TOKYO J. MATH.
VOL. 33, NO. 1, 2010

On Limit Sets of 4-dimensional Kleinian Groups with 3 Generators

Keita SAKUGAWA

Meiji University

(Communicated by H. Morimoto)

Abstract. In this paper, we consider a quaternionic representation of a 4-dimensional Kleinian group G with 3 generators f, g, and h, where g and h are simple parabolic, $[g, h] = id$, and $[f, g], [f, h]$ are order-2 elliptic elements. We parameterize such f, g and h up to conjugacy and we simulate the shape of the limit set $\Lambda(G)$ using computer.

1. Introduction

Let G be a d-dimensional Kleinian group, a discrete subgroup of the orientation preserving isometry group $\mathrm{Isom}^+(\mathbf{H}^d)$ of the d-dimensional hyperbolic space $\mathbf{H}^d$. The set $\Lambda(G) \subset \partial\mathbf{H}^d$ is called *the limit set* of G, the accumulation point set of any G-orbits in $\mathbf{H}^d$. The set $\Omega(G) = \partial\mathbf{H}^d \setminus \Lambda(G)$ is called *the discontinuity set* of G.

In $d = 3$ case, a 2-generator subgroup $G = \langle f, g\rangle$ of $\mathrm{PSL}_2\mathbb{C} \simeq \mathrm{Isom}^+(\mathbf{H}^3)$ such that the commutator $[f, g]$ is parabolic is called a *once punctured torus group*. When g is parabolic, the family of once punctured torus groups is called *Maskit slice*. The limit set of G consists of infinite number of mutually tangential circles, because such G has a Fuchsian subgroup of first-kind. In the same way when $[f, g]$ is elliptic and g is parabolic, the limit set also consists of mutually tangential circles.

In this paper, we discuss 4-dimensional Kleinian groups with 3 generators such that the limit sets consist of infinite number of mutually tangential spheres in $\mathbb{R}^3 = \partial\mathbf{H}^4$. Suppose that a 4-dimensional Kleinian group G is generated by 3 elements $f, g, h \in \mathrm{Isom}^+(\mathbf{H}^4)$ such that $[f, g]$ and $[f, h]$ are elliptic with simple parabolic g and h. To simplify the problem, we add algebraic assumptions that $[f, g]$ and $[f, h]$ are order 2, and $[g, h] = 1$.

From the assumption that g and h are simple parabolic, and $[g, h] = 1$, we may set $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in \mathrm{Isom}^+(\mathbf{H}^4) \simeq \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3) \subset \mathrm{GL}(2, \mathbb{H})/\{\pm I\}$, where p is a complex number. Let $f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be another generator. We have a parameterization

Received July 31, 2008; revised January 26, 2009; revised February 2, 2009

theorem as follows.

THEOREM 1.1 (Theorem 3.12). *Suppose that* $G = \langle f, g, h\rangle \subset Isom^+(\mathbf{H}^4)$ *satisfies* (i) $[f, g], [g, h]$ *are order* 2 *elliptic*, (ii) g, h *are simple parabolic*, *and* (iii) $[g, h] = 1$. *G is parameterized up to conjugacy in* $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ *as follows*:

$$G(t, p) = \left\langle \begin{pmatrix} t & (1-t^2)\boldsymbol{j}/\sqrt{2} \\ \sqrt{2}\boldsymbol{j} & t^* \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \right\rangle,$$

where $t = t_1 + t_2\boldsymbol{j} + t_3\boldsymbol{k}$, $(t_1, t_2, t_3 \in \mathbb{R})$, $t^* = t_1 + t_2\boldsymbol{j} - t_3\boldsymbol{k}$ *is the Clifford transpose of* t, (*see section* 2,) $p \in \mathbb{C}$ *and* $|p| = 1$. *Especially*, *if G is faithful and discrete*, *then* $p \notin \mathbb{R}$.

In this situation, we show that if $p = \pm\boldsymbol{i}$, $p = \pm\exp(\pi\boldsymbol{i}/3)$ or $p = \pm\exp(2\pi\boldsymbol{i}/3)$ then a subgroup $H = \langle g, h, f^{-1}gf, f^{-1}hf\rangle$ is a Kleinian group of first kind. Using this subgroup H, we show that the limit set $\Lambda(G)$ consists of infinite number of mutually tangential spheres in $\hat{\mathbb{R}}^3$ as follows.

THEOREM 1.2 (Lemma 3.8, Proposition 3.10, Theorem 3.12). (1) *Let* $H := \langle g, h, f^{-1}gf, f^{-1}hf\rangle$. *There exists an euclidean sphere* $P \subset \hat{\mathbb{R}}^3 \simeq \partial\mathbf{H}^4$ *such that P is invariant under the action of H*.

(2) *H is discrete if* $p = \pm\boldsymbol{i}$, $p = \pm exp(\pi\boldsymbol{i}/3)$ *or* $p = \pm exp(2\pi\boldsymbol{i}/3)$. *For these p*, $\Lambda(H) = P$

(3) *For* $p = \pm\boldsymbol{i}$, $p = \pm exp(\pi\boldsymbol{i}/3)$ *or* $p = \pm exp(2\pi\boldsymbol{i}/3)$,

$$\Lambda(G) = \overline{\bigcup_{aH\in G/H} aP}\,.$$

When $t \in \mathbb{R}$, $p = \pm\boldsymbol{i}$, there exists a plane $Q_g, Q_h \subset \hat{\mathbb{R}}^3$ such that Q_g (resp. Q_h) is invariant under the action of $\langle f, g\rangle$ (resp. $\langle f, h\rangle$.) So, when $p = \pm\boldsymbol{i}$ (t is general), $\langle f, g\rangle$, and $\langle f, h\rangle$ are extensions of some 3-dimensional Kleinian groups. Araki and Ito [6] found a similar family of groups in a geometrical way. In this paper, we find a family which includes the Araki-Ito's family using quaternionic matrices.

This paper is organized as follows. In section 2, we restate a classification theorem of $\mathrm{Isom}^+(\mathbf{H}^4)$ due to [8] in terms of the upper half space model. Here we refer [15]. In section 3, we show Theorem 1.1 and Theorem 1.2. In section 4, we observe computer experiments and introduce computer graphics of some limit sets.

2. Classification of $\mathrm{Isom}^+(\mathbf{H}^4)$ due to [8]

Let $\mathbf{H}^4$ be the hyperbolic space of upper half space model, and let $\hat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ be its boundary. It is well-known that an orientation preserving isometry of $\mathbf{H}^4$ is obtained from a Möbius transformation in $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ by Poincaré expansion. (See [11].) In this section, we introduce a presentation of $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ to quaternionic 2×2 matrices, and restate the classification theorem by Cao, Parker, and Wang in terms of $\mathrm{GL}_2\mathbb{H}$.

2.1. Quaternion field and Möbius transformation. Let $\mathbb{H}$ be the *quaternion field*. That is,

$$\mathbb{H} = \{x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\} ,$$

where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{i}\boldsymbol{j}\boldsymbol{k} = -1$. $\mathbb{H}$ is a non-commutative field and contains the complex number field $\mathbb{C}$. As usual, for a quaternion $x = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} \in \mathbb{H}$, we define *conjugate* of x by $\overline{x} = x_0 - x_1\boldsymbol{i} - x_2\boldsymbol{j} - x_3\boldsymbol{k}$. Moreover, we define *Clifford transpose* by $x^* = -\boldsymbol{k}\overline{x}\boldsymbol{k} = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} - x_3\boldsymbol{k}$.

