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On the Multiplicity of Multigraded Modules Over Artinian Local Rings

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Abstract. Let S be a finitely generated standard multigraded algebra over an Artinian local ring A; M a finitely generated multigraded S-module. This paper first investigates the relationship between the multiplicity and mixed multiplicities of M. Next, we give some applications to multigraded fiber cones.

1. Introduction

Throughout this paper, let $(A, \mathfrak{m})$ denote an Artinian local ring with maximal ideal $\mathfrak{m}$; $S = \bigoplus_{n_1,\ldots,n_d \geq 0} S_{(n_1,\ldots,n_d)}$ a finitely generated standard d-graded algebra over A (i.e., S is generated over A by elements of total degree 1), where $d \geq 2$ is a positive integer. Let $M = \bigoplus_{n_1,\ldots,n_d \geq 0} M_{(n_1,\ldots,n_d)}$ be a finitely generated d-graded S-module. Set $\mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n \geq 0} (\mathfrak{a} : \mathfrak{b}^n)$,

$$S^\triangle = \bigoplus_{n \geq 0} S_{(n,\ldots,n)},\ S_i = S_{(0,\ldots,\underbrace{1}_{i},\ldots,0)},$$

$$S_{(i+)} = S_i S = \bigoplus_{n_1 \geq 0,\ldots,\mathbf{n_i > 0},\ldots,n_d \geq 0} S_{(n_1,\ldots,n_d)}\ (i = 1, \ldots, d)\,,$$

$$S_{++} = \bigcap_{i=1}^d S_{(i+)} = \bigoplus_{n_1,\ldots,n_d > 0} S_{(n_1,\ldots,n_d)}\,,$$

$$S_+ = S_{(1+)} + \cdots + S_{(d+)} = \bigoplus_{n_1+\cdots+n_d > 0} S_{(n_1,\ldots,n_d)}\,,$$

$$S_+^\triangle = \bigoplus_{n > 0} S_{(n,\ldots,n)},\ M^\triangle = \bigoplus_{n \geq 0} M_{(n,\ldots,n)},\ \ell = \dim M^\triangle\,.$$

Denote by Proj S the set of the homogeneous prime ideals of S which do not contain S_{++}. Set $\mathrm{Supp}_{++}M = \{P \in \mathrm{Proj}\ S \mid M_P \neq 0\}$. By [HHRT, Theorem 4.1] and Remark 2.1(ii), $\dim \mathrm{Supp}_{++}M = \ell - 1$ and $l_A[M_{(n_1,\ldots,n_d)}]$ is a numerical polynomial of degree $\ell - 1$ for all large $n_1, \ldots, n_d$ (see Section 2, Remark 2.1). The terms of total degree $\ell - 1$ in this

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polynomial have the form

$$\sum_{k_1+\cdots+k_d=\ell-1} e(M; k_1, \ldots, k_d)\frac{n_1^{k_1}\cdots n_d^{k_d}}{k_1!\cdots k_d!}.$$

Then $e(M; k_1, \ldots, k_d)$ are non-negative integers not all zero, called the *mixed multiplicity of type* $(k_1, \ldots, k_d)$ of M [HHRT].

Set $\mathcal{M} = \mathfrak{m} \oplus S_+$. It is clear that $\mathcal{M}$ is the homogeneous maximal ideal of S. If I is a homogeneous $\mathcal{M}$-primary ideal of S, denote by $e(IS_{\mathcal{M}}; M_{\mathcal{M}})$ the Hilbert-Samuel multiplicity of $M_{\mathcal{M}}$ with respect to $IS_{\mathcal{M}}$. Set

$$e(I; M) = e(IS_{\mathcal{M}}; M_{\mathcal{M}}),\ e(M) = e(\mathcal{M}S_{\mathcal{M}}; M_{\mathcal{M}}).$$

We call $e(M)$ the *multiplicity* of M [HHRT]. It can be verified that S_+ is a reduction of $\mathcal{M}$. This implies that $S_{\mathcal{M}+} = (S_+)S_{\mathcal{M}}$ is a reduction of $\mathcal{M}S_{\mathcal{M}}$. So

$$e(M) = e(\mathcal{M}S_{\mathcal{M}}; M_{\mathcal{M}}) = e(S_{\mathcal{M}+}; M_{\mathcal{M}}) = e(S_+; M).$$

Expressing the multiplicity of multigraded rings in terms of mixed multiplicities was mentioned by authors: Verma in [Ve1, Ve2] for Rees algebras and multigraded Rees algebras ; Katz and Verma in [KV] for extended Rees algebras; P. Roberts in [Ro] for local Chern classes; D'Cruz in [CD] for multigraded extended Rees algebras; Herrmann et al. in [HHRT] for finitely generated standard multigraded algebras over an Artinian local ring.

The relationship between the multiplicity and mixed multiplicities of finitely generated standard multigraded algebras was showed by the authors in [HHRT] as follows.

THEOREM [HHRT, Theorem 4.3]. *Let S be a finitely generated standard d-graded algebra of dimension $d+q-1$ over an Artinian local ring A. Suppose that*

$$\dim\left(\frac{S}{S_{(i_1+)}+\cdots+S_{(i_r+)}}\right) \leq d+q-1-r$$

for all $1 \leq i_1 < \cdots < i_r \leq d$. Then

$$e(S) = \sum_{k_1+\cdots+k_d=q-1} e(S; k_1, \ldots, k_d).$$

It is clear that this result is general and important. It expresses the multiplicity of multigraded rings as a sum of mixed multiplicities. By applying the above theorem, the authors in [HHRT] expressed the multiplicity of associated multigraded rings and the multiplicity of multigraded Rees algebras in terms of mixed multiplicities (see [HHRT, Theorem 4.4, Corollary 4.7]). The aim of this paper is to give a perfect version of [HHRT, Theorem 4.3] and some applications to multigraded fiber cones.

Then our purpose is achieved by the following theorem that is the main result of this paper.

MAIN THEOREM (Theorem 2.4). *Let S be a finitely generated standard d-graded algebra over an Artinian local ring A and M a finitely generated d-graded S-module of dimension $d + q - 1$ such that $M_{(n_1,\ldots,n_d)} = S_{(n_1,\ldots,n_d)}M_{(0,\ldots,0)}$ for all $n_1, \ldots, n_d$. Set $\ell = \dim M^{\triangle}$. Then the following statements are equivalent.*

(i) $\dim M/S_{(i+)}M \leq d + q - 2$ *for all* $i = 1, \ldots, d$.

(ii) $\ell = q > 0$ *and* $e(M) = \sum_{k_1 + \cdots + k_d = q-1} e(M; k_1, \ldots, k_d)$.

So we not only obtain a generalized result of [HHRT, Theorem 4.3] to multigraded modules but also give a necessary and sufficient condition for the simpler condition. As consequences, we get Theorem 2.5 for multigraded algebras; Corollary 2.6 for the dimension of multigraded modules; and some applications to multigraded fiber cones (Corollary 3.1, Corollary 3.3, Corollary 3.7, Corollary 3.8).

This paper is divided into three sections. In Section 2, we investigate the relationship between the multiplicity and mixed multiplicities of multigraded modules. The main result of this section is Theorem 2.4 that expresses the multiplicity of multigraded modules as a sum of its mixed multiplicities. Section 3 gives some applications of Sections 2 to multigraded fiber cones.

2. The Multiplicity of Multigraded Modules

Let S be a finitely generated standard d-graded algebra over an Artinian local ring A and M a finitely generated d-graded S-module such that

$$M_{(n_1,\ldots,n_d)} = S_{(n_1,\ldots,n_d)}M_{(0,\ldots,0)}$$

for all $n_1, \ldots, n_d$. In this section we will express the multiplicity of M as a sum of its mixed multiplicities.

REMARK 2.1.

(i) Recall that a polynomial $F(t_1, \ldots, t_d) \in \mathbf{Q}[t_1, \ldots, t_d]$ is called a *numerical polynomial* if $F(n_1, \ldots, n_d) \in \mathbf{Z}$ for all $n_1, \ldots, n_d \in \mathbf{Z}$.

