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On a Differential Equation Characterizing a Riemannian Structure of a Manifold

Masahiko KANAI

Keio University
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It often happens that the existence of a function on a riemannian manifold satisfying some condition gives informations about the topological, differentiable or riemannian structure of the manifold. In fact, in 1962, Obata characterized the euclidean sphere of radius $1/\sqrt{k}$ as the only complete riemannian manifold which has a nontrivial solution for the differential equation

$$(1) \qquad \mathrm{Hess}\, f + kfg = 0$$

with a positive constant k, where $\mathrm{Hess}\, f$ is a symmetric (0, 2)-tensor called the hessian of f defined by $(\mathrm{Hess}\, f)(X, Y) = (\nabla_X df)(Y) = XYf - (\nabla_X Y)f$ for any vector fields X and Y, and g is the metric tensor: that is

THEOREM A (Obata [3, 4]). *Let $k>0$. For a C^∞ complete riemannian manifold (M, g) of dimension $n(\geqq 2)$, there is a C^∞ nontrivial function f on M satisfying* (1), *if and only if (M, g) is isometric to the euclidean n-sphere $(S^n, (1/k)g_0)$ of radius $1/\sqrt{k}$, where g_0 denotes the canonical metric on S^n with constant curvature* 1.

Also there is a work by Tanno [5] in which he investigated effects of some differential equations of order three on riemannian and kählerian manifolds.

In this article we give necessary and sufficient conditions for the existence of a nontrivial function f on (M, g) which satisfies (1) with a nonpositive constant k. A manifold is assumed to be of C^∞ and connected, unless otherwise indicated. Also all tensors (including functions, vector fields, etc.) are assumed to be C^∞, unless otherwise indicated.

The case $k=0$ is reduced to the following trivial theorem:

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THEOREM B. *A complete riemannian manifold* (M, g) *of dimension* $n(\geqq 2)$ *has a nontrivial function* f *on* M *satisfying*

$$\mathrm{Hess}\, f = 0\ ,$$

if and only if (M, g) *is the riemannian product* $(\bar{M}, \bar{g}) \times (\boldsymbol{R}, g_0)$ *of a complete riemannian manifold* $(\bar{M}, \bar{g})$ *and the real line* $(\boldsymbol{R}, g_0)$, *where* g_0 *denotes the canonical metric of* $\boldsymbol{R}$.

A proof of Theorem B will be found in e.g. [7].

It is easily verified that every nontrivial function satisfying (1) with $k>0$ has critical points and that any nontrivial function satisfying (1) with $k=0$ does not have critical points. But the case $k<0$ is divided into following two theorems:

THEOREM C. *Suppose that* (M, g) *is a complete riemannian manifold of dimension* $n(\geqq 2)$, *and that* $k<0$. *Then there is a nontrivial function* f *on* M *with a critical point which satisfies* (1), *if and only if* (M, g) *is the simply connected complete riemannian manifold* $(H^n, -(1/k)g_0)$ *of constant curvature* k, *where* g_0 *is the canonical metric on the hyperbolic space of constant curvature* -1.

THEOREM D. *Let* (M, g) *and* k *be as in Theorem* C. *Then there is a function* f *on* M *without critical points which satisfies* (1), *if and only if* (M, g) *is the warped product* $(\bar{M}, \bar{g})_\xi \times (\boldsymbol{R}, g_0)$ *of a complete riemannian manifold* $(\bar{M}, \bar{g})$ *and the real line* $(\boldsymbol{R}, g_0)$, *warped by a function* $\xi\colon \boldsymbol{R} \to \boldsymbol{R}$ *such that* $\ddot{\xi} + k\xi = 0$, $\xi > 0$, *where* g_0 *denotes the canonical metric on* $\boldsymbol{R}$; $g_0 = dt^2$.

The notion of warped products shall be reviewed in § 1, and proofs of Theorem C and Theorem D are given in § 2 and § 3, respectively.

By noting that the simply connected complete n-dimensional riemannian manifold $(H^n, -(1/k)g_0)$ of constant curvature k $(n \geqq 2, k<0)$ is constructed as the warped product $(\boldsymbol{R}^{n-1}, g_0)_\xi \times (\boldsymbol{R}, g_0)$ with $\xi = e^{\pm\sqrt{-k}\,t}$ (see Lemma 3 in § 1), we conclude from Theorem C and Theorem D.