Let $\mathbf{H}^4$ be the upper half space in the quaternionic right projective line $\mathbf{P}^1(\mathbb{H})$, where $\mathbf{P}^1(\mathbb{H})$ is a set of right $\mathbb{H}$ lines in $\mathbb{H}^2$,

$$\mathbf{P}^1(\mathbb{H}) := \left\{ [x : y] = \begin{pmatrix} x \\ y \end{pmatrix} \mathbb{H} \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{H}^2 - \{0\} \right\} .$$

Let $\mathrm{Sp}^K(1,1)$ be a subgroup of $\mathrm{GL}_2\mathbb{H}$ acting on $\mathbf{H}^4$, $\partial\mathbf{H}^4$. That is,

$$\begin{aligned} \mathbf{H}^4 &= \{v_0 + v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} \mid v_3 > 0\} \\ &\simeq \left\{ v \in \mathbf{P}^1(\mathbb{H}) \;\middle|\; {}^t\overline{v} \begin{pmatrix} 0 & -\boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix} v > 0 \right\} , \end{aligned}$$

$$\begin{aligned} \partial\mathbf{H}^4 = \hat{\mathbb{R}}^3 &= \{v_0 + v_1\boldsymbol{i} + v_2\boldsymbol{j}\} \cup \{\infty\} \\ &\simeq \left\{ v \in \mathbf{P}^1(\mathbb{H}) \;\middle|\; {}^t\overline{v} \begin{pmatrix} 0 & -\boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix} v = 0 \right\} , \end{aligned}$$

$$\mathrm{Sp}^K(1,1) = \left\{ M \in \mathrm{GL}_2\mathbb{H} \;\middle|\; {}^t\overline{M}KM = K,\;\; K = \begin{pmatrix} 0 & -\boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix} \right\} .$$

Here we identify $v_0 + v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ in $\mathbf{H}^4$ with $v = [v_0 + v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} : 1] \in \mathbf{P}^1(\mathbb{H})$. We remark that in $\mathbf{P}^1(\mathbb{H})$, $[u_1 : u_2] = [u_1\lambda : u_2\lambda]$ for non-zero $\lambda \in \mathbb{H}$. Using this identification, we represent $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ in $\mathrm{GL}_2\mathbb{H}$ as follows.

LEMMA 2.1.

$$\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3) \simeq \mathrm{Sp}^K(1,1) \big/ \{\pm I\} .$$

PROOF. We omit the proof. See [15] for detail. We remark that for $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $u \in \hat{\mathbb{R}}^3 \subset \mathbb{H}$,

$$M[u:1] = [xu + y : zu + w] = \begin{cases} [(xu+y)(zu+y)^{-1} : 1] & (zu + w \neq 0) \\ \infty & (zu + w = 0) \end{cases} .$$

This gives a quaternionic linear fractional transformation on $\hat{\mathbb{R}}^3$. $\square$

LEMMA 2.2 (Properties of $\mathrm{Sp}^K(1,1)$). *For* $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2\mathbb{H}$, *the following conditions are equivalent.*

(1) $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{Sp}^K(1,1)$.

(2) $xw^* - yz^* = w^*x - y^*z = 1, xy^* = yx^*, zw^* = wz^*, z^*x = x^*z, w^*y = y^*w$.

(3) $\begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} = \begin{pmatrix} w^* & -y^* \\ -z^* & x^* \end{pmatrix}$.

PROOF. (2) and (3) are equivalent trivially. (1) $\Rightarrow$ (3). Calculating

$$ {}^t\overline{\begin{pmatrix} x & y \\ z & w \end{pmatrix}} K \begin{pmatrix} x & y \\ z & w \end{pmatrix} = K\,, $$

we have $w^*x - y^*z = 1$, $z^*x = x^*z$, and $w^*y = y^*w$. Hence

$$ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} w^* & -y^* \\ -z^* & x^* \end{pmatrix} = 1\,. $$

The inverse is trivial. $\square$

As well known in $\mathrm{PSL}_2\mathbb{C}$, we define 3 types of Möbius transformations of $\hat{\mathbb{R}}^3$.

DEFINITION 2.3. Let $g \in \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ with $g \neq \mathrm{id}$. We define type of g by its action on $\mathbf{H}^4$ as following.

1. g is called *elliptic* if g has fixed points in $\mathbf{H}^4$.
2. g is called *parabolic* if g does not have fixed points in $\mathbf{H}^4$ and it has exactly one fixed point in $\hat{\mathbb{R}}^3$.
3. g is called *loxodromic* if g does not have fixed points in $\mathbf{H}^4$ and it has exactly two fixed points in $\hat{\mathbb{R}}^3$.

NOTE 2.4. If g has more than two fixed points in $\hat{\mathbb{R}}^3$, then g is elliptic or identical.

A Classification by the trace of an element of $\mathrm{PSL}_2\mathbb{C} \simeq \mathrm{M\ddot{o}b}(\hat{\mathbb{R}}^2)$ is a well known fact. Let $f \in \mathrm{PSL}_2\mathbb{C}$ be a nontrivial element. If $\mathrm{tr}(f) \in (-2,2)$, then f is elliptic. If $\mathrm{tr}(f) = \pm 2$, then f is parabolic. If $\mathrm{tr}(f) \notin [-2,2]$, then f is loxodromic.

Since $\mathrm{SL}_2\mathbb{C} \subset \mathrm{Sp}^K(1,1)$, it is natural that the trace of a matrix is useful for 3-dimensional Möbius transformations. But $\mathbb{H}$ is not commutative, the trace (in a usual way) of a matrix in $\mathrm{Sp}^K(1,1)$ is not conjugacy invariant. Only the real part of the trace is conjugacy invariant.

LEMMA 2.5. $\mathrm{Re}\,\mathrm{tr}(AB) = \mathrm{Re}\,\mathrm{tr}(BA)$ *for* $A, B \in \mathrm{M}_2\mathbb{H}$.

PROOF. For any quaternions $x, y \in \mathbb{H}$, $\mathrm{Re}(xy) = \mathrm{Re}(yx)$. □

We can define (the square of the absolute value of) "imaginary part" of the trace as following.

LEMMA 2.6. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}^K(1,1)$. $|\mathrm{Im}(a+d^*)|^2 + 4b_3c_3$ *is conjugacy invariant, where* b_3, c_3 *is* $\boldsymbol{k}$*-part of* b, c.

PROOF. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}^K(1,1)$, $A + A^{-1} = \begin{pmatrix} a+d^* & b-b^* \\ c-c^* & a^*+d \end{pmatrix} \in \mathrm{M}_2\mathbb{H}$. By Lemma 2.5, $\mathrm{Re}\,\mathrm{tr}((A+A^{-1})^2)$ is also conjugacy invariant for A. A direct calculation shows

$$\mathrm{Re}\,\mathrm{tr}((A+A^{-1})^2) = 4((\mathrm{Re}\,\mathrm{tr}\,A)^2 - (|\mathrm{Im}(a+d^*)|^2 + 4b_3c_3)) .$$

Since $\mathrm{Re}\,\mathrm{tr}\,A$ is a conjugacy invariant for A, $|\mathrm{Im}(a+d^*)|^2 + 4b_3c_3$ is also conjugacy invariant for A. □

Kido [15] also shows that $|\mathrm{Im}(a+d^*)|^2 + 4b_3c_3$ is conjugacy invariant by calculating Jacobian of quaternionic function determined by fixed points equation of Möbius transformations.