(ii) Remember that a polynomial $P(n_1, \ldots, n_d)$ is called the *Hilbert-Samuel polynomial* of $l_A[M_{(n_1,\ldots,n_d)}]$ if $P(n_1, \ldots, n_d) = l_A[M_{(n_1,\ldots,n_d)}]$ for all large $n_1, \ldots, n_d$. Set $\ell = \dim M^{\triangle}$. Assume that $\ell > 0$. By [HHRT, Theorem 4.1], $P(n_1, \ldots, n_d)$ is a numerical polynomial and

$$\deg P(n_1, \ldots, n_d) = \dim \operatorname{Supp}_{++} M\,.$$

Moreover, all coefficients of monomials of highest degree in $P(n_1, \ldots, n_d)$ are non-negative integers not all zero. So $\deg P(n_1, \ldots, n_d) = \deg P(n, \ldots, n)$. Since

$$P(n, \ldots, n) = l_A[M_{(n,\ldots,n)}] = l_A(M_n^{\triangle})$$

for all large n, we have

$$\deg P(n, \ldots, n) = \dim M^{\triangle} - 1 = \ell - 1\,.$$

Hence $\deg P(n_1, \ldots, n_d) = \dim \operatorname{Supp}_{++} M = \ell - 1$.

(iii) Note that a map

$$\begin{aligned} f:\ \mathbf{N}^d &\longrightarrow \mathbf{Q} \\ (n_1, \ldots, n_d) &\longmapsto f(n_1, \ldots, n_d) \end{aligned}$$

is called a polynomial function of degree r if there exists

$$g(X_1, \ldots, X_d) \in \mathbf{Q}[X_1, \ldots, X_d], \deg g = r$$

such that $f(n_1, \ldots, n_d) = g(n_1, \ldots, n_d)$ for all large $n_1, \ldots, n_d$. The degree and leading coefficients of $g(X_1, \ldots, X_d)$ are also called the degree and leading coefficients of the polynomial function f, respectively. Denote by $\deg f$ the degree of f. Hence we have $\deg f = \deg g = r$.

By the same argument as in [HHRT, Lemma 4.2], we have the following lemma.

LEMMA 2.2 [HHRT, Lemma 4.2]. *Let $F(n_1, \ldots, n_d)$ be a numerical polynomial of degree p in $n_1, \ldots, n_d$ and $u_1, \ldots, u_d$ non-negative integers. Then the function*

$$G(n) = \sum_{n_1+\cdots+n_d=n,\ n_1 \geq u_1, \ldots, n_d \geq u_d} F(n_1, \ldots, n_d)$$

is a numerical polynomial of degree $\leq\ p + d - 1$ in n for large n and the coefficient of n^{p+d-1} in this polynomial is $\dfrac{1}{(p+d-1)!} \sum_{k_1+\cdots+k_d=p} e(k_1, \ldots, k_d)$, where $\dfrac{e(k_1, \ldots, k_d)}{k_1! \cdots k_d!}$ is the coefficient of $n_1^{k_1} \cdots n_d^{k_d}$ in $F(n_1, \ldots, n_d)$.

REMARK 2.3. Let $1 \leq r \leq d-1$ and $i_1, \ldots, i_d$ positive integers such that

$$1 \leq i_1 < \cdots < i_r \leq d, 1 \leq i_{r+1} < \cdots < i_d \leq d, \{1, 2, \ldots, d\} = \{i_1, i_2, \ldots, i_d\}\,.$$

Set

$$\alpha^{(i_1,\ldots,i_r)}_{(n_1,\ldots,n_r)} = (0, \ldots, 0, \underbrace{n_1}_{i_1}, 0, \ldots, 0, \underbrace{n_j}_{i_j}, 0, \ldots, 0, \underbrace{n_r}_{i_r}, 0, \ldots, 0) \in \mathbf{Z}^d\,,$$

$$S_{i_1,\ldots,i_r} = \bigoplus_{n_1,\ldots,n_r \geq 0} S_{\alpha^{(i_1,\ldots,i_r)}_{(n_1,\ldots,n_r)}}\,,\ M_{i_1,\ldots,i_r} = \bigoplus_{n_1,\ldots,n_r \geq 0} M_{\alpha^{(i_1,\ldots,i_r)}_{(n_1,\ldots,n_r)}}\,.$$

Since

$$M_{i_1,\ldots,i_r} \simeq_{S_{i_1,\ldots,i_r}} \frac{M}{S_{(i_{r+1}+)}M + \cdots + S_{(i_d+)}M}\,,$$

we have

$$\begin{aligned}\dim_{S_{i_1,\dots,i_r}} M_{i_1,\dots,i_r} &= \dim_{S_{i_1,\dots,i_r}}\left[\frac{M}{S_{(i_{r+1}+)}M+\cdots+S_{(i_d+)}M}\right]\\ &= \dim_{\frac{S}{S_{(i_{r+1}+)}+\cdots+S_{(i_d+)}}}\left[\frac{M}{S_{(i_{r+1}+)}M+\cdots+S_{(i_d+)}M}\right]\\ &= \dim_S\left[\frac{M}{S_{(i_{r+1}+)}M+\cdots+S_{(i_d+)}M}\right]\leq \dim M/S_{(i_j+)}M\end{aligned}$$

for all $j = r+1,\dots,d$.

The relationship between the multiplicity and mixed multiplicities of M is determined as follows.

THEOREM 2.4. *Let S be a finitely generated standard d-graded algebra over an Artinian local ring A and M a finitely generated d-graded S-module of dimension $d+q-1$ such that $M_{(n_1,\dots,n_d)} = S_{(n_1,\dots,n_d)}M_{(0,\dots,0)}$ for all $n_1,\dots,n_d$. Set $\ell = \dim M^{\triangle}$. Then the following statements are equivalent.*

(i) $\dim M/S_{(i+)}M \leq d+q-2$ *for all* $i = 1,\dots,d$.

(ii) $\ell = q > 0$ *and* $e(M) = \sum_{k_1+\cdots+k_d=q-1} e(M;k_1,\dots,k_d)$.

PROOF. Set $F(n) = l_S\left[\dfrac{(S_+)^n M}{(S_+)^{n+1}M}\right]$. Then $F(n)$ is a polynomial of degree $\dim M - 1$ for all large n. Remember that

$$e(M) = \lim_{n\to\infty}\frac{(\dim M-1)!F(n)}{n^{\dim M-1}}\,.$$

Since $M_{(n_1,\dots,n_d)} = S_{(n_1,\dots,n_d)}M_{(0,\dots,0)}$ for all $n_1,\dots,n_d$, it is easily seen that

$$F(n) = \sum_{n_1+\cdots+n_d=n} l_A[M_{(n_1,\dots,n_d)}]\,.$$

Assume that u is a positive integer such that $l_A[M_{(n_1,\dots,n_d)}]$ is a polynomial for all $n_1,\dots,n_d \geq u$. Set

$$D_n = \left\{(n_1,\dots,n_d)\,\middle|\,\sum_{i=1}^d n_i = n\right\},\ E_{(n,u)} = \{(n_1,\dots,n_d)\in D_n | n_1,\dots,n_d \geq u\}\,.$$

For every $1 \leq i_1 < \cdots < i_r \leq d$, $1 \leq r \leq d-1$ and non-negative integers $u_{r+1},\dots,u_d < u$, set

$$E_{i_1,\dots,i_r}^{(n,u,u_{r+1},\dots,u_d)} = \{(n_1,\dots,n_d)\in D_n | n_{i_1},\dots,n_{i_r} \geq u, n_{i_{r+1}} = u_{r+1},\dots,n_{i_d} = u_d\}\,,$$

where $1 \leq i_{r+1} < \cdots < i_d \leq d$ and

$$\{i_{r+1},\dots,i_d\} = \{1,\dots,d\}\setminus\{i_1,\dots,i_r\}\,.$$

Then for all $n \geq du$, we have

$$D_n = E_{(n,u)} \bigcup \Big\{ \bigcup_{r=1}^{d-1} \Big[\bigcup_{1\leq i_1<\cdots<i_r\leq d} \Big(\bigcup_{0\leq u_{r+1},\ldots,u_d<u} E_{i_1,\ldots,i_r}^{(n,u,u_{r+1},\ldots,u_d)} \Big) \Big] \Big\}.$$