COROLLARY E. *Let* (M, g) *and* k *be as in Theorem* C. *Then there is a nontrivial function* f *on* M *satisfying* (1), *if and only if* (M, g) *is the warped product* $(\bar{M}, \bar{g})_\xi \times (\boldsymbol{R}, g_0)$ *of a complete riemannian manifold* $(\bar{M}, \bar{g})$ *and the real line* $(\boldsymbol{R}, g_0)$, *warped by a function* $\xi\colon \boldsymbol{R} \to \boldsymbol{R}$ *such that* $\ddot{\xi} + k\xi = 0$, $\xi > 0$.

Combining the theorems mentioned above and Lemma 3 in § 1, we have

COROLLARY F. *Suppose that* (M, g) *is a complete riemannian manifold of dimension* 2 *and that* k *is any constant. If there is a nontrivial function* f *on* M *which satisfies* (1), *then* (M, g) *is of constant curvature* k.

In § 4, we will briefly discuss conformal vector fields on a riemannian manifold as an application of the above theorems, and generalize a theorem of Yano and Nagano [7] for (not necessarily complete) conformal vector fields on a complete Einstein manifold.

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§ 1. Warped products.

In this section, we review briefly the fundamental formulas for warped products (see, for detail, Bishop-O'Neill [1]).

Let I be an open interval in $\boldsymbol{R}$. For a riemannian manifold $(\bar{M}, \bar{g})$ and a function $\xi: I \to \boldsymbol{R}$ with positive values, the *warped product* $(M, g) = (\bar{M}, \bar{g})_\xi \times (I, g_0)$ of $(\bar{M}, \bar{g})$ and the interval (I, g_0) warped by ξ, is defined by

$$M = \bar{M} \times I, \qquad g = \xi^2 \bar{g} + g_0,$$

where g_0 is the canonical metric of I. Note that any vector field X on $\bar{M}$ is lifted onto M uniquely. Unless confusions may happen, the lift of X is also denoted by the same symbol X. Let ∇ and $\bar{\nabla}$ be the riemannian connections of (M, g) and $(\bar{M}, \bar{g})$, respectively. Then the following lemma is easily verified, where T denotes the lift of the vector field d/dt on I.

LEMMA 1. *For any vector fields* X, Y on $\bar{M}$,

(a) $\nabla_X Y = \bar{\nabla}_X Y - \xi\dot{\xi}\bar{g}(X, Y)T$

(b) $\nabla_T X = \nabla_X T = (\dot{\xi}/\xi)X$

(c) $\nabla_T T = 0$.

A proof is found in [1]. This lemma implies

COROLLARY 2. *For any riemannian manifold* $(\bar{M}, \bar{g})$, *the warped product* $(M, g) = (\bar{M}, \bar{g})_\xi \times (\boldsymbol{R}, g_0)$ *with* $\ddot{\xi} - \xi = 0$, $\xi > 0$, *has a solution of the equation* $\operatorname{Hess} f - fg = 0$ *without critical points.*

PROOF. Take a function $F: \boldsymbol{R} \to \boldsymbol{R}$ as $\dot{\xi}\dot{F} - \xi F = 0$. Then the function $f: \bar{M} \times \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(\bar{p}, t) = F(t)$ $(\bar{p} \in \bar{M}, t \in \boldsymbol{R})$ is a solution of the above equation without critical points.

Next, we calculate the curvature tensor and the Ricci curvature of a warped product. Let R be the curvature tensor of $(M, g)=(\bar{M}, \bar{g})_{\xi}\times(I, g_0)$ and $\bar{R}$ the curvature tensor of $(\bar{M}, \bar{g})$ (here we adopt the sign convention $R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$). Also let Ric and $\overline{Ric}$ be the Ricci tensors of (M, g) and $(\bar{M}, \bar{g})$, respectively.

LEMMA 3. *For any vector fields X, Y and Z on $\bar{M}$, the following hold:*

(a) $R(X, Y)Z=\bar{R}(X, Y)Z+\dot{\xi}^2\{\bar{g}(X, Z)Y-\bar{g}(Y, Z)X\}$

(b) $R(X, Y)T=0$

(c) $g(R(T, X)T, Y)=\ddot{\xi}\xi\bar{g}(X, Y)$.