2.2. The Classification of $\mathrm{Möb}^+(\hat{\mathbb{R}}^3)$ in the unit ball model. Cao, Parker, and Wang [8] define more precise classification. They give simple-type and compound-type for each 3 type geometrically. They show in [[8], Theorem 1.1] that an equivalent condition of these 6 types in terms of the Poincaré-disk model $\mathbf{B}^4$. Here we introduce the geometric definitions and the equivalent conditions (Proposition 2.9).

DEFINITION 2.7. (1) g is simple elliptic if g is elliptic and conjugate to an element in $\mathrm{SL}_2\mathbb{R}$. g is compound elliptic if g is elliptic but not simple.

(2) g is simple parabolic if g is parabolic and conjugate to an element in $\mathrm{SL}_2\mathbb{R}$. g is compound parabolic if g is parabolic but not simple.

(3) g is simple loxodromic if g is loxodromic and conjugate to an element in $\mathrm{SL}_2\mathbb{R}$. g is compound loxodromic if g is loxodromic but not simple.

We introduce basic properties of the Poincaré-disk model.

PROPOSITION 2.8. (1) *Let* $\mathbf{B}^4$ *be the Poincaré-disk model of* 4 *dimensional hyperbolic space. then*,

$$\mathbf{B}^4 = \{v \in \mathbb{H} \mid |v| < 1\} = \left\{ v \in \mathbf{P}^1(\mathbb{H}) \;\middle|\; {}^t\overline{v} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v < 0 \right\} .$$

The isometry group of $\mathbf{B}^4$ *is given by*

$$\mathit{Isom}(\mathbf{B}^4) \simeq \left\{ M \in \mathrm{GL}_2\mathbb{H} \;\middle|\; {}^t\overline{M} J M = J,\ J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} / \{\pm I\} .$$

(2) *For* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $Isom(\mathbf{B}^4)$,

(i) $g^{-1} = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}$.

(ii) $|a|^2 - |b|^2 = 1$, $|a| = |d|$, $|b| = |c|$.

(iii) $\overline{a}b = \overline{c}d$, $a\overline{c} = b\overline{d}$.

Cao, Parker, and Wang show the following proposition.

PROPOSITION 2.9. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Isom(\mathbf{B}^4)$.

(1) *Case* $c = b = 0$.
   (i) *If* $\mathrm{Re}(a) = \mathrm{Re}(d)$ *then* g *is simple elliptic*,
   (ii) *if* $\mathrm{Re}(a) \neq \mathrm{Re}(d)$ *then* g *is compound elliptic.*

(2) *Case* $c \neq 0, \overline{c} = b$.
   (i) *If* $\mathrm{Re}(d)^2 < 1$ *then* g *is simple elliptic*,
   (ii) *if* $\mathrm{Re}(d)^2 = 1$ *then* g *is simple parabolic*, *and*
   (iii) *if* $\mathrm{Re}(d)^2 > 1$ *then* g *is simple loxodromic.*

(3) *Case* $c \neq 0, \overline{c} \neq b$. *Let* $\Delta = |\mathrm{Im}((\overline{c}^{-1}b - 1)\overline{d})|^2 - |\overline{c}^{-1}b - 1|^2$.
   (i) *If* $\Delta < 0$ *then* g *is compound elliptic*,
   (ii) *if* $\Delta = 0$ *then* g *is compound parabolic*, *and*
   (iii) *if* $\Delta > 0$ *then* g *is compound loxodromic.*

NOTE 2.10. (1) From (2)(ii) of Proposition 2.8, the case $c = 0$ and $b \neq 0$ never happens. The cases (1), (2), and (3) of Proposition 2.9 are all possibilities for g in $\mathrm{Isom}(\mathbf{B}^4)$.

(2) We have $\Delta = |b - \overline{c}|^2 - |\mathrm{Re}(a - d)|^2$. In fact,

$$\begin{aligned} \Delta &= |\mathrm{Im}((\overline{c}^{-1}b - 1)\overline{d})|^2 - |\overline{c}^{-1}b - 1|^2 \\ &= |(\overline{c}^{-1}b - 1)\overline{d}|^2 - |\mathrm{Re}((\overline{c}^{-1}b - 1)\overline{d})|^2 - |\overline{c}^{-1}b - 1|^2 \\ &= |\overline{c}^{-1}b - 1|^2(|\overline{d}|^2 - 1) - |\mathrm{Re}(\overline{c}^{-1}b\overline{d} - \overline{d})|^2 \\ &= |\overline{c}^{-1}(b - \overline{c})|^2|c|^2 - |\mathrm{Re}(\overline{c}^{-1}a\overline{c} - \overline{d})|^2 \\ &= |b - \overline{c}|^2 - |\mathrm{Re}(a - d)|^2 . \end{aligned}$$

In Proposition 2.9, $\Delta = 0$ for the case (1)(i), the case (2), or the case (3)(ii). $\Delta < 0$ for the case (1)(ii). Therefore we have the following corollary.

COROLLARY 2.11. (1) *g is compound elliptic if and only if* $\Delta < 0$.
(2) *g is compound loxodromic if and only if* $\Delta > 0$.

If g is simple elliptic, simple parabolic, or simple loxodromic then we call g is *simple*. We have the following condition that g is simple.

COROLLARY 2.12. *g is simple if and only if $\Delta = \mathrm{Re}(a-d) = 0$.*

PROOF. From Proposition 2.9, g is simple (i) if $c = 0, \mathrm{Re}(a) = \mathrm{Re}(d)$, or (ii) if $c \neq 0, b = \overline{c}$. We remark that if $c = 0$ and $\mathrm{Re}(a) = \mathrm{Re}(d)$ then $\Delta = 0$. When $b = \overline{c}$, we have $\Delta = 0$. $a\overline{b} = c\overline{d}$ follows $a = c\overline{d}c^{-1}$. Hence $\mathrm{Re}(a) = \mathrm{Re}(d)$. □

COROLLARY 2.13. *If g is simple elliptic then* $\mathrm{Re}(d)^2 < 1$.

PROOF. It is sufficient to show in case $c = 0$ and $\mathrm{Re}(a) = \mathrm{Re}(d)$. From the condition $|a|^2 - |c|^2 = 1$ and $c = 0$, we have $|a| = |d| = 1$ and $|\mathrm{Re}(d)| \leq 1$. If $|\mathrm{Re}(d)| = 1$ then g is an identity, so $\mathrm{Re}(d)^2 < 1$. □

From the above three corollaries, we can restate Proposition 2.9 as follows.