From this it follows that

$$F(n) = \sum_{n_1+\cdots+n_d=n;\ n_1,\ldots,n_d\geq u} l_A[M_{(n_1,\ldots,n_d)}]$$

$$+ \sum_{r=1}^{d-1} \Big\{ \sum_{1\leq i_1<\cdots<i_r\leq d} \Big[\sum_{0\leq u_{r+1},\ldots,u_d<u} \Big(\sum_{(n_1,\ldots,n_d)\in E_{i_1,\ldots,i_r}^{(n,u,u_{r+1},\ldots,u_d)}} l_A[M_{(n_1,\ldots,n_d)}] \Big) \Big] \Big\}.$$

Set

$$F_u(n) = \sum_{n_1+\cdots+n_d=n;\ n_1,\ldots,n_d\geq u} l_A[M_{(n_1,\ldots,n_d)}];$$

$$F_{i_1,\ldots,i_r}^{(u,u_{r+1},\ldots,u_d)}(n) = \sum_{(n_1,\ldots,n_d)\in E_{i_1,\ldots,i_r}^{(n,u,u_{r+1},\ldots,u_d)}} l_A[M_{(n_1,\ldots,n_d)}];$$

$$G_u(n) = \sum_{r=1}^{d-1} \Big[\sum_{1\leq i_1<\cdots<i_r\leq d} \Big(\sum_{0\leq u_{r+1},\ldots,u_d<u} F_{i_1,\ldots,i_r}^{(u,u_{r+1},\ldots,u_d)}(n) \Big) \Big].$$

Now, we will adhere to the notations of the proof for Theorem 2.4.

CLAIM 1. *If $\ell > 0$ then $F_u(n)$ is a polynomial of degree $\ell + d - 2$ for large n and the coefficient of $n^{\ell+d-2}$ in this polynomial is*

$$\frac{1}{(\ell+d-2)!} \sum_{k_1+\cdots+k_d=\ell-1} e(M; k_1, \ldots, k_d).$$

By Remark 2.1(ii), there exists a positive integer u such that $l_A[M_{(n_1,\ldots,n_d)}]$ is a numerical polynomial of degree $\ell - 1$ for all $n_1, \ldots, n_d \geq u$. Moreover, since $\ell > 0$, it implies that the elements of

$$\{e(M; k_1, \ldots, k_d) \mid k_1 + \cdots + k_d = \ell - 1\}$$

are non-negative integers not all zero. Thus

$$\sum_{k_1+\cdots+k_d=\ell-1} e(M; k_1, \ldots, k_d) > 0.$$

Denote by $f(n_1, \dots, n_d)$ this polynomial. We have

$$F_u(n) = \sum_{n_1+\dots+n_d=n;\ n_1,\dots,n_d \geq u} f(n_1, \dots, n_d)\,.$$

By Lemma 2.2, $F_u(n)$ is a polynomial of degree $\leq \ell + d - 2$ for large n and the coefficient of $n^{\ell+d-2}$ in this polynomial is

$$\frac{1}{(\ell+d-2)!} \sum_{k_1 + \dots + k_d = \ell-1} e(M; k_1, \dots, k_d) > 0\,.$$

Hence $\deg F_u(n) = \ell + d - 2$.

CLAIM 2. *Set $b = u_{r+1} + \dots + u_d$ and*

$$M_{i_1,\dots,i_r}^{(u,u_{r+1},\dots,u_d)} = \bigoplus_{n_{i_1},\dots,n_{i_r} \geq u;\ n_{i_{r+1}}=u_{r+1},\dots,n_{i_d}=u_d} M_{(n_1,\dots,n_d)}\,.$$

Then

$$F_{i_1,\dots,i_r}^{(u,u_{r+1},\dots,u_d)}(n) = l_{S_{i_1,\dots,i_r}} \left[\frac{(S_{i_1,\dots,i_r+})^{n-b-ru} M_{i_1,\dots,i_r}^{(u,u_{r+1},\dots,u_d)}}{(S_{i_1,\dots,i_r+})^{n-b-ru+1} M_{i_1,\dots,i_r}^{(u,u_{r+1},\dots,u_d)}} \right].$$

For simplicity of exposition, we can assume that $i_1 = 1, \dots, i_j = j, \dots, i_d = d$. Then

$$S_{i_1,\dots,i_r} = S_{1,\dots,r} = \bigoplus_{n_1,\dots,n_r \geq 0} S_{(n_1,\dots,n_r,0,\dots,0)},$$

$$S_{1,\dots,r+} = \bigoplus_{n_1+\dots+n_r > 0} S_{(n_1,\dots,n_r,0,\dots,0)},$$

$$M_{1,\dots,r}^{(u,u_{r+1},\dots,u_d)} = \bigoplus_{n_1,\dots,n_r \geq u} M_{(n_1,\dots,n_r,u_{r+1},\dots,u_d)}\,.$$

It is clear that $M_{1,\dots,r}^{(u,u_{r+1},\dots,u_d)}$ is an r-graded $S_{1,\dots,r}$-module. Since

$$M_{(n_1,\dots,n_d)} = S_{(n_1,\dots,n_d)} M_{(0,\dots,0)}$$

for all $n_1, \dots, n_d$, it can be verified that

$$F_{1,\dots,r}^{(u,u_{r+1},\dots,u_d)}(n) = l_{S_{1,\dots,r}} \left[\frac{(S_{1,\dots,r+})^{n-b-ru} M_{1,\dots,r}^{(u,u_{r+1},\dots,u_d)}}{(S_{1,\dots,r+})^{n-b-ru+1} M_{1,\dots,r}^{(u,u_{r+1},\dots,u_d)}} \right].$$

Claim 2 follows.

CLAIM 3.
(i) $F^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}(n)$ *is a polynomial of degree* $\dim_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r} - 1$ *for all large* n.
(ii) $G_u(n)$ *is a polynomial for all large* n *and*

$$\deg G_u(n) = \max_{1\le i_1<\cdots<i_r\le d,\ 1\le r\le d-1,\ 0\le u_{r+1},\dots,u_d<u}\{\dim_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}\} - 1\,.$$

By Claim 2, $F^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}(n)$ is a polynomial of degree

$$\dim_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r} - 1$$

for all large n. We get (i).

By (i) and note that

$$G_u(n) = \sum_{r=1}^{d-1}\left[\sum_{1\le i_1<\cdots<i_r\le d}\left(\sum_{0\le u_{r+1},\dots,u_d<u} F^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}(n)\right)\right],$$

$G_u(n)$ is a polynomial for all large n. Since the leading coefficient of $F^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}(n)$ is non-negative for all $1\le i_1<\cdots<i_r\le d,\ 1\le r\le d-1,\ 0\le u_{r+1},\dots,u_d<u$ and by (i),

$$\deg G_u(n) = \max_{1\le i_1<\cdots<i_r\le d,\ 1\le r\le d-1,\ 0\le u_{r+1},\dots,u_d<u}\{\dim_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}\} - 1\,.$$

We get (ii).

CLAIM 4.
(i) $\deg F(n) = \max\{\deg F_u(n),\ \deg G_u(n)\}$.
(ii) $\deg G_u(n) \le \max\{\dim M/S_{(i+)}M \mid i = 1,\dots,d\} - 1$.

Since $F(n) = F_u(n) + G_u(n)$ and the leading coefficients of $F_u(n), G_u(n)$ are non-negative, we immediately get (i).

It is easy to see that

$$\mathrm{Ann}_{S_{i_1,\dots,i_r}} M_{i_1,\dots,i_r} \subseteq \mathrm{Ann}_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r}\,.$$

By Remark 2.3,

$$\dim_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r} \le \dim_{S_{i_1,\dots,i_r}} M_{i_1,\dots,i_r} \le \dim M/S_{(i_j+)}M$$

for all $j = r+1,\dots,d$. Hence

$$\dim_{S_{i_1,\dots,i_r}} M^{(u,u_{r+1},\dots,u_d)}_{i_1,\dots,i_r} \le \max\{\dim M/S_{(i+)}M \mid i = 1,\dots,d\}$$

for all $1 \le i_1 < \cdots < i_r \le d,\ 1 \le r \le d-1,\ 0 \le u_{r+1}, \ldots, u_d < u$. From this fact and by Claim 3(ii),

$$\deg G_u(n) = \max_{1\le i_1<\cdots<i_r\le d,\ 1\le r\le d-1,\ 0\le u_{r+1},\ldots,u_d<u} \{\dim_{S_{i_1,\ldots,i_r}} M^{(u,u_{r+1},\ldots,u_d)}_{i_1,\ldots,i_r}\} - 1$$

$$\le \max\{\dim M/S_{(i+)}M | i = 1, \ldots, d\} - 1\,.$$

We get (ii).

