbf{N}) \tag{2}$$

$$\vartheta(\tau+1; \mathbf{h}, A_1, N) = \exp\left(\frac{\mathbf{h}\mathrm{A}_1^{\mathrm{t}}\mathbf{h}}{2\mathrm{N}^2}\right)\vartheta(\tau; \mathbf{h}, \mathrm{A}_1, \mathrm{N}) \tag{3}$$

where

$$\vartheta(\tau; \mathbf{h}, A_1, N) = \sum_{\substack{\mathbf{x}\equiv \mathbf{h} \pmod N \\ \mathbf{x}\in \mathbf{Z}^2}} q^{\frac{Q_1(\mathbf{x})}{N^2}}.$$

Since $c(\gamma\tau) = 1 - (c\tau+1)^{-1}$, we obtain by applying (2),(3) and the transformation formula (cf.[7] Lemma 4.9.1)

$$\vartheta \mid [\gamma]_1(\tau; A_1, N) = (\det A_1)^{-\frac{1}{2}} c^{-1}(-\sqrt{-1}) \sum_{\substack{\mathbf{m}\in \mathbf{Z}^2/N\mathbf{Z}^2 \\ A_1\mathbf{m}\equiv 0 \pmod N}} \Phi(\mathbf{m})\vartheta(\tau; A_1, \mathbf{m}, N)\,,$$

where

$$\Phi(\mathbf{m}) = \sum_{\substack{\mathbf{g}\equiv 0 \pmod N \\ \mathbf{g}\in \mathbf{Z}^2/cN\mathbf{Z}^2}} \mathbf{e}\left(\frac{1}{cN^2}\left\{\frac{1}{2}\mathbf{g}A_1{}^t\mathbf{g} + \mathbf{m}A_1{}^t\mathbf{g} + \frac{1}{2}\mathbf{m}A_1{}^t\mathbf{m}\right\}\right).$$

Hence $\vartheta(\tau; A_1, N)$ has a q_{h_c} expansion ($q_{h_c} = q^{\frac{1}{h_c}}$)

$$\vartheta \mid [\gamma]_1(\tau; A_1, N) = (\det A_1)^{-\frac{1}{2}} c^{-1}(-\sqrt{-1}) \sum_{\substack{\mathbf{m}\in \mathbf{Z}^2/N\mathbf{Z}^2 \\ A_1\mathbf{m}\equiv 0 \pmod N}} \Phi(\mathbf{m}) q_{h_c}^{\frac{Q_1(\mathbf{m})h_c}{N^2}}.$$

LEMMA 1. *For $i = 1, 2$ we have*

$$\min\left\{\frac{Q_i(\mathbf{m})h_c}{N^2} \,\middle|\, \mathbf{m}\in \mathbf{Z}^2\backslash\{\mathbf{0}\},\ A_i\mathbf{m}\equiv \mathbf{0} \ (\mathrm{mod} N)\right\} \geq \frac{N+c^2}{24c}.$$

PROOF. We put $\mu_i = \mu_i(\mathbf{m}) := \frac{Q_i(\mathbf{m})h_c}{N^2}$. Then the equation

$$6x^2 + yx + \left(\frac{N+1}{24}y^2 - \frac{\mu_i N^2}{h_c}\right) = 0$$

has integral solutions. We put

$$f(x) := 6x^2 + yx + \left(\frac{N+1}{24}y^2 - \frac{\mu_i N^2}{h_c}\right).$$

Then the discriminant of $f(x)$

$$\begin{aligned}\mathrm{disc(f(x))} &= y^2 - 24\left(\frac{N+1}{24}y^2 - \frac{\mu_i N^2}{h_c}\right) \\ &= N(-y^2 + 24\mu_i c)\end{aligned}$$

is a square. Since N is square-free, there exist $\alpha \in 2\mathbf{N}+1$ and $s \in \mathbf{N}$ such that

$$-y^2 + 24\mu_i c = N^\alpha s^2 .$$

Then it follows that

$$\begin{aligned}y^2 &= 24\mu_i c - N^\alpha s^2 \\ &= c(24\mu_i - h_c N^{\alpha-1} s^2).\end{aligned}$$

Therefore there exist $\beta \in 2\mathbf{N}+1$ and $t \in \mathbf{N}$ such that

$$24\mu_i - h_c N^{\alpha-1} s^2 = c^\beta t^2 .$$

Thus we obtain

$$\mu_i = \frac{h_c N^{\alpha-1} s^2 + c^\beta t^2}{24} \geq \frac{h_c + c}{24} = \frac{N + c^2}{24c} .$$

□

PROOF OF THEOREM 1. By Lemma 1 we have

$$\frac{\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)}{\eta(\tau)\eta(N\tau)} \in \mathcal{M}_0(\Gamma_0(N)).$$

We note that there are no non-constant modular forms of weight zero, i.e.

$$\mathcal{M}_0(\Gamma) = \mathbf{C}$$

for any congruence subgroup Γ (cf.[5] p. 129 proposition 18). Hence we have

$$\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau) = A\eta(\tau)\eta(N\tau)$$

for some $A \in \mathbf{C}$. Comparing the coefficient of $q^{\frac{N+1}{24}}$, we obtain $A = 2$. This completes the proof of Theorem 1 □

4. Application of Theorem 1

We recall the notion of genus of quadratic forms. Two quadratic forms Q_1 and Q_2 are in the same genus, if they are equivalent over $\mathbf{R}$ and $\mathbf{Z}_p$ for all primes p. This definition depends only on equivalence classes and so we can define the genera of classes of quadratic forms. For example, in Theorem 1, Q_1 and Q_2 are in the same genus. In general, Siegel showed the following.