PROPOSITION 2.14. *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathit{Isom}(\mathbf{B}^4)$ $(g \neq id,)$ let $\Delta = |b - \overline{c}|^2 - |\mathrm{Re}(a-d)|^2$. Then we have*

(1) *g is simple elliptic if and only if $\Delta = \mathrm{Re}(a-d) = 0,\ \mathrm{Re}(d)^2 < 1$.*
(2) *g is simple parabolic if and only if $\Delta = \mathrm{Re}(a-d) = 0,\ \mathrm{Re}(d)^2 = 1$.*
(3) *g is simple loxodromic if and only if $\Delta = \mathrm{Re}(a-d) = 0,\ \mathrm{Re}(d)^2 > 1$.*
(4) *g is compound elliptic if and only if $\Delta < 0$.*
(5) *g is compound parabolic if and only if $\Delta = 0,\ \mathrm{Re}(a-d) \neq 0$.*
(6) *g is compound loxodromic if and only if $\Delta > 0$.*

2.3. Classification of $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ in the upper half space model. We have an easy converting formula between $\mathrm{Isom}(\mathbf{B}^4)$ and $\mathrm{Isom}(\mathbf{H}^4) \simeq \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ as follows.

LEMMA 2.15. (1) $\begin{pmatrix} 1 & -\boldsymbol{k} \\ 1 & \boldsymbol{k} \end{pmatrix} \begin{pmatrix} 0 & -\boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \boldsymbol{k} & -\boldsymbol{k} \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(2) $\xi : \mathit{Isom}(\mathbf{H}^4) \to \mathit{Isom}(\mathbf{B}^4)$:

$$\xi \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -\boldsymbol{k} \\ 1 & \boldsymbol{k} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \boldsymbol{k} & -\boldsymbol{k} \end{pmatrix} \text{ is isomorphism.}$$

PROOF. (1) is obtained by direct calculations. From (1), the relation in $\mathrm{Isom}(\mathbf{H}^4)$ is transformed to the relation in $\mathrm{Isom}(\mathbf{B}^4)$ (Proposition 2.8, (1)). □

NOTE 2.16.

$$\xi \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x + y\boldsymbol{k} - \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k} & x - y\boldsymbol{k} - \boldsymbol{k}z + \boldsymbol{k}w\boldsymbol{k} \\ x + y\boldsymbol{k} + \boldsymbol{k}z + \boldsymbol{k}w\boldsymbol{k} & x - y\boldsymbol{k} + \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k} \end{pmatrix}.$$

From this lemma, we can restate Proposition 2.14 in terms of $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$.

THEOREM 2.17 (Classification of $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$). *For* $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$, *let* $\mathrm{tr}^*(g) = x + w^*$. *Then the following statements hold.*

(1) $\Delta(g) = |\mathrm{Im}\,\mathrm{tr}^*(g)|^2 + 4y_3z_3$, *where* y_3, z_3 *are* $\boldsymbol{k}$*-part of* y, z.

(2) *For* $g \neq id$, *the following statements hold.*

(a) *If* $g + g^{-1}$ *is a diagonal matrix with real coefficients, then* $\Delta = y_3 = z_3 = 0$ *and* g *is simple.*

i. g *is simple elliptic if and only if* $|\mathrm{Re}\,\mathrm{tr}^*\, g| < 2$.

ii. g *is simple parabolic if and only if* $|\mathrm{Re}\,\mathrm{tr}^*\, g| = 2$.

iii. g *is simple loxodromic if and only if* $|\mathrm{Re}\,\mathrm{tr}^*\, g| > 2$.

(b) *Otherwise*, g *is compound.*

i. g *is compound elliptic if and only if* $\Delta < 0$.

ii. g *is compound parabolic if and only if* $\Delta = 0$.

iii. g *is compound loxodromic if and only if* $\Delta > 0$.

PROOF. Suppose $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ is corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Isom}(\mathbf{B}^4)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x + y\boldsymbol{k} - \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k} & x - y\boldsymbol{k} - \boldsymbol{k}z + \boldsymbol{k}w\boldsymbol{k} \\ x + y\boldsymbol{k} + \boldsymbol{k}z + \boldsymbol{k}w\boldsymbol{k} & x - y\boldsymbol{k} + \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k} \end{pmatrix}.$$

Therefore we have formulae for $b - \overline{c}$ and $\mathrm{Re}(a - d)$ as

$$\begin{aligned} b - \overline{c} &= \frac{1}{2}\left(x - y\boldsymbol{k} - \boldsymbol{k}z + \boldsymbol{k}w\boldsymbol{k} - \overline{(x + y\boldsymbol{k} + \boldsymbol{k}z + \boldsymbol{k}w\boldsymbol{k})}\right) \\ &= \mathrm{Im}(x + w^*) + y_3 + z_3\,, \\ \mathrm{Re}(a - d) &= \frac{1}{2}\mathrm{Re}(x + y\boldsymbol{k} - \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k} - (x - y\boldsymbol{k} + \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k})) \\ &= z_3 - y_3\,. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \Delta &= |b - \overline{c}|^2 - |\mathrm{Re}(a - d)|^2 \\ &= |\mathrm{Im}(x + w^*) + y_3 + z_3|^2 - |z_3 - y_3|^2 \\ &= |\mathrm{Im}(x + w^*)|^2 + 4y_3z_3\,. \end{aligned}$$

Suppose that g is simple. If $\Delta = 0$ and $y_3 = z_3$ then $y_3 = z_3 = 0$.

Next we calculate $\mathrm{Re}(d)^2$.

$$\begin{aligned} \mathrm{Re}(d)^2 &= \frac{1}{2}\mathrm{Re}(x - y\boldsymbol{k} + \boldsymbol{k}z - \boldsymbol{k}w\boldsymbol{k})^2 \\ &= \frac{1}{4}(\mathrm{Re}(x + w^*) + y_3 - z_3)^2\,. \end{aligned}$$

$y_3 = z_3$ follows $\mathrm{Re}(d)^2 = \dfrac{1}{4}(\mathrm{Re}\,\mathrm{tr}^*\, g)^2$.

This completes the proof of Theorem 2.17. $\square$

The following is an easy corollary.

COROLLARY 2.18. *For a nontrivial* $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$, *if* $y_3 = 0$ *or* $z_3 = 0$ *then* g *is* not *compound elliptic.*

PROOF. If $y_3 = 0$ or $z_3 = 0$ then $\Delta \geq 0$. $\square$

3. Proofs of Theorem 1.1 and Theorem 1.2

Let $G' = \langle \alpha, \beta, \gamma \mid [\alpha, \beta]^2 = [\alpha, \gamma]^2 = [\beta, \gamma] = 1 \rangle$. Consider a faithful representation ρ from G' to $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$. Let $f = \rho(\alpha)$, $g = \rho(\beta)$, and $h = \rho(\gamma)$, $G = \langle f, g, h \rangle$ and suppose that g, h are simple parabolic.

As a typical example of G, the case $f = \begin{pmatrix} t & (1-t^2)\boldsymbol{j}/\sqrt{2} \\ \sqrt{2}\boldsymbol{j} & t \end{pmatrix}$, $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & \boldsymbol{i} \\ 0 & 1 \end{pmatrix}$ $(t \in \mathbb{R}, t > 1)$ is introduced in Araki and Ito's paper [6].

LEMMA 3.1. *Assume that* $\rho|\langle \beta, \gamma \rangle$ *is faithful and* $\rho(\langle \beta, \gamma \rangle)$ *is discrete. The fixed points of* g *and* h *coincide. After taking a conjugate in* $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$, *we take that* $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, *and* $h = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$, $p \in \mathbb{C} \setminus \mathbb{R}$.