We now return to the proof of Theorem 2.4.

For $i = 1, \ldots, d$, set

$$D_n^i = \{(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_d) | \sum_{j=1, j\neq i}^{d} n_j = n\},$$

$$F_i(n) = \sum_{(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_d)\in D_n^i} l_A[M_{(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_d)}]\,.$$

Set

$$H_{i,u}(n) = \sum_{(n_1,\ldots,n_d)\in D\setminus[D_n^i \bigcup E_{(n,u)}]} l_A[M_{(n_1,\ldots,n_d)}]\,.$$

Since $u > 0$, $D_i(n) \bigcap E_{(n,u)} = \emptyset$. From this fact and note that $D_i(n)$ and $E_{(n,u)}$ are subsets of D_n, we have

$$F(n) = F_u(n) + F_i(n) + H_{i,u}(n) \text{ and } G_u(n) = F_i(n) + H_{i,u}(n)\,.$$

Since $M_{(n_1,\ldots,n_d)} = S_{(n_1,\ldots,n_d)}M_{(0,\ldots,0)}$ for all $n_1, \ldots, n_d$, it can be verified that

$$F_i(n) = l_S\left[\frac{(S_+)^n M/S_{(i+)}M}{(S_+)^{n+1} M/S_{(i+)}M}\right].$$

Thus $F_i(n)$ is a polynomial of degree $\dim M/S_{(i+)}M - 1$ for all large n. Since $G_u(n)$ and $F_i(n)$ are polynomials for all large n, it follows that $H_{i,u}(n)$ is also a polynomial for all large n. Moreover since $H_{i,u}(n) \ge 0$ for all n, the leading coefficient of $H_{i,u}(n)$ is non-negative. Note that the leading coefficient of $F_i(n)$ is also non-negative. Hence

$$\deg G_u(n) = \max\{\deg F_i(n), \deg H_{i,u}(n)\} = \max\{\dim M/S_{(i+)}M - 1, \deg H_{i,u}(n)\}\,.$$

(i) $\Rightarrow$ (ii): Since $F(n) = F_u(n) + G_u(n)$ and note that

$$e(M) = \lim_{n\to\infty} \frac{(\dim M - 1)!F(n)}{n^{\dim M - 1}}\,,$$

we have

$$e(M) = \lim_{n\to\infty} \frac{(\dim M - 1)!F_u(n)}{n^{\dim M - 1}} + \lim_{n\to\infty} \frac{(\dim M - 1)!G_u(n)}{n^{\dim M - 1}}\,.$$

Since $\dim M/S_{(i+)}M < \dim M$ for all $i = 1, \ldots, d$ by Claim 4(ii),

$$\deg G_u(n) < \dim M - 1 .$$

This implies that $\lim_{n\to\infty} \dfrac{(\dim M - 1)!G_u(n)}{n^{\dim M - 1}} = 0$. Thus

$$e(M) = \lim_{n\to\infty} \frac{(\dim M - 1)!F(n)}{n^{\dim M-1}} = \lim_{n\to\infty} \frac{(\dim M - 1)!F_u(n)}{n^{\dim M-1}} .$$

Since $\deg F(n) = \dim M - 1 > \deg G_u(n)$, $\deg F(n) > \deg G_u(n)$. By Claim 4(i),

$$\deg F(n) = \deg F_u(n) > \deg G_u(n) .$$

It follows that $F_u(n) \neq 0$. Hence $\ell > 0$ for if $\ell = 0$ then $F_u(n) = 0$. By Claim 1,

$$\dim M - 1 = \deg F_u(n) = \ell + d - 2 .$$

Hence $\dim M = d + \ell - 1$. From this fact and note that $\dim M = d + q - 1$, we get $\ell = q$. Since $\dim M = d + \ell - 1$,

$$e(M) = \lim_{n\to\infty} \frac{(\dim M - 1)!F_u(n)}{n^{\dim M-1}} = \lim_{n\to\infty} \frac{(\ell + d - 2)!F_u(n)}{n^{\ell+d-2}} .$$

Hence by Claim 1,

$$e(M) = \sum_{k_1 + \cdots + k_d = \ell-1} e(M; k_1, \ldots, k_d) = \sum_{k_1 + \cdots + k_d = q-1} e(M; k_1, \ldots, k_d) .$$

(ii) $\Rightarrow$ (i): Since $\dim M = d + q - 1$ and $0 < \ell = q$, we have $\dim M - 1 = d + \ell - 2$ and

$$\sum_{k_1 + \cdots + k_d = \ell-1} e(M; k_1, \ldots, k_d) = \sum_{k_1 + \cdots + k_d = q-1} e(M; k_1, \ldots, k_d) .$$

Since $e(M) = \lim_{n\to\infty} \dfrac{(\dim M - 1)!F(n)}{n^{\dim M-1}}$, we have

$$\lim_{n\to\infty} \frac{(\dim M - 1)!F(n)}{n^{\dim M-1}} = \sum_{k_1 + \cdots + k_d = \ell-1} e(M; k_1, \ldots, k_d) .$$

Note that $F(n) = F_u(n) + G_u(n)$,

$$\lim_{n\to\infty} \frac{(\dim M - 1)!F(n)}{n^{\dim M-1}} = \lim_{n\to\infty} \frac{(\dim M - 1)!F_u(n)}{n^{\dim M-1}} + \lim_{n\to\infty} \frac{(\dim M - 1)!G_u(n)}{n^{\dim M-1}} .$$

By Claim 1,

$$\lim_{n\to\infty} \frac{(\dim M - 1)!F_u(n)}{n^{\dim M-1}} = \lim_{n\to\infty} \frac{(\ell + d - 2)!F_u(n)}{n^{\ell+d-2}} = \sum_{k_1 + \cdots + k_d = \ell-1} e(M; k_1, \ldots, k_d) .$$

Hence

$$\lim_{n\to\infty} \frac{(\dim M - 1)!G_u(n)}{n^{\dim M-1}} = 0\,.$$

It follows that $\deg G_u(n) < \dim M - 1$. From this fact and since

$$\deg G_u(n) = \max\{\dim M/S_{(i+)}M - 1, \deg H_{i,u}(n)\}$$

for all $i = 1, \dots, d$, we get $\dim M/S_{(i+)}M < \dim M$ for all $i = 1, \dots, d$. Theorem 2.4 has been proved. ∎

As an immediate consequence of Theorem 2.4, we have the following theorem.

THEOREM 2.5. *Let S be a finitely generated standard d-graded algebra of dimension $d + q - 1$ over an Artinian local ring A. Set $\ell = \dim S^{\triangle}$. Then the following statements are equivalent.*

(i) $\dim S/S_{(i+)} \le d + q - 2$ *for all* $i = 1, \dots, d$.

(ii) $\ell = q > 0$ *and* $e(S) = \sum_{k_1+\cdots+k_d=q-1} e(S; k_1, \dots, k_d)$.

So we obtain with Theorem 2.5 as a replacement of the condition

$$\dim\left(\frac{S}{S_{(i_1+)} + \cdots + S_{(i_r+)}}\right) \le \dim S - r$$

for all $1 \le i_1 < \cdots < i_r \le d$ in [HHRT, Theorem 4.3] by the weaker condition

$$\dim S/S_{(i+)} < \dim S \text{ for all } 1 \le i \le d\,.$$

From the proof of Theorem 2.4, we also get the result on the dimension of multigraded modules as follows.