LEMMA 4. *If X is a vector field on $\bar{M}$, then the following hold:*

(a) $Ric(X, X)=\overline{Ric}(X, X)-\{\ddot{\xi}/\xi+(n-1)(\dot{\xi}/\xi)^2\}g(X, X)$

(b) $Ric(T, T)=-n(\ddot{\xi}/\xi)$

where n is the dimension of $\bar{M}$.

Proofs are found in [1].

§ 2. Proof of Theorem C.

In this section, we give a proof of Theorem C. We begin with the following lemma.

LEMMA 5. *Suppose that (N, g) is a (not necessarily complete) riemannian manifold of dimension $n(\geqq 2)$, and that f is a function on (N, g) without critical points. If f satisfies the equation*

$$\operatorname{Hess} f-fg=0\ , \tag{2}$$

then the following hold:

(a) *If ν is the vector field on N defined by $\nu=(1/|\operatorname{grad} f|)\operatorname{grad} f$, then $\nabla_\nu\nu=0$, i.e., any integral curve of ν is a geodesic.*

(b) *Every hypersurface $f^{-1}(a)$ is totally umbilic; in fact, if h is the second fundamental form of $f^{-1}(a)$ with ν as its unit normal vector field, then $h=-(f/|\operatorname{grad} f|)g$.*

PROOF. To prove (a) it suffices to show that $\nabla_{\operatorname{grad} f}\operatorname{grad} f$ is linearly dependent on $\operatorname{grad} f$. By the definition of the hessian, we have $(\operatorname{Hess} f)(X, \operatorname{grad} f)=\langle\nabla_{\operatorname{grad} f}\operatorname{grad} f, X\rangle$ for any tangent vector X, where $g=\langle\cdot\,,\cdot\rangle$. This implies, with (2), that $\nabla_{\operatorname{grad} f}\operatorname{grad} f=f\operatorname{grad} f$. Thus we have (a).

Next we prove (b). For any vector fields X, Y tangent to $f^{-1}(a)$, we have, by Weingarten's formula for hypersurfaces, that

$$-h(X, Y)=\langle\nabla_X\nu, Y\rangle=-\langle\nu, \nabla_X Y\rangle=-\frac{1}{|\operatorname{grad} f|}(\nabla_X Y)f .$$

On the other hand, (2) implies that $(\nabla_X Y)f+f\langle X, Y\rangle=0$. Thus we conclude (b).

PROOF OF THEOREM C. It suffices to prove Theorem C with $k=-1$. Also it is obvious that (H^n, g_0), the simply connected n-dimensional complete riemannian manifold of constant curvature -1, has a nontrivial function f with a critical point which satisfies $\operatorname{Hess} f-fg_0=0$. In fact, if we choose any point p in H^n, the function f on H^n defined by $f(q)=\cosh r$, where $r=\operatorname{dist}(p, q)$, is a nontrivial solution of the above equation with a critical point p. Now we shall show that the converse is also true. Let (M, g) be a complete riemannian manifold of dimension $n(\geqq 2)$, and f a nontrivial function on (M, g) with a critical point which satisfies

$$\operatorname{Hess} f-fg=0 . \tag{3}$$

Note that if $\gamma: \boldsymbol{R}\to M$ is a geodesic with unit speed, then the equation (3) is written down, on γ, as

$$\frac{d^2}{dt^2}(f\circ\gamma)-f\circ\gamma=0 . \tag{4}$$

This is a second order ordinary differential equation, so $f\circ\gamma$ is determined uniquely by the values $f\circ\gamma(0)$ and $df\circ\gamma(0)$. Without loss of generality, we may assume $f(p)=1$ and $df(p)=0$. Also we have easily that

$$f(q)=\cosh|X| \quad \text{for} \quad q=\exp_p X \tag{5}$$

by (4). On the other hand, by joining p and q by a minimizing geodesic, we have

$$f(q)=\cosh r , \quad \text{where} \quad r=\operatorname{dist}(p, q) . \tag{6}$$