PROOF. First, we may take $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ without loss of generality. Put $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$ and solve the equation $gh = hg$.

$$\begin{pmatrix} h_1 + h_3 & h_2 + h_4 \\ h_3 & h_4 \end{pmatrix} = \begin{pmatrix} h_1 & h_1 + h_2 \\ h_3 & h_3 + h_4 \end{pmatrix}.$$

Hence we have $h_1 = h_4, h_3 = 0$. From the assumption that h is parabolic, $\mathrm{fix}(h) = \{\infty\} = \mathrm{fix}(g)$ and $h_1 = h_4 = 1$.

From Lemma 2.2, $h_1 h_2^* = h_2 h_1^*$ and the $\boldsymbol{k}$-part of h_2 is 0. Using a conjugation by a rotation around the real number axis, we may take $h = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$, where p is a complex number.

We exclude the case $p \in \mathbb{R}$. Because if p is a rational number, g, h has another relation than $[g, h] = 1$ and ρ is not faithful. If p is an irrational number then the orbit of the origin is not discrete on the real axis. Therefore $p \in \mathbb{C} \setminus \mathbb{R}$. □

Next, we solve the equation $[f, g]^2 = [f, h]^2 = 1$.

LEMMA 3.2. *For* $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$, *the trace will be denoted by* $tr^*(M) = x + w^*$. *Suppose that* M *is nontrivial and simple.* M *is an order-2 element if and only if* $tr^*(M) = 0$.

PROOF. $M^2 = id$ follows $M = M^{-1}$. There are 2 cases, $M = \pm M^{-1}$.

(case 1) $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} w^* & -y^* \\ -z^* & x^* \end{pmatrix}$

Comparing entries, we have $w = x^*$, $y = y_3\boldsymbol{k}$, and $z = z_3\boldsymbol{k}$ for real numbers y_3, z_3. From Theorem 2.17, $y_3 = z_3 = 0$ and hence $y = z = 0$. But this means $x^2 = 1$ because of $xw^* - yz^* = 1$. This contradicts the condition that M is nontrivial.

(case 2) $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} -w^* & y^* \\ z^* & -x^* \end{pmatrix}$

Comparing entries, we have $x = -w^*$ and $tr^*(M) = 0$.

Conversely, suppose that M is simple (especially, $y_3 = z_3 = 0$) and $tr^*(M) = x + w^* = 0$. Using $xw^* - yz^* = 1$, we obtain

$$\begin{aligned} \begin{pmatrix} x & y \\ z & w \end{pmatrix}^2 &= \begin{pmatrix} x^2 + yz & x(y - y^*) \\ (z - z^*)x & w^2 + zy \end{pmatrix} \\ &= \begin{pmatrix} -xw^* + yz^* & 2xy_3\boldsymbol{k} \\ 2z_3x\boldsymbol{k} & (-w^*x + z^*y)^* \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

□

NOTE 3.3. In Lemma 3.2, the assumption that M is simple is essential. There are many counter-examples if M is compound. For example, the trace of

$$\begin{pmatrix} 0 & \cos(\pi/n) + \boldsymbol{k}\sin(\pi/n) \\ -\cos(\pi/n) - \boldsymbol{k}\sin(\pi/n) & 0 \end{pmatrix}$$

is zero but this is compound elliptic and the order is n. The trace of $\begin{pmatrix} \boldsymbol{k} & \boldsymbol{k} \\ 0 & \boldsymbol{k} \end{pmatrix}$ is also zero, but this is compound parabolic.

We show that $[f, g]$ and $[f, h]$ are simple elliptic.

LEMMA 3.4. (1) *For* $f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, *and* $h = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in$ $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$, *where* $p \in \mathbb{C} \backslash \mathbb{R}$, *traces of commutators are* $tr^*(g^{-1}f^{-1}gf) = z^*z+2$ *and* $tr^*(h^{-1}f^{-1}hf) = pz^*pz + 2$.

(2) $[f, g]$ *and* $[f, h]$ *are simple elliptic.*

PROOF. (1) We show the second formula directly. The first formula is obtained by substitution $p = 1$ into the second one.

$$h^{-1}f^{-1}hf = \begin{pmatrix} 1 + w^*pz + pz^*pz & w^*pw - p + pz^*pw \\ -z^*pz & 1 - z^*pw \end{pmatrix},$$

$$tr^*(h^{-1}f^{-1}hf) = (1 + w^*pz + pz^*pz) + (1 - z^*pw)^* = pz^*pz + 2\,.$$

Because $(-z^*pz)^* = -z^*pz$, the $\boldsymbol{k}$-part of the (2, 1)-entry is zero. From Corollary 2.18, $[f, h]$ and $[f, g]$ are not compound elliptic but simple elliptic. $\square$

From Lemma 3.2 and Lemma 3.4, the condition $[f, g]^2 = [f, h]^2 = id$ is equivalent to $z^*z + 2 = pz^*pz + 2 = 0$. Solving this equation, we have the following.

LEMMA 3.5.

$$\begin{cases} z^*z + 2 = 0 \\ pz^*pz + 2 = 0 \end{cases} \quad \textit{if and only if} \quad \begin{cases} z = \pm\sqrt{2}\boldsymbol{j} \\ |p| = 1 \end{cases}$$

PROOF. Suppose that $z = z_0 + z_1\boldsymbol{i} + z_2\boldsymbol{j} + z_3\boldsymbol{k} \in \mathbb{H}$, and $p = p_0 + p_1\boldsymbol{i} \in \mathbb{C}$, where $z_0, z_1, z_2, z_3, p_0, p_1 \in \mathbb{R}$, and $p_1 \neq 0$.

$$\begin{aligned} z^*z + 2 &= (z_0 + z_1\boldsymbol{i} + z_2\boldsymbol{j} - z_3\boldsymbol{k})(z_0 + z_1\boldsymbol{i} + z_2\boldsymbol{j} + z_3\boldsymbol{k}) + 2 \\ &= z_0^2 - z_1^2 - z_2^2 + z_3^2 + 2 + 2(-z_0z_1 + z_2z_3)\boldsymbol{i} - 2(z_0z_2 + z_1z_3)\boldsymbol{j} \\ &= 0\,, \end{aligned}$$

and we have

$$z_0^2 - z_1^2 - z_2^2 + z_3^2 + 2 = 0\,, \tag{3.1}$$

$$z_0z_1 - z_2z_3 = 0\,, \tag{3.2}$$

$$z_0z_2 + z_1z_3 = 0\,. \tag{3.3}$$

Calculating (3.2)$\times z_1$+(3.3)$\times z_2$ and (3.3)$\times z_2$−(3.2)$\times z_1$, we obtain

$$z_0(z_1^2 + z_2^2) = 0\,,$$

$$z_3(z_1^2 + z_2^2) = 0\,.$$

If $z_1^2 + z_2^2 = 0$ then it contradicts (3.1). Hence

$$z_0 = z_3 = 0\,. \tag{3.4}$$

Using (3.1), we obtain

$$z_1^2 + z_2^2 = 2\,. \tag{3.5}$$

Next we calculate $pz^*pz + 2$, remarking that $z^* = z$ from (3.4).