COROLLARY 2.6. *Let S be a finitely generated standard d-graded algebra over an Artinian local ring A $(d > 1)$ and M a finitely generated d-graded S-module such that $M_{(n_1,\dots,n_d)} = S_{(n_1,\dots,n_d)}M_{(0,\dots,0)}$ for all $n_1, \dots, n_d$. Set $\ell = \dim M^{\triangle}$. Then the following statements hold.*

(i) *If* $\ell > 0$ *then* $\dim M = \max\{d + \ell - 1, \dim M/S_{(i+)}M \mid i = 1, \dots, d\}$.

(ii) *If* $\ell = 0$ *then* $\dim M = \max\{\dim M/S_{(i+)}M \mid i = 1, \dots, d\}$.

PROOF. (i) Since $\dim M = \deg F(n) + 1$ and by Claim 4(i),

$$\dim M = \max\{\deg F_u(n), \deg G_u(n)\} + 1\,.$$

By Claim 1, $\deg F_u(n) = d + \ell - 2$. By Claim 4(ii),

$$\deg G_u(n) \le \max\{\dim M/S_{(i+)}M \mid i = 1, \dots, d\} - 1\,.$$

From the above facts, we have

$$\dim M \le \max\{d + \ell - 1, \dim M/S_{(i+)}M \mid i = 1, \dots, d\}\,.$$

Clearly we also have

$$\max\{d+\ell-1, \dim M/S_{(i+)}M \mid i=1,\dots,d\} \leq \dim M\,.$$

Hence we get (i).

(ii) If $\ell = 0$ then $F_u(n) = 0$. By Claim 4(i),

$$\dim M = \max\{\deg F_u(n), \deg G_u(n)\} + 1 = \deg G_u(n) + 1\,.$$

By Claim 4(ii),

$$\deg G_u(n) \leq \max\{\dim M/S_{(i+)}M \mid i=1,\dots,d\} - 1\,.$$

Thus

$$\dim M \leq \max\{\dim M/S_{(i+)}M \mid i=1,\dots,d\}\,.$$

Clearly we also have

$$\max\{\dim M/S_{(i+)}M \mid i=1,\dots,d\} \leq \dim M\,.$$

Hence we get (ii). ■

3. Some Applications to Multigraded Fiber cones

Let $(B, \mathfrak{n})$ denote a Noetherian local ring with maximal ideal $\mathfrak{n}$;
$R = \bigoplus_{n_1,\dots,n_d \geq 0} R_{(n_1,\dots,n_d)}$
a finitely generated standard d-graded algebra over B (i.e., R is generated over B by elements of total degree 1), where d is a positive integer; $N = \bigoplus_{n_1,\dots,n_d\geq 0} N_{(n_1,\dots,n_d)}$ a finitely generated d-graded R-module such that

$$N_{(n_1,\dots,n_d)} = R_{(n_1,\dots,n_d)} N_{(0,\dots,0)}$$

for all $n_1,\dots,n_d$. Let J be an $\mathfrak{n}$-primary ideal of B. Define

$$F_J(R) = R/JR = \bigoplus_{n_1,\dots,n_d\geq 0} \frac{R_{(n_1,\dots,n_d)}}{JR_{(n_1,\dots,n_d)}},\ \ F_J(N) = N/JN = \bigoplus_{n_1,\dots,n_d\geq 0} \frac{N_{(n_1,\dots,n_d)}}{JN_{(n_1,\dots,n_d)}}$$

to be the *d-graded fiber cone* of R and N with respect to J, respectively. Then $F_J(R)$ is a finitely generated standard d-graded algebra over Artinian local ring B/J and $F_J(N)$ is a finitely generated d-graded $F_J(R)$-module. By applying the results in Section 2, this section gives some results on the multiplicity of the fiber cone $F_J(N)$.

Set $N^{\triangle} = \bigoplus_{n\geq 0} N_{(n,\dots,n)}$, $R_{(i+)} = \bigoplus_{n_1\geq 0,\dots,\mathbf{n_i}>\mathbf{0},\dots,n_d\geq 0} R_{(n_1,\dots,n_d)}$ for $i = 1,\dots,d$. It is easily seen that

$$F_J(N)^{\triangle} = \bigoplus_{n\geq 0} \frac{N_{(n,\dots,n)}}{JN_{(n,\dots,n)}} = N^{\triangle}/JN^{\triangle} = F_J(N^{\triangle})\,,$$

$$\frac{F_J(N)}{F_J(R)_{(i+)}F_J(N)} \simeq F_J(N/R_{(i+)}N),\ i = 1, \ldots, d\,.$$

Denote by $e(F_J(N); k_1, \ldots, k_d)$ the mixed multiplicity of type $(k_1, \ldots, k_d)$ of $F_J(N)$. By Theorem 2.4, we get the following result.

COROLLARY 3.1. *Let R be a finitely generated standard d-graded algebra over a Noetherian local ring B and N a finitely generated d-graded R-module such that $N_{(n_1,\ldots,n_d)} = R_{(n_1,\ldots,n_d)}N_{(0,\ldots,0)}$ for all $n_1, \ldots, n_d$. Let J be an $\mathfrak{n}$-primary ideal of B. Set $\ell = \dim F_J(N^{\triangle})$. Assume that $\dim F_J(N) = d + q - 1$. Then the following statements are equivalent.*

(i) $\dim F_J(N/R_{(i+)}N) \leq d + q - 2$ *for all* $i = 1, \ldots, d$.

(ii) $\ell = q > 0$ *and*

$$e(F_J(N)) = \sum_{k_1 + \cdots + k_d = q-1} e(F_J(N); k_1, \ldots, k_d)\,.$$

Let $I_1, \ldots, I_d$ be ideals of B and let K be a finitely generated B-module with Krull dimension $\dim K > 0$. Define

$$F(J, I_1, \ldots, I_d) = \bigoplus_{n_1,\ldots,n_d \geq 0} \frac{I_1^{n_1} \cdots I_d^{n_d}}{JI_1^{n_1} \cdots I_d^{n_d}},\ F_K(J, I_1, \ldots, I_d) = \bigoplus_{n_1,\ldots,n_d \geq 0} \frac{I_1^{n_1} \cdots I_d^{n_d}K}{JI_1^{n_1} \cdots I_d^{n_d}K}$$

to be *the d-graded fiber cone* of B and K with respect to $J, I_1, \ldots, I_d$, respectively. Let $t_1, \ldots, t_d$ be indeterminates. Set

$$R(I_1, \ldots, I_d) = \bigoplus_{n_1,\ldots,n_d \geq 0} I_1^{n_1} \cdots I_d^{n_d} t_1^{n_1} \cdots t_d^{n_d},$$

$$R_K(I_1, \ldots, I_d) = \bigoplus_{n_1,\ldots,n_d \geq 0} I_1^{n_1} \cdots I_d^{n_d} K t_1^{n_1} \cdots t_d^{n_d}\,.$$

$R(I_1, \ldots, I_d)$ and $R_K(I_1, \ldots, I_d)$ are called the d-graded Rees algebra of $I_1, \ldots, I_d$ and the d-graded Rees module of $I_1, \ldots, I_d$ with respect to K, respectively. Then clearly $F(J, I_1, \ldots, I_d) \simeq F_J(R(I_1, \ldots, I_d))$ and $F_K(J, I_1, \ldots, I_d) \simeq F_J(R_K(I_1, \ldots, I_d))$.

Then we have the following remark.

REMARK 3.2. Set $I = I_1 \cdots I_d,\ \ell = \dim\Big(\bigoplus_{n \geq 0} \frac{I^n K}{\mathfrak{n} I^n K}\Big)$. We call ℓ *the analytic spread* of I with respect to K. Since $\sqrt{J} = \mathfrak{n}$,

$$\ell = \dim\Big(\bigoplus_{n \geq 0} \frac{I^n K}{\mathfrak{n} I^n K}\Big) = \dim\Big(\bigoplus_{n \geq 0} \frac{I^n K}{J I^n K}\Big)\,.$$

From this fact and note that

$$F_K(J, I_1, \ldots, I_d)^{\triangle} \simeq F_J(R_K(I_1, \ldots, I_d))^{\triangle} \simeq \bigoplus_{n \geq 0} \frac{I^n K}{J I^n K}\,,$$

we get $\ell = \dim F_K(J, I_1, \ldots, I_d)^{\triangle}$. Hence by Remark 2.1(ii), $l_A\left(\dfrac{I_1^{n_1}\cdots I_d^{n_d}K}{JI_1^{n_1}\cdots I_d^{n_d}K}\right)$ is a polynomial of degree $\ell - 1$ for all large $n_1, \ldots, n_d$.