By (5) and (6), any geodesic through p is minimizing, and therefore,

$$\exp_p\colon T_pM\longrightarrow M \quad \text{is bijective.} \tag{7}$$

Let $\gamma: [0, \infty)\to M$ be an arbitrary geodesic such that $\gamma(0)=p$, $|\dot\gamma|\equiv 1$, and J a Jacobi field along γ such that $J(0)=0$, $|\dot J(0)|=1$ and that $\langle J, \dot\gamma\rangle\equiv 0$. We shall show that

$$|J(r)|=\sinh r , \qquad \text{for all} \quad r\in[0, \infty) . \tag{8}$$

Fix $r\in(0, \infty)$. Then $\bar M=f^{-1}(\cosh r)$ is a hypersurface in M, and ν, the restriction of $(1/|\text{grad}\, f|)\text{grad}\, f$ to $\bar{M}$, is a unit normal vector field of $\bar{M}$. Note that $J(r)$ is tangent to $\bar{M}$. Let h be the second fundamental form of $\bar{M}$. Then, by Lemma 5 and Weingarten's formula, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=r}\langle J, J\rangle &= \langle J, \nabla_\nu J\rangle|_{t=r} = \langle J, \nabla_J \nu\rangle|_{t=r}\\ &= -h(J(r), J(r)) = \frac{f}{|\text{grad}\, f|}\langle J, J\rangle|_{t=r} = \frac{\cosh r}{\sinh r}\langle J, J\rangle|_{t=r}\,.\end{aligned}$$

Thus (8) holds. By (8), we conclude that γ has no conjugate points of $p=\gamma(0)$. Combining this fact with (7), we have

(9) $\qquad \exp_p\colon T_pM \longrightarrow M$ is a diffeomorphism.

(8) and (9) show that $M\backslash\{p\}$ is isometric to the warped product $(S^{n-1}, g_0)_\xi \times (I, g_0)$ with $I=(0, \infty)$, $\xi(t)=\sinh t$. In fact, identifying (S^{n-1}, g_0) with $\{X\in T_pM\colon |X|=1\}$, we can construct an isometry of $(S^{n-1}, g_0)_\xi\times(I, g_0)$ onto (M, g) by $(X, t)\mapsto \exp_p tX$, $X\in T_pM$, $|X|=1$; $t\in I$. But, by Lemma 3, $(S^{n-1}, g_0)_\xi\times(I, g_0)$ is of constant curvature -1, and therefore, $M\backslash\{p\}$ is of constant curvature -1. Because of the continuity of the curvature, M is of constant curvature -1. Thus, with (9), we conclude that (M, g) is isometric to (H^n, g_0). This completes the proof of Theorem C.

§ 3. Proof of Theorem D.

In this section, we give a proof of Theorem D. Without loss of generality, we may assume $k=-1$.

PROOF OF THEOREM D. We have already seen in Corollary 2 that a half of Theorem D holds. So it is sufficient to show that the converse is also true.

Suppose that (M, g) is a complete riemannian manifold of dimension $n(\geqq 2)$, and that f is a function on M without critical points which satisfies (3). We shall show that (M, g) is isometric to the warped product $(\bar{M}, \bar{g})_\xi\times(\boldsymbol{R}, g_0)$ of some complete riemannian manifold $(\bar{M}, \bar{g})$ and the real line $(\boldsymbol{R}, g_0)$ warped by a function $\xi\colon \boldsymbol{R}\to\boldsymbol{R}$ with $\xi>0$, $\ddot{\xi}-\xi=0$. Put $\nu=(1/|\text{grad}\, f|)\text{grad}\, f$. Fix $a\in\boldsymbol{R}$ so that $\bar{M}=f^{-1}(a)\neq\emptyset$, and let $\bar{g}$ be the induced metric of $\bar{M}$. Also let φ be the flow of ν and define the map $\Psi\colon \bar{M}\times\boldsymbol{R}\to M$ by $\Psi(\bar{p}, t)=\varphi_t(\bar{p})$ for $\bar{p}\in\bar{M}$ and $t\in\boldsymbol{R}$. It is obvious that Ψ is a diffeomorphism. So $\bar{M}$ is connected. Moreover $X\cdot\langle\text{grad}\, f, \text{grad}\, f\rangle = 2(\text{Hess}\, f)(X, \text{grad}\, f)=0$ for any tangent vector X of $\bar{M}$, so $|\text{grad}\, f|$ is constant on $\bar{M}$. Since the value of f at $\Psi(\bar{p}, t)$ is determined by the values of f and df at $\bar{p}$, f is given by