$$\begin{aligned}
pz^*pz + 2 &= (pz)^2 + 2 \\
&= (-z_1p_1 + z_1p_0\boldsymbol{i} + z_2p_0\boldsymbol{j} + z_2p_1\boldsymbol{k})^2 + 2 \\
&= (z_1p_1)^2 - (z_1p_0)^2 - (z_2p_0)^2 - (z_2p_1)^2 \\
&\quad + 2 - 2z_1p_1(z_1p_0\boldsymbol{i} + z_2p_0\boldsymbol{j} + z_2p_1\boldsymbol{k}) \\
&= -2p_0^2 + (2z_1^2 - 2)p_1^2 + 2 - 2z_1p_1(z_1p_0\boldsymbol{i} + z_2p_0\boldsymbol{j} + z_2p_1\boldsymbol{k}) \\
&= 0\,.
\end{aligned}$$

Hence

$$p_0^2 + (1 - z_1^2)p_1^2 - 1 = 0\,, \tag{3.6}$$

$$p_0{z_1}^2 = 0\,, \tag{3.7}$$

$$p_0z_1z_2 = 0\,,$$

$$z_1z_2 = 0\,. \tag{3.8}$$

we have $z_1 = 0$ or $z_2 = 0$ by (3.8).

(i) Case $z_1 = 0$. From (3.5), $z = \pm\sqrt{2}\boldsymbol{j}$. From (3.6), ${p_0}^2 + {p_1}^2 = 1$, that is, $|p| = 1$.

(ii) Case $z_2 = 0$. From (3.5), $z_1^2 = 2 \neq 0$. From (3.7), $p_0 = 0$. But from (3.6), $p_1^2 = -1$. This is a contradiction.

We obtain $z = \pm\sqrt{2}\boldsymbol{j}$ and $|p| = 1$. This completes the proof. □

NOTE 3.6. We suppose that $z = \sqrt{2}\boldsymbol{j}$. (An arbitrary element in $\mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$ has an ambiguity of the multiplication by $\pm I$.)

Here we put off the proof of Theorem 1.1, we show Theorem 1.2 first. We consider a sphere in $\hat{\mathbb{R}}^3$ such that it is a part of $\Lambda(G)$. Clearly any plane in $\mathbb{R}^3$ parallel to $\mathbb{C} \subset \mathbb{R}^3$ is invariant under the action of $\langle g, h\rangle$.

LEMMA 3.7. *Let H be $\langle g, h, f^{-1}gf, f^{-1}hf\rangle$. Then the sphere*

$$P := \{f^{-1}(\infty) + v \,\big|\, v \in \mathbb{C}\} \cup \{\infty\} \subset \hat{\mathbb{R}}^3$$

is H-invariant.

PROOF. Clearly g and h preserve P. We check that $f^{-1}hf$ preserves P. For $f^{-1}gf$, we can show it easily after showing on $f^{-1}hf$, substituting $p = 1$ in the calculation on $f^{-1}hf$. We have $f^{-1}(\infty) = -w^*(z^*)^{-1}$ and

$$f^{-1}hf(-w^*(z^*)^{-1}) = f^{-1}h(\infty) = f^{-1}(\infty) = -w^*(z^*)^{-1}\,.$$

We identify $u = \begin{pmatrix} u \\ 1 \end{pmatrix}$ with $[u : 1] \in \mathbf{P}^1(\mathbb{H})$. For any $-w^*(z^*)^{-1} + v \in P$

$$
\begin{aligned}
f^{-1}hf \begin{pmatrix} -w^*(z^*)^{-1} + v \\ 1 \end{pmatrix} &= \begin{pmatrix} -w^*(z^*)^{-1} \\ 1 \end{pmatrix} + f^{-1}hf \begin{pmatrix} v \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -w^*(z^*)^{-1} + (1 + w^*pz)v \\ 1 - z^*pzv \end{pmatrix} \\
&= \begin{pmatrix} (-w^*(z^*)^{-1} + (1 + w^*pz)v)(1 - z^*pzv)^{-1} \\ 1 \end{pmatrix} .
\end{aligned}
$$

Here,

$$
\begin{aligned}
&\big(- w^*(z^*)^{-1} + (1 + w^*pz)v\big)(1 - z^*pzv)^{-1} \\
&\quad = \big(- w^*(z^*)^{-1}(1 - z^*pzv) + v\big)(1 - z^*pzv)^{-1} \\
&\quad = -w^*(z^*)^{-1} + v(1 - z^*pzv)^{-1} .
\end{aligned}
$$

From Lemma 3.4, $pz^*pz = -2$. $z^*pz = -2/p \in \mathbb{C}$ and hence $v(1 - z^*pzv)^{-1} \in \mathbb{C}$. $f^{-1}(\infty) + v(1 - z^*pzv)^{-1}$ is contained in P. □

The action of the subgroup H on $P \simeq \{v \in \mathbb{C}\} \cup \{\infty\} = \hat{\mathbb{C}}$ is the following. (We remark $z = \sqrt{2}\boldsymbol{j}$.)

$$
\begin{aligned}
g &: v \mapsto v + 1 \\
h &: v \mapsto v + p \\
f^{-1}gf &: v \mapsto v(1 - \sqrt{2}\,\boldsymbol{j}\sqrt{2}\,\boldsymbol{j}v)^{-1} = v(1 + 2v)^{-1} \\
f^{-1}hf &: v \mapsto v(1 - \sqrt{2}\,\boldsymbol{j}\,p\sqrt{2}\,\boldsymbol{j}v)^{-1} = v(1 + 2\overline{p}v)^{-1}
\end{aligned}
$$

These transformations are Möbius transformations and we represent H into $\mathrm{PSL}_2\mathbb{C}$ as

$$
H = H(p) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2\overline{p} & 1 \end{pmatrix} \right\rangle \subset \mathrm{PSL}_2\mathbb{C} .
$$

LEMMA 3.8. *$H(\pm\boldsymbol{i}), H(\pm\omega)$ and $H(\pm\omega^2)$ are discrete in* $\mathrm{PSL}_2\mathbb{C}$, *where* $\omega = \dfrac{-1 + \sqrt{3}}{2}$.

PROOF. When p is $\pm\boldsymbol{i}$, $\pm\omega$ or $\pm\omega^2$, $\mathbb{Z} + p\mathbb{Z}$ is closed with respect to summation and multiplication. Any entries of any elements in $H(p)$ are in $\mathbb{Z} + p\mathbb{Z}$. Hence $H(p)$ is discrete. □

Next, we determine the shape of the limit set $\Lambda(G)$ for $\pm\boldsymbol{i}$, $\pm\omega$, or $\pm\omega^2$. From $H(p) = H(-p)$ and $H(\omega) = H(\omega^2)$, it is sufficient to show in case of $p = \boldsymbol{i}$ or ω.

PROPOSITION 3.9. *$\Lambda(H) = P$ in case of $p = \boldsymbol{i}$ or ω.*

In the case $p = \boldsymbol{i}$, $H(p)$ is a finite index subgroup of Picard group $\mathrm{PSL}_2(\mathbb{Z} + \boldsymbol{i}\mathbb{Z})$. Therefore, it is clear that $\Lambda(H) = P$. In the case $p = \omega$, we show the following.