Denote by $E_J(I_1^{[k_1]}, \ldots, I_d^{[k_d]}; K)$ the mixed multiplicity of type $(k_1, \ldots, k_d)$ of $F_K(J, I_1, \ldots, I_d)$ for all non-negative integers $k_1, \ldots, k_d$ such that $k_1 + \cdots + k_d = \ell - 1$. The authors in [MV] answered when mixed multiplicities of $F_K(J, I_1, \ldots, I_d)$ are positive and expressed them in terms of the length of modules (see [MV, Theorem 3.5]). For $i = 1, \ldots, d$, set

$$F_K(J, I_1, \ldots, I_{i-1}, I_{i+1}, \ldots, I_d) = \bigoplus_{n_1,\ldots,n_{i-1},n_{i+1},\ldots,n_d \geq 0} \frac{I_1^{n_1}\cdots I_{i-1}^{n_{i-1}} I_{i+1}^{n_{i+1}}\cdots I_d^{n_d}K}{JI_1^{n_1}\cdots I_{i-1}^{n_{i-1}} I_{i+1}^{n_{i+1}}\cdots I_d^{n_d}K}\,.$$

By Corollary 3.1, we get the following result that expresses the multiplicity of $F_K(J, I_1, \ldots, I_d)$ as a sum of its mixed multiplicities.

COROLLARY 3.3. *Let J be an $\mathfrak{n}$-primary ideal and let $I_1, \ldots, I_d$ be ideals of B. Set* $I = I_1 \cdots I_d$, $\ell = \dim\Big(\bigoplus_{n\geq 0} \dfrac{I^nK}{\mathfrak{n}I^nK}\Big)$. *Suppose that* $\dim F_K(J, I_1, \ldots, I_d) = d + q - 1$. *Then the following statements are equivalent.*

(i) $\dim F_K(J, I_1, \ldots, I_{i-1}, I_{i+1}, \ldots, I_d) \leq d + q - 2$ *for all* $i = 1, \ldots, d$.

(ii) $\ell = q > 0$ *and*

$$e(F_K(J, I_1, \ldots, I_d)) = \sum_{k_1+\cdots+k_d = q-1} E_J(I_1^{[k_1]}, \ldots, I_d^{[k_d]}; K)\,.$$

Now, we investigate the multiplicity of $F_K(J, I_1, \ldots, I_d)$ in the case that $I_1, \ldots, I_d$ satisfy

$$\mathrm{ht}\Big(\frac{I_1\cdots I_d + \mathrm{Ann}_B K}{\mathrm{Ann}_B K}\Big) > 0\,.$$

REMARK 3.4. Let $\Im, \Im_1, \Im_2$ be ideals of B such that $\mathrm{ht}\Big(\dfrac{\Im_1\Im_2 + \mathrm{Ann}_B K}{\mathrm{Ann}_B K}\Big) > 0$. Set

$$R(\Im) = \bigoplus_{n\geq 0} \Im^n t^n,\ \ R_K(\Im) = \bigoplus_{n\geq 0} \Im^n K t^n\,,$$

where t is an indeterminate. We have

$$\begin{aligned}\dim\Big(\frac{\Im_2R_K(\Im)}{\Im_1\Im_2R_K(\Im)}\Big) &= \dim\Big(\frac{R(\Im)}{\Im_1\Im_2R_K(\Im) : \Im_2R_K(\Im)}\Big)\\ &= \dim\Big(\frac{R(\Im)}{\Im_1R(\Im) + \mathrm{Ann}_{R(\Im)}(\Im_2R_K(\Im))}\Big)\\ &= \dim\Big(\frac{R(\Im)}{\Im_1R(\Im) + \sqrt{\mathrm{Ann}_{R(\Im)}(\Im_2R_K(\Im))}}\Big)\,.\end{aligned}$$

On the other hand,

$$\sqrt{\mathrm{Ann}_{R(\Im)}(\Im_2 R_K(\Im))} = \bigoplus_{n\geq 0}(\Im^n \bigcap \sqrt{\mathrm{Ann}_B(\Im_2 K)})t^n\,.$$

Since $\mathrm{ht}\left(\dfrac{\Im_1\Im_2 + \mathrm{Ann}_B K}{\mathrm{Ann}_B K}\right) > 0$, it follows that $\mathrm{ht}\left(\dfrac{\Im_2 + \mathrm{Ann}_B K}{\mathrm{Ann}_B K}\right) > 0$. This implies that

$$\sqrt{\mathrm{Ann}_B(\Im_2 K)} = \sqrt{\mathrm{Ann}_B K}\,.$$

Thus

$$\sqrt{\mathrm{Ann}_{R(\Im)}(\Im_2 R_K(\Im))} = \bigoplus_{n\geq 0}(\Im^n \bigcap \sqrt{\mathrm{Ann}_B K})t^n = \sqrt{\mathrm{Ann}_{R(\Im)}(R_K(\Im))}\,.$$

From the above facts, we get

$$\begin{aligned}\dim\left(\frac{\Im_2 R_K(\Im)}{\Im_1\Im_2 R_K(\Im)}\right) &= \dim\left(\frac{R(\Im)}{\Im_1 R(\Im) + \sqrt{\mathrm{Ann}_{R(\Im)}(R_K(\Im))}}\right)\\ &= \dim\left(\frac{R(\Im)}{\Im_1 R(\Im) + \mathrm{Ann}_{R(\Im)}(R_K(\Im))}\right)\\ &= \dim\left(\frac{R(\Im)}{\Im_1 R_K(\Im) : R_K(\Im)}\right)\\ &= \dim\left(\frac{R_K(\Im)}{\Im_1 R_K(\Im)}\right).\end{aligned}$$

Hence $\dim\left(\dfrac{\Im_2 R_K(\Im)}{\Im_1\Im_2 R_K(\Im)}\right) = \dim\left(\dfrac{R_K(\Im)}{\Im_1 R_K(\Im)}\right)$.

REMARK 3.5. Let $\Im_1, \Im_2$ be ideals of B such that $\mathrm{ht}\left(\dfrac{\Im_1\Im_2 + \mathrm{Ann}K}{\mathrm{Ann}K}\right) > 0$. Set

$$\begin{aligned}\ell_K(\Im_1) &= \dim\left(\bigoplus_{n\geq 0}\frac{\Im_1^n K}{\mathfrak{n}\Im_1^n K}\right),\\ \ell_K(\Im_2) &= \dim\left(\bigoplus_{n\geq 0}\frac{\Im_2^n K}{\mathfrak{n}\Im_2^n K}\right),\\ \ell_K(\Im_1\Im_2) &= \dim\left[\bigoplus_{n\geq 0}\frac{(\Im_1\Im_2)^n K}{\mathfrak{n}(\Im_1\Im_2)^n K}\right].\end{aligned}$$

Denote by $f(n_1, n_2)$ the Hilbert-Samuel polynomial of the function $l_B\left(\dfrac{\Im_1^{n_1}\Im_2^{n_2}K}{\mathfrak{n}\Im_1^{n_1}\Im_2^{n_2}K}\right)$. By Remark 3.2, $\deg f(n_1, n_2) = \ell_K(\Im_1\Im_2) - 1$. Assume that u is a non-negative integer such that

$$f(n_1, n_2) = l_B\left(\frac{\Im_1^{n_1}\Im_2^{n_2}K}{\mathfrak{n}\Im_1^{n_1}\Im_2^{n_2}K}\right)$$

for all $n_1, n_2 \geq u$. Then $\deg f(n_1, n_2) \geq \deg f(n_1, u)$. Since

$$f(n_1, u) = l_B\left(\frac{\Im_2^{u}\Im_1^{n_1}K}{\mathfrak{n}\Im_2^{u}\Im_1^{n_1}K}\right)$$

for all $n_1 \geq u$, we have

$$\deg f(n_1, u) = \dim\left(\bigoplus_{n_1\geq 0}\frac{\Im_2^{u}\Im_1^{n_1}K}{\mathfrak{n}\Im_2^{u}\Im_1^{n_1}K}\right)-1 = \dim\left[\frac{\Im_2^{u}R_K(\Im_1)}{\mathfrak{n}\Im_2^{u}R_K(\Im_1)}\right]-1\,.$$