$$f\circ\Psi(\bar{p}, t)=Ae^{t}-Be^{-t}\,, \qquad \bar{p}\in\bar{M}\,,\quad t\in\boldsymbol{R}\,, \tag{10}$$

where A and B are constants such that $A, B\geqq 0$, $A^2+B^2\neq 0$. Put $\xi(t)=(Ae^t+Be^{-t})/(A+B)$. Now we prove that Ψ is an isometry from $(\bar{M}, \bar{g})_\xi\times(\boldsymbol{R}, g_0)$ onto (M, g). To see this, it suffices to show that for each fixed $r\in\boldsymbol{R}$, $\psi=\Psi|_{\bar{M}\times\{r\}}$ maps $(\bar{M}, \bar{g})$ into (M, g) in such a way that

$$\{\xi(r)\}^2\bar{g}=\psi^*g\,. \tag{11}$$

Fix $\bar{p}\in\bar{M}$ and put $\gamma(t)=\Psi(\bar{p}, t)$, $\bar{p}\in\bar{M}$, $t\in\boldsymbol{R}$. Recall that γ is a geodesic in M. Let J be a Jacobi field along γ such that $J(0)$ is tangent to $\bar{M}$. Then, by Weingarten's formula, Lemma 5 and (10), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=r}\langle J, J\rangle &=\langle J, \nabla_J\nu\rangle|_{t=r}=-h(J(r), J(r))\\ &=\frac{f}{|\operatorname{grad} f|}\langle J, J\rangle|_{t=r}=\frac{Ae^r-Be^{-r}}{Ae^r+Be^{-r}}\langle J, J\rangle|_{t=r}\,,\end{aligned}$$

where h is the second fundamental form of the hypersurface $f^{-1}(f\circ\gamma(r))$. Thus $|J(r)|^2=\{\xi(r)\}^2|J(0)|^2$, and, therefore, $|\psi_*J(0)|^2=|J(r)|^2=\{\xi(r)\}^2|J(0)|^2$. Thus (11) holds. This completes the proof of Theorem D.

§ 4. Conformal vector fields.

In [7], Yano and Nagano established the fact that if a complete Einstein manifold of dimension $n(\geqq 3)$ admits a complete nonhomothetic conformal vector field then the manifold is isometric to a sphere of constant curvature. Recall that a vector field V on a riemannian manifold (M, g) is said to be *conformal* if it satisfies

$$L_Vg=2fg \tag{12}$$

with some function f on M, where L_Vg denotes the Lie derivative of g with respect to V. Especially if (12) holds for a constant function f (resp. $f\equiv 0$), then V is said to be *homothetic* (resp. *isometric*). Also a vector field on a manifold M is said to be *complete* if its flow $\varphi_t(p)$, $p\in M$, is defined for all $t\in\boldsymbol{R}$ and $p\in M$. In this section, we investigate conformal vector fields which is not necessarily complete. We begin with some examples.

EXAMPLES. (1) Let f be a nontrivial function on a riemannian manifold (M, g) satisfying (1) with $k\neq 0$. Since $L_{\operatorname{grad} f}\,g=2\operatorname{Hess} f$, the gradient vector field $\operatorname{grad} f$ is nonhomothetic and conformal. But for $k<0$, $\operatorname{grad} f$ is not complete even if (M, g) is complete (cf. [7]).

(2) A vector field V on the $n(\geqq 2)$ dimensional euclidean space $(\boldsymbol{R}^n, g_0)$ defined by

$$V=x^n\left(x^1\frac{\partial}{\partial x^1}+\cdots+x^{n-1}\frac{\partial}{\partial x^{n-1}}\right)+\frac{1}{2}(x^n)^2\frac{\partial}{\partial x^n}$$

is nonhomothetic and conformal, where $(x^1, \cdots, x^n)$ is the canonical coordinates. But V is not complete.