LEMMA 3.10. *$\mathbf{H}^3/H(\omega)$ has finite volume, hence $H(\omega)$ is a Kleinian group of first kind and $\Lambda(H(\omega)) = P$.*

PROOF. Let D_1, D_2, D_3, D_4 be domains as follows.

$$D_1 = \left\{ (z,t) \in \mathbf{H}^3, z \in \mathbb{C} \;\middle|\; -\frac{1}{2} \le \mathrm{Re}(z) \le \frac{1}{2} \right\},$$

$$D_2 = \left\{ (z,t) \in \mathbf{H}^3, z \in \mathbb{C} \mid -1 \le z\overline{\omega} + \overline{z}\omega \le 1,\ -1 \le z\omega + \overline{z}\,\overline{\omega} \le 1 \right\},$$

$$D_3 = \left\{ (z,t) \in \mathbf{H}^3, z \in \mathbb{C} \;\middle|\; |z - \frac{1}{2}|^2 + t^2 \ge 1/4,\ |z + \frac{1}{2}|^2 + t^2 \ge \frac{1}{4} \right\},$$

$$D_4 = \left\{ (z,t) \in \mathbf{H}^3, z \in \mathbb{C} \;\middle|\; |z - \frac{\overline{\omega}}{2}|^2 + t^2 \ge 1/4,\ |z + \frac{\overline{\omega}}{2}|^2 + t^2 \ge \frac{1}{4} \right\}.$$

$D_1 \cap D_2$ is a fundamental domain of $\langle g, h \rangle$. D_3 is a fundamental domain of $\langle f^{-1} g f \rangle$. D_4 is a fundamental domain of $\langle f^{-1} h f \rangle$. Since a fundamental domain of $H(\omega)$ is $D_1 \cap D_2 \cap D_3 \cap D_4$, $\mathbf{H}^3/H(\omega)$ has finite volume. □

Therefore, in case of $p = \pm\boldsymbol{i}$, $\pm\omega$ or $\pm\omega^2$, we have the conclusion.

THEOREM 3.11. *In case p is $\pm\boldsymbol{i}$, $\pm\omega$ or $\pm\omega^2$, if G is Kleinian,*

$$\Lambda(G) = \overline{\bigcup_{aH \in G/H} aP}\,.$$

PROOF. For any coset $aH \in G/H$, we have $\Lambda(aHa^{-1}) = aP$, $aP \subset \Lambda(G)$ and hence $\overline{\bigcup_{aH \in G/H} aP} \subset \Lambda(G)$. On the other hand, $\overline{\bigcup_{aH \in G/H} aP}$ is closed and G-invariant. Hence $\Lambda(G) \subset \overline{\bigcup_{aH \in G/H} aP}$. □

We complete the proof of Theorem 1.2.

We resume the proof Theorem 1.1. Consider a parameterization of G up to conjugacy. Let $f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $z = \sqrt{2}\boldsymbol{j}$. (from Lemma 3.5.) We can determine y uniquely from x, w, using $xw^* - yz^* = 1$. Thus we will parameterize x and w.

From $z^*x = x^*z$, $zw^* = wz^*$, and $z = \sqrt{2}\boldsymbol{j}$, we have $\boldsymbol{j}x = x^*\boldsymbol{j}$, $\boldsymbol{j}w = w^*\boldsymbol{j}$. Hence the $\boldsymbol{i}$-parts of x and w are zero.

We may take $x = w^*$ by a conjugation. In fact, let u be a quaternion such that $u = u^*$ and let $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in \mathbf{M\ddot{o}b}^+(\hat{\mathbb{R}}^3)$. Since U is a translation on $\hat{\mathbb{R}}^3$, $U^{-1}gU = g$, $U^{-1}hU = h$.

By a calculation,

$$U^{-1} f U = \begin{pmatrix} x - \sqrt{2}u\boldsymbol{j} & * \\ \sqrt{2}\boldsymbol{j} & w + \sqrt{2}\boldsymbol{j}u \end{pmatrix} .$$

If we put $u = \dfrac{(w^* - x)\boldsymbol{j}}{2\sqrt{2}}$ then $u = u^*$ and $x - \sqrt{2}u\boldsymbol{j} = (w + \sqrt{2}\boldsymbol{j}u)^*$. Hence we have the following theorem.

THEOREM 3.12. *$G = \langle f, g, h\rangle$ is parameterized up to conjugacy by (t, p). In fact,*

$$G(t, p) = \left\langle \begin{pmatrix} t & (1 - t^2)\boldsymbol{j}/\sqrt{2} \\ \sqrt{2}\boldsymbol{j} & t^* \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \right\rangle ,$$

where $t = t_1 + t_2\boldsymbol{j} + t_3\boldsymbol{k}$, $(t_1, t_2, t_3 \in \mathbb{R})$, $p \in \mathbb{C} \setminus \mathbb{R}$, and $|p| = 1$.

For any t and p, representation may not be faithful or discrete.

LEMMA 3.13. *Let p be $\boldsymbol{i}$ or ω. If $t \in \mathbb{R}$ and $|t| \geq 1$ then G is discrete.*

PROOF. Suppose $p = \boldsymbol{i}$. Let t be a real number such that $|t| \geq 1$. Let C_1 and C_2 be two spheres such that their centers are $\pm\dfrac{t\boldsymbol{j}}{\sqrt{2}}$ and the radii are both $\dfrac{1}{\sqrt{2}}$. f maps the interior of C_1 to the exterior of C_2. Let F be the intersection of the exterior of C_1, C_2 (that is, the part with the infinity point,) and a regular prism with a square section with length 1 edges and with center in the $\boldsymbol{j}$-axis. All of dihedral angles of F are $\pi/2$ or $\pi/4$. F is a fundamental domain of G by Poincaré's theorem. Hence G is discrete in $\hat{\mathbb{R}}^3$. In the similar way, if $p = \omega$, using a regular hexagonal prism instead of a square prism, we can obtain the same conclusion. □

Let $\mathcal{M}$ be the parameter space of discrete G. In [7], Araki and Ito make computer graphics of $\mathcal{M}$ for $p = \boldsymbol{i}$.

4. Computer simulation of the limit set

The author developed software *Norio* [22], where we are allowed to see computer graphics of $\bigcup_{aH\in G/H} aP$ for given $p \in \mathbb{C}$ $(|p| = 1)$ and $t = t_1 + t_2\boldsymbol{i} + t_3\boldsymbol{j} \in \mathbb{R}^3$. Figure 1, 2, 3, and 4 are the pictures of the simulation for some parameters. The following pictures are given by POV-Ray (Mac OS version) [21]. In one picture, we draw about 1, 000, 000 spheres of aP, $(aH \in G/H.)$

We try the software for many parameters and we have the following observation.