Since $\operatorname{ht}\left(\dfrac{\Im_1\Im_2 + \operatorname{Ann}K}{\operatorname{Ann}K}\right) > 0$ and $\operatorname{ht}\left(\dfrac{\mathfrak{n} + \operatorname{Ann}K}{\operatorname{Ann}K}\right) > 0$, it follows that

$$\operatorname{ht}\left(\frac{\mathfrak{n}\Im_2^{u} + \operatorname{Ann}K}{\operatorname{Ann}K}\right) > 0\,.$$

Hence by Remark 3.4,

$$\dim\left[\frac{\Im_2^{u}R_K(\Im_1)}{\mathfrak{n}\Im_2^{u}R_K(\Im_1)}\right] = \dim\left[\frac{R_K(\Im_1)}{\mathfrak{n}R_K(\Im_1)}\right] = \ell_K(\Im_1)\,.$$

Thus

$$\deg f(n_1, u) = \dim\left[\frac{R_K(\Im_1)}{\mathfrak{n}R_K(\Im_1)}\right]-1 = \ell_K(\Im_1) - 1\,.$$

From the above facts, we get $\ell_K(\Im_1\Im_2) \geq \ell_K(\Im_1)$. By symmetry, we also have $\ell_K(\Im_1\Im_2) \geq \ell_K(\Im_2)$.

REMARK 3.6. Let $I_1, \dots, I_d$ be ideals of B such that $\operatorname{ht}\left(\dfrac{I + \operatorname{Ann}K}{\operatorname{Ann}K}\right) > 0$, where $I = I_1 \cdots I_d$. Set

$$\ell = \dim\left(\bigoplus_{n\geq 0}\frac{I^nK}{\mathfrak{n}I^nK}\right), \ell_K(I_1) = \dim\left(\bigoplus_{n\geq 0}\frac{I_1^nK}{\mathfrak{n}I_1^nK}\right),$$

$$\ell_K(I_2) = \dim\left(\bigoplus_{n\geq 0}\frac{I_2^nK}{\mathfrak{n}I_2^nK}\right), \ell_K(I_1I_2) = \dim\left[\bigoplus_{n\geq 0}\frac{(I_1I_2)^nK}{\mathfrak{n}(I_1I_2)^nK}\right].$$

By Remark 3.2,

$$\ell = \dim F_K(J, I_1, \dots, I_d)^{\triangle}, \ell_K(I_1) = \dim F_K(J, I_1)\,,$$

$$\ell_K(I_2) = \dim F_K(J, I_2), \ell_K(I_1I_2) = \dim F_K(J, I_1, I_2)^{\triangle}\,.$$

Since $\mathrm{ht}\left(\dfrac{I+\mathrm{Ann}K}{\mathrm{Ann}K}\right)>0$, we have $\ell_K(I_1I_2)>0$. Hence by Corollary 2.6,

$$\begin{aligned}\dim F_K(J,I_1,I_2)&=\max\{\ell_K(I_1I_2)+1,\dim F_K(J,I_1),\dim F_K(J,I_2)\}\\&=\max\{\ell_K(I_1I_2)+1,\ell_K(I_1),\ell_K(I_2)\}\,.\end{aligned}$$

By Remark 3.5,

$$\max\{\ell_K(I_1I_2)+1,\ell_K(I_2),\ell_K(I_2)\}=\ell_K(I_1I_2)+1\,.$$

Hence $\dim F_K(J,I_1,I_2)=\ell_K(I_1I_2)+1$. By induction, assume that

$$\dim F_K(J,I_1,\dots,I_{i-1},I_{i+1},\dots,I_d)=\ell_K(I_1\cdots I_{i-1}I_{i+1}\cdots I_d)+d-2 \qquad (*)$$

for all $i=1,\dots,d$, where

$$\begin{aligned}\ell_K(I_1\cdots I_{i-1}I_{i+1}\cdots I_d)&=\dim\left[\bigoplus_{n\geq 0}\frac{(I_1\cdots I_{i-1}I_{i+1}\cdots I_d)^nK}{\mathfrak{n}(I_1\cdots I_{i-1}I_{i+1}\cdots I_d)^nK}\right]\\&=\dim F_K(J,I_1,\dots,I_{i-1},I_{i+1},\dots,I_d)^{\triangle}\,.\end{aligned}$$

Since $\mathrm{ht}\left(\dfrac{I+\mathrm{Ann}K}{\mathrm{Ann}K}\right)>0$, we have $\ell>0$. Hence by Corollary 2.6,

$$\begin{aligned}&\dim F_K(J,I_1,\dots,I_d)\\&\quad=\max\{d+\ell-1,\dim F_K(J,I_1,\dots,I_{i-1},I_{i+1},\dots,I_d)|i=1,2,\dots,d\}\,.\end{aligned}$$

By (*),

$$\dim F_K(J,I_1,\dots,I_d)=\max\{d+\ell-1,d+\ell_K(I_1\cdots I_{i-1}I_{i+1}\cdots I_d)-2|i=1,2,\dots,d\}\,.$$

Since $\mathrm{ht}\left(\dfrac{I+\mathrm{Ann}K}{\mathrm{Ann}K}\right)>0$, $\ell_K(I_1\cdots I_{i-1}I_{i+1}\cdots I_d)\leq\ell$ by Remark 3.5. Hence we get $\dim F_K(J,I_1,\dots,I_d)=d+\ell-1$ and

$$\dim F_K(J,I_1,\dots,I_{i-1},I_{i+1},\dots,I_d)<\dim F_K(J,I_1,\dots,I_d)$$

for all $i=1,2,\dots,d$.

By Corollary 3.3 and Remark 3.6, we get an interesting result as follows.

COROLLARY 3.7. *Let J be an $\mathfrak{n}$-primary ideal and let $I_1,\dots,I_d$ be ideals of B such that* $\mathrm{ht}\left(\dfrac{I+\mathrm{Ann}K}{\mathrm{Ann}K}\right)>0$, *where $I=I_1\cdots I_d$. Set $\ell=\dim\left(\bigoplus_{n\geq 0}\dfrac{I^nK}{\mathfrak{n}I^nK}\right)$. Then*

(i) $\dim F_K(J,I_1,\dots,I_d)=d+\ell-1$.

(ii) $e(F_K(J,I_1,\dots,I_d))=\sum_{k_1+\cdots+k_d=\ell-1}E_J(I_1^{[k_1]},\dots,I_d^{[k_d]};K)$.

In the case where $I_1,\dots,I_d$ are $\mathfrak{n}$-primary ideals, it is easily seen that

$$\ell=\dim\left(\bigoplus_{n\geq 0}\frac{I^nK}{\mathfrak{n}I^nK}\right)=\mathrm{ht}\left(\frac{I+\mathrm{Ann}K}{\mathrm{Ann}K}\right)=\dim K>0\,,$$

where $I = I_1 \dots I_d$. By Corollary 3.7, we obtain the following result.

COROLLARY 3.8. *Let $J, I_1, \dots, I_d$ be $\mathfrak{n}$-primary ideals of B. Then*
(i) $\dim F_K(J, I_1, \dots, I_d) = \dim K + d - 1$.
(ii) $e(F_K(J, I_1, \dots, I_d)) = \sum_{k_1 + \dots + k_d = \dim K - 1} E_J(I_1^{[k_1]}, \dots, I_d^{[k_d]}; K)$.