THEOREM G. *An $n(\geqq 3)$ dimensional complete Einstein manifold (M, g) with scalar curvature $n(n-1)k$ admits a (not necessarily complete) nonhomothetic comformal vector field if and only if one of the following conditions holds*:

(i) *$k>0$ and (M, g) is isometric to the euclidean n-sphere $(S^n, (1/k)g_0)$ of radius $1/\sqrt{k}$.*

(ii) *$k=0$ and (M, g) is isometric to the n-dimensional euclidean space $(\boldsymbol{R}^n, g_0)$.*

(iii) *$k<0$ and (M, g) is isometric to the warped product $(\bar{M}, \bar{g})_\xi \times (\boldsymbol{R}, g_0)$ of a complete Einstein manifold $(\bar{M}, \bar{g})$ of scalar curvature $4(n-1)(n-2)kC_1C_2$ and the real line $(\boldsymbol{R}, g_0)$, warped by $\xi(t)=C_1e^{\sqrt{-k}t}+C_2e^{-\sqrt{-k}t}$, where C_1 and C_2 are nonnegative constants.*

PROOF. It is an immediate consequence of Theorem A, Corollary E and the above examples that (M, g) admits a nonhomothetic conformal vector field if (i), (ii) or (iii) holds. So we shall prove that the converse is also true. Let V be a nonhomothetic conformal vector field on (M, g) with $L_V g=2fg$. Since (M, g) is an Einstein manifold with $Ric=(n-1)kg$, we have $\operatorname{Hess} f+kfg=0$ (see [6], pp. 160–161). We continue the proof dividing into three cases.

(a) $k>0$: In this case, (M, g) is isometric to $(S^n, (1/k)g_0)$, by Theorem A.

(b) $k<0$: If f has a critical point, then Theorem C implies that (M, g) is isometric to $(H^n, -(1/k)g_0)=(\boldsymbol{R}^{n-1}, g_0)_\xi \times (\boldsymbol{R}, g_0)$, $\xi(t)=e^{\pm\sqrt{-k}t}$, and therefore, the condition (iii) holds. Now, suppose that f has no critical points. Since (M, g) has a nontrivial function f satisfying $\operatorname{Hess} f+kfg=0$, (M, g) is isometric to a warped product $(\bar{M}, \bar{g})_\xi \times (\boldsymbol{R}, g_0)$ with $\ddot{\xi}+k\xi=0$, by Theorem D. Let Ric and $\overline{Ric}$ be the Ricci tensors of (M, g) and $(\bar{M}, \bar{g})$, respectively. (M, g) is an Einstein manifold with $Ric=(n-1)kg$ and, hence, we have $\overline{Ric}=(n-2)(k\xi^2+\dot{\xi}^2)\bar{g}$, by Lemma 4.

(c) $k=0$: Since $\operatorname{Hess} f=0$ holds, (M, g) is isometric to a riemannian product $(\bar{M}, \bar{g})\times(\boldsymbol{R}, g_0)$ and f is defined by $f(\bar{p}, t)=At+B$, $\bar{p}\in\bar{M}$, $t\in\boldsymbol{R}$, by Theorem B. Decompose V as $V=\bar{V}+W$, where $V|_{\bar{M}\times\{t\}}$ is tangent to $\bar{M}\times\{t\}$ and $W|_{\{\bar{p}\}\times\boldsymbol{R}}$ is tangent to $\{\bar{p}\}\times\boldsymbol{R}$ for each fixed $\bar{p}\in\bar{M}$, $t\in\boldsymbol{R}$. Then for each fixed t and any vector fields X, Y tangent to $\bar{M}\times\{t\}$, $(L_{\bar{V}}\bar{g})(X, Y)=2(At+B)\bar{g}(X, Y)$. Hence $\bar{M}\times\{t\}$ admits a nonisometric homothetic vector field $\bar{V}|_{\bar{M}\times\{t\}}$, for each fixed t such that $At+B\neq 0$. It is known that a complete riemannian manifold which admits a nonisometric homothetic vector field is isometric to the euclidean space ([2]). So (M, g) is isometric to $(\boldsymbol{R}^{n-1}, g_0)\times(\boldsymbol{R}, g_0)=(\boldsymbol{R}^n, g_0)$. This completes the proof of Theorem G.

ADDED IN PROOF. Just before this article comes to be published, I was announced by Prof. Y. Tashiro that our results (especially Theorem C and Theorem D mentioned in the introduction) had been already established in his paper [8; Theorem 2] where he investigated equations more general than those considered here. But his treatments seem to be different from ours in some points. I express my gratitude to Prof. Tashiro for his kind suggestions.

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Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU,
YOKOHAMA 223
JAPAN