OBSERVATION 4.1. 1. If $p \neq \pm\boldsymbol{i}$, $p \neq \pm\omega$ or $p \neq \pm\omega^2$ then

$$\bigcup_{aH\in G/H} aP = \mathbb{R}^3 .$$

2. If $p = \omega$, the parameter space $\mathcal{M}$ is three dimensional and has a fractal boundary.

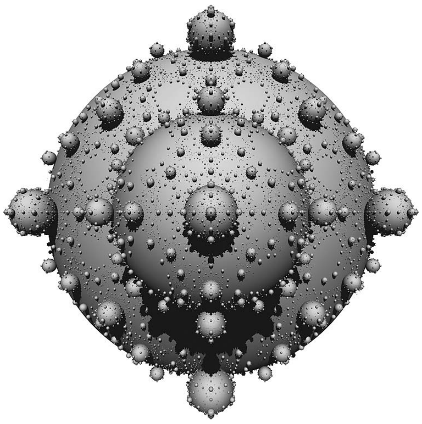

FIGURE 1. $p = i, t = 2.8$, see from the view direction of the fixed point of f

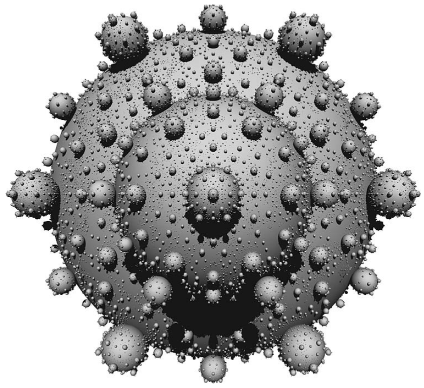

FIGURE 2. $p = \omega, t = 2.8$, see from the view direction of the fixed point of f

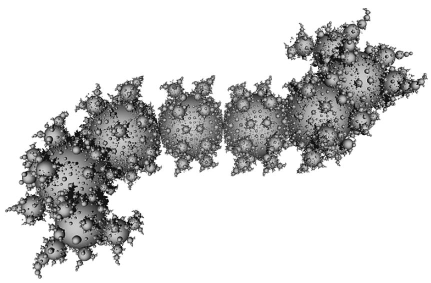

FIGURE 3. $p = \omega, t = 1.95 + 0.15j + 0.15k$

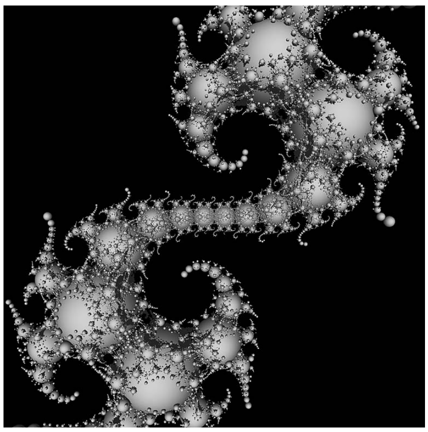

FIGURE 4. $p = \boldsymbol{i}, t = 1.93 + 0.05j$

ACKNOWLEGEMENT. The author would like to thank Prof. Kentaro Ito, Prof. Ken'ichi Ohshika, and Prof. Nariya Kawazumi for their useful advice and encouragement. He thanks to Prof. Yoshihide Okumura for encouragement through messages. He thanks also Prof. Kazushi Ahara for his coaching of computer programming and for direction in the seminars.

References

[1] AHARA K., Sphairahedra and 4-dimensional quasi-fuchsian groups (in Japanese), RIMS proceedings, 1329 (2003), 109–114.

[2] AHARA K. and ARAKI Y., Spheirahedral approach to parameterize visible three dimensional quasi-Fuchsian fractals, Proc. of the CGI (2003), 226–229.

[3] AKIYOSHI H., SAKUMA M., WADA M. and YAMASHITA Y., Jorgensen's picture of quasifuchsian punctured torus groups, *Kleinian groups and hyperbolic 3-manifolds*, London Math. Soc., Lecture Note Series. 299 (2003), 247–273.

[4] AKIYOSHI H. and SAKUMA M., Comparing two convex hull constructions of cusped hyperbolic manifolds, *Kleinian groups and hyperbolic 3-manifolds*, London Math. Soc., Lecture Note Series. 299 (2003), 209–246.

[5] AKIYOSHI H., MIYACHI H. and SAKUMA M., A refinement of McShane's identity for quasifuchsian punctured torus groups, *Kleinian groups and hyperbolic 3-manifolds*, London Math. Soc., Lecture Note Series. 329 (2006), 151–185.

[6] ARAKI Y. and ITO K., An extension of the Maskit slice for 4-dimensional Kleinian groups, Conform. Geom. Dyn. 12 (2008), 199–226.

[7] ARAKI Y., ITO K. and KOMORI Y., A note on a 3-dimensional extension of the Maskit slice (in Japanese), RIMS Proceedings, 1571 (2007), 172–192.

[8] CAO W., PARKER J. and WANG X., On the classification of quaternionic Möbius transformations, Math. Proc. Cambridge Philosophical Soc., **137**:2 (2004), 349–362.

[9] EPSTEIN D. B. A., MARDEN A. and MARKOVIC V., Quasiconformal homeomorphisms and the convex hull boundary, Ann. of Math., **159** (2004), 305–336.

[10] HERSONSKY S., A generalization of the Shimizu-Leutbecher and Jørgensen inequalities to Möbius transformations in $\mathbf{R}^{\mathbf{N}}$, Proc. of the American Math. Soc., **121**:1 (1994), 209–215.

[11] HERTRICH-JEROMIN U., *Introduction to Möbius Differential Geometry*, London Math. Soc., Lecture Note Series. 300 (2003).

[12] HIDALGO R. and MASKIT B., On neoclassical schottky groups, Trans. of the American Math. Soc., **358**:11 (2006), 4765–4792.

[13] KAPOVICH M., *Hyperbolic Manifolds and Discrete Groups*, Birkhäuser, 2001.

[14] KAPOVICH M., Topological aspects of Kleinian groups in several dimensions, Proc. of 3-rd Ahlfors-Bers Colloquium (2002).

[15] KIDO T., Möbius transformations on quaternion, Tohoku Math. J., to appear.

[16] KOMORI Y., SUGAWA T., WADA M. and YAMASHITA Y., Drawing Bers Embeddings of the Teichmuller Space of Once-Punctured Tori, Exper. Math., **15** (2006), 51–60.

[17] LANPHIER D. and ROSENHOUSE J., A decomposition theorem for Cayley graph of Picard group quotients, The Journal of Combinatorial Math. and Combinatorial Computing, **50** (2004), 95–104.

[18] MASKIT B., *Kleinian Groups*, Springer (1988).

[19] MIYACHI H., Private introduction of 3-dimensional hyperbolic geometry, preprint.

[20] MUMFORD D., SERIES C. and WRIGHT D., *Indra's Pearls*, Cambridge Press, 2003.

[21] POV-Ray, http://www.povray.org/

[22] SAKUGAWA K., "Norio", software.

[23] TANIGUCHI M. and MATSUZAKI K., *Hyperbolic manifolds and Kleinian groups* (in Japanese), Nihon-Hyoronsha, 1993.

[24] TANIGUCHI M. and OKUMURA Y., *Introduction of hyperbolic geometry* (in Japanese), Baifu-kan, 1996.

[25] WADA M., OPTi's Algorithm for Discreteness Determination, Exper. Math., **15** (2006), 61–66.

[26] WADA M., OPTi's Algorithm for Drawing the Limit Set, Interdisciplinary Information Sciences, **9** (2003), 183–187.

[27] WRIGHT D., Searching for the cusp, *Spaces of Kleinian Groups*, Cambridge University Press Lond. Math. Soc. Lec. Notes 329 (2005), 301–336.

Present Address:
DEPARTMENT OF MATHEMATICS,
MEIJI UNIVERSITY,
HIGASHIMITA, TAMA, KAWASAKI, KANAGAWA, 214–8571 JAPAN.