EXAMPLE 3.9. Let k be a field and let $x_1, x_2, x_3, x_4, x_5, x_6$ be indeterminates. Set

$$B = k[[x_1, x_2, x_3, x_4, x_5, x_6]], \mathfrak{n} = (x_1, x_2, x_3, x_4, x_5, x_6)\,,$$

$$I_1 = (x_1, x_2, x_3, x_4, x_5), I_2 = (x_1, x_2, x_3, x_4), I_3 = (x_1, x_2, x_3)\,.$$

Consider 3-graded fiber cone of B with respect to $\mathfrak{n}, I_1, I_2, I_3$:

$$F(\mathfrak{n}, I_1, I_2, I_3) = \bigoplus_{n_1, n_2, n_3 \geq 0} \frac{I_1^{n_1} I_2^{n_2} I_3^{n_3}}{\mathfrak{n} I_1^{n_1} I_2^{n_2} I_3^{n_3}}\,.$$

Set

$$C = \left\{ \prod_{i=1}^{5} x_i^{\alpha_i} | 0 \leq \alpha_i \in \mathbf{Z}, i = 1, \dots, 5, \alpha_5 \leq n_1, \alpha_4 + \alpha_5 \leq n_1 + n_2, \sum_{i=1}^{5} \alpha_i = \sum_{i=1}^{3} n_i \right\}.$$

Denote by V the k-vector space generated by C. It can be verified that

$$V \simeq_k \frac{I_1^{n_1} I_2^{n_2} I_3^{n_3}}{\mathfrak{n} I_1^{n_1} I_2^{n_2} I_3^{n_3}}\,.$$

Thus

$$l_B\left(\frac{I_1^{n_1} I_2^{n_2} I_3^{n_3}}{\mathfrak{n} I_1^{n_1} I_2^{n_2} I_3^{n_3}} \right) = l_k\left(\frac{I_1^{n_1} I_2^{n_2} I_3^{n_3}}{\mathfrak{n} I_1^{n_1} I_2^{n_2} I_3^{n_3}} \right) = \dim_k(V) = \mathrm{Card}(C)\,.$$

Set

$$D = \left\{ \prod_{i=1}^{3} x_i^{\alpha_i} | 0 \leq \alpha_1, \alpha_2, \alpha_3 \in \mathbf{Z}, \sum_{i=1}^{3} \alpha_i = n_1 + n_2 + n_3 - (\alpha_4 + \alpha_5) \right\}.$$

Then we have

$$\begin{aligned} l_B\left(\frac{I_1^{n_1} I_2^{n_2} I_3^{n_3}}{\mathfrak{n} I_1^{n_1} I_2^{n_2} I_3^{n_3}} \right) &= \mathrm{Card}(C) = \sum_{\alpha_5=0}^{n_1} \sum_{\alpha_4=0}^{n_1+n_2-\alpha_5} \mathrm{Card}(D) \\ &= \sum_{\alpha_5=0}^{n_1} \sum_{\alpha_4=0}^{n_1+n_2-\alpha_5} \binom{n_1 + n_2 + n_3 - (\alpha_4 + \alpha_5) + 2}{2} \\ &= \sum_{\alpha_5=0}^{n_1} \left[\binom{n_1 + n_2 + n_3 - \alpha_5 + 3}{3} - \binom{n_3 + 2}{3} \right]. \end{aligned}$$

By direct computing, we have

$$\begin{aligned}\dim_k(V) = &\frac{n_1^4 + 4n_1^3 n_2 + 4n_1^3 n_3 + 6n_1^2 n_2^2 + 12n_1^2 n_2 n_3}{24} \\ &+ \frac{6n_1^2 n_3^2 + 4n_1 n_2^3 + 12n_1 n_2^2 n_3 + 12n_1 n_2 n_3^2}{24} + g(n_1, n_2, n_3)\,,\end{aligned}$$

where $g(n_1, n_2, n_3)$ is a polynomial and $\deg g(n_1, n_2, n_3) < 4$. From this fact, we get

$$\begin{aligned}&E_{\mathfrak{n}}(I_1^{[4]}, I_2^{[0]}, I_3^{[0]}; B) = E_{\mathfrak{n}}(I_1^{[3]}, I_2^{[1]}, I_3^{[0]}; B) = E_{\mathfrak{n}}(I_1^{[3]}, I_2^{[0]}, I_3^{[1]}; B) \\ &\quad = E_{\mathfrak{n}}(I_1^{[2]}, I_2^{[2]}, I_3^{[0]}; B) = E_{\mathfrak{n}}(I_1^{[2]}, I_2^{[1]}, I_3^{[1]}; B) = E_{\mathfrak{n}}(I_1^{[2]}, I_2^{[0]}, I_3^{[2]}; B) \\ &\quad = E_{\mathfrak{n}}(I_1^{[1]}, I_2^{[3]}, I_3^{[0]}; B) = E_{\mathfrak{n}}(I_1^{[1]}, I_2^{[2]}, I_3^{[1]}; B) = E_{\mathfrak{n}}(I_1^{[1]}, I_2^{[1]}, I_3^{[2]}; B) = 1\,.\end{aligned}$$

The others are zero. Since $\mathrm{ht}(I_1) = 5$, $\mathrm{ht}(I_2) = 4$ and $\mathrm{ht}(I_3) = 3$, by Corollary 3.7 we obtain $\dim F(\mathfrak{n}, I_1, I_2, I_3) = 7$ and

$$e(F(\mathfrak{n}, I_1, I_2, I_3)) = \sum_{k_1+k_2+k_3=4} E_{\mathfrak{n}}(I_1^{[k_1]}, I_2^{[k_2]}, I_3^{[k_3]}; B) = 9\,.$$

EXAMPLE 3.10. Let k be a field and $B = k[[x, y, z, t]]/(x) \cap (y, z, t)$, where x, y, z, t are indeterminates. Set

$$\mathfrak{n} = (x, y, z, t)/(x) \cap (y, z, t),\ I = (x)/(x) \cap (y, z, t).$$

Clearly $\mathrm{ht}(I) = 0$ and $\dim B = 3$. Consider 2-graded fiber cone

$$F(\mathfrak{n}, \mathfrak{n}^2, I) = \bigoplus_{n_1, n_2 \geq 0} \frac{\mathfrak{n}^{2n_1} I^{n_2}}{\mathfrak{n}^{2n_1+1} I^{n_2}}\,.$$

Set

$$\ell = \dim \bigoplus_{n \geq 0} \frac{(\mathfrak{n}^2 I)^n}{\mathfrak{n}(\mathfrak{n}^2 I)^n},\ \ f(n_1, n_2) = l_B\left(\frac{\mathfrak{n}^{2n_1} I^{n_2}}{\mathfrak{n}^{2n_1+1} I^{n_2}}\right).$$

Direct computation shows that $f(n_1, n_2) = 1$ for all $n_1, n_2 \geq 1$. Hence by Remark 3.2,

$$\ell = \deg f(n_1, n_2) + 1 = 1 \text{ and } E_{\mathfrak{n}}(\mathfrak{n}^{2[0]}, I^{[0]}; B) = 1\,.$$

Clearly $f(0, 0) = 1$. Set $F(n) = \sum_{n_1+n_2=n} f(n_1, n_2)$. Then

$$F(n) = f(0, n) + f(n, 0) + \sum_{n_1+n_2=n, n_1, n_2 \geq 1} f(n_1, n_2)$$

for all $n \geq 1$. We have

$$\sum_{n_1+n_2=n, n_1, n_2 \geq 1} f(n_1, n_2) = \sum_{n_1+n_2=n, n_1, n_2 \geq 1} 1 = n - 1,\ f(0, n) = 1\,.$$

By direct computing, $f(n,0) = \binom{2n+2}{2} + 1 = 2n^2 + 3n + 2$. Thus

$$F(n) = n - 1 + 1 + 2n^2 + 3n + 2 = 2n^2 + 4n + 2$$

is a polynomial of degree 2 for all $n \geq 1$. Hence

(i) $\dim F(\mathfrak{n}, \mathfrak{n}^2, I) = 3 > 2 = \ell + d - 1$ ($d = 2, \ell = 1$).

(ii) $e(F(\mathfrak{n}, \mathfrak{n}^2, I)) = 4 \neq 1 = E_{\mathfrak{n}}(\mathfrak{n}^{2[0]}, I^{[0]}; B) = \sum_{k_1+k_2=0} E_{\mathfrak{n}}(\mathfrak{n}^{2[k_1]}, I^{[k_2]}; B)$.

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