PROPOSITION 3 ([13], p. 577). *If Q_1 and Q_2 are classes of positive definite quadratic forms in even variables which are in the same genus, then $\vartheta_{Q_1}(\tau) - \vartheta_{Q_2}(\tau)$ is a cusp form.*

We note that the above claim is obtained by theta transformation formula. Here, we give examples of Siegel's theorem. Let p be a prime number, and $f = \sum_{n=1}^{\infty} a_n q^n$ a modular form on $\mathcal{M}_k(\Gamma_0(N), \chi)$. The Hecke operators U_p ($p \mid N$) and T_p ($p \nmid N$) are defined by

$$f \mid U_p = \sum_{n=1}^{\infty} a_{pn} q^n ,$$

$$f \mid T_p = \sum_{n=1}^{\infty} a_{pn} q^n + \chi(p) p^{k-1} \sum_{n=1}^{\infty} a_n q^{pn}$$

(cf. [9] [12]). Let $H(-N)$ be the group of equivalent classes of primitive positive definite binary quadratic forms with discriminant $-N$. There is an algorithm for computing equivalent classes of primitive positive definite binary quadratic forms (see for example [1] Theorem 2.8). Moreover, the group law of $H(-N)$ see for example [1] Proposition 3.8 and Theorem 3.9. For simplify, put $[a, b, c] := ax^2 + bxy + cy^2$.

EXAMPLE 1. $N = 47$

In this case, we see that $H(-47)$ is isomorphic to the cyclic group of order 5. We put $R_0 := [1, 1, 12]$, $R_1 := [2, 1, 6]$, $R_2 := [3, 1, 4]$, $R_3 := [3, -1, 4]$, $R_4 := [2, -1, 6]$. Note that the quadratic form R_0 is the identity element of $H(-47)$, R_1 is a generator of $H(-47)$ and $R_i = (R_1)^i (1 \le i \le 5)$. The R_i are in the same genus, and then we have the following equalities:

$$\frac{1}{2}(\vartheta_{R_1}(\tau) - \vartheta_{R_2}(\tau)) = \eta(\tau)\eta(47\tau) ,$$

$$\frac{1}{2}(\vartheta_{R_0}(\tau) - \vartheta_{R_2}(\tau)) = \eta(\tau)\eta(47\tau) \mid T_2 ,$$

$$\frac{1}{2}(\vartheta_{R_0}(\tau) - \vartheta_{R_1}(\tau)) = -\eta(\tau)\eta(47\tau) \mid T_4 .$$

EXAMPLE 2. $N = 95$

In this case, we see that $H(-95)$ is isomorphic to the cyclic group of order 8, and consists of two genera. We put $R_0 := [1, 1, 24]$, $R_1 := [2, 1, 12]$, $R_2 := [4, 1, 6]$, $R_3 :=$

$[3, 1, 8]$, $R_4 := [5, 5, 6]$, $R_5 := [3, -1, 8]$, $R_6 := [4, -1, 6]$, $R_7 := [2, -1, 12]$. Then we see that R_0, R_2, R_4, R_6 are in the same genus and R_1, R_3, R_5, R_7 are in the same genus. We have the following equalities:

$$\frac{1}{2}(\vartheta_{R_2}(\tau) - \vartheta_{R_4}(\tau)) = \eta(\tau)\eta(95\tau)\,,$$
$$\frac{1}{2}(\vartheta_{R_0}(\tau) - \vartheta_{R_2}(\tau)) = \eta(\tau)\eta(95\tau) \mid U_5 = \eta(5\tau)\eta(19\tau)\,,$$
$$\frac{1}{2}(\vartheta_{R_1}(\tau) - \vartheta_{R_3}(\tau)) = -\eta(\tau)\eta(95\tau) \mid T_3\,,$$
$$\frac{1}{2}(\vartheta_{R_0}(\tau) - \vartheta_{R_4}(\tau)) = -\eta(\tau)\eta(95\tau) \mid T_6\,.$$

We can check the equalities of example 1 and 2 by Petersson's valence principle. We omit a proof, because it is just a comparison of Fourier coefficients.

LEMMA 2 (Petersson's valence principle). *Assume that* $f = \sum_{n=1}^{\infty} a_n q^n$ *is a modular form on* $\mathcal{M}_k(\Gamma_0(N), \chi)$. *Put* $\mu := N\prod_{p|N}(1+\frac{1}{p}) = [SL_2(\mathbf{Z}) : \Gamma_0(N)]$. *If* $a_n = 0$ *for* $0 \le n \le \frac{\mu k}{12}$, *then* $f = 0$.

PROOF. Let m be the order of χ. Then $f^m \in \mathcal{M}_{km}(\Gamma_0(N), \chi_0)$ (χ_0 is the trivial character) and has $\frac{\mu k m}{12}$ zeros in $\mathcal{H}^*/\Gamma_0(N)$ (cf. [11], Chapter V, Theorem 8). Hence the order of f at $(i\infty)$ is at most $\frac{\mu k}{12}$. □

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Present Address:
e-mail: prime@toki.waseda.jp