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Inner Extensions of Automorphisms of Irrational Rotation Algebras to AF-Algebras

Kazunori KODAKA

Keio University

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Abstract. Let A_θ be an irrational rotation algebra. In the present paper we will show that automorphisms of A_θ with some properties can be extended to inner automorphisms of an AF-algebra. In other words, there are a monomorphism ρ of A_θ into an AF-algebra B and a unitary element $w \in B$ such that $\rho(\alpha(x)) = w\rho(x)w^*$ for any $x \in A_\theta$.

§1. Introduction.

Let θ be an irrational number in [0, 1] and let σ be the rotation by the angle $2\pi\theta$ on the circle $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$. Let $C(\boldsymbol{T})$ be the abelian C^*-algebra of all complex valued continuous functions on $\boldsymbol{T}$. Then we can regard σ as an automorphism of $C(\boldsymbol{T})$. Hence we can consider the crossed product $C(\boldsymbol{T}) \times_\sigma \boldsymbol{Z}$ of $C(\boldsymbol{T})$ by σ and we donote it by A_θ, which is called the *irrational rotation algebra by* θ. It is well known that A_θ has two generators u and v with $vu = e^{2\pi i\theta}uv$. Let $\mathrm{Aut}(A_\theta)$ be the group of all automorphisms of A_θ and $C^*(v)$ be the abelian C^*-subalgebra of A_θ generated by v. Furthermore throughout this paper we mean a unital $*$-monomorphism by a monomorphism.

DEFINITION. *Let* $\alpha \in \mathrm{Aut}(A_\theta)$. We say that α *can be extended to an inner automorphism of an AF-algebra* if there are a monomorphism ρ of A_θ into an AF-algebra B and a unitary element $w \in B$ such that $\rho(\alpha(x)) = w\rho(x)w^*$ for any $x \in A_\theta$.

Now generally let A be a unital C^*-algebra and for each $n \in \boldsymbol{N}$ let M_n be the $n \times n$ matrix algebra. We identify $A \otimes M_n$ with the $n \times n$ matrix algebra $M_n(A)$ over A. Let α be an automorphism of A. For $i = 0, 1$ we denote the K_i-group of A by $K_i(A)$ and for any projection $p \in A \otimes M_n$ (resp. any unitary element $x \in A \otimes M_n$) $[p]$ (resp. $[x]$) denote the corresponding class in $K_0(A)$ (resp. $K_1(A)$). Let ∂ be the connecting map

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of $K_1(A\times_\alpha \boldsymbol{Z})$ into $K_0(A)$.

LEMMA 1. *With the above notations if* $p\in A\otimes M_n$ *satisfies* $\alpha(p)=xpx^*$ *for some unitary element* $x\in A\otimes M_n$, *then an element* $w=(1-p)+px^*yp$ $\in(A\times_\alpha \boldsymbol{Z})\otimes M_n$ *is a unitary element with* $\partial([w])=[p]$ *where* y *is a unitary element in* $A\times_\alpha \boldsymbol{Z}$ *satisfying that* $\alpha=\mathrm{Ad}(y)$ *and* A *and* y *generate* $A\times_\alpha \boldsymbol{Z}$.

PROOF. We will use the notations in Pimsner and Voiculescu [6]. Let K be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space and T be the Toeplitz algebra for (A, α). Let J be a closed two sided ideal generated by a projection $Q=1\otimes I-(y\otimes S)(y\otimes S)^*=1\otimes P$. Then we obtain the connecting map d of $K_1(T/J)$ into $K_0(J)$. By Pimsner and Voiculescu [6], J is isomorphic to $A\otimes K$ and T/J is isomorphic to $A\times_\alpha \boldsymbol{Z}$. We denote the isomorphism of $A\otimes K$ onto J by ψ and the isomorphism of $A\times_\alpha \boldsymbol{Z}$ onto T/J by ϕ. Then it is sufficient to show that $d([\phi(w)])=[\psi(p)]$. By the definitions of ϕ and ψ, we have

$$\phi(w)=(1-p)\otimes I+px^*yp\otimes S^*$$

and

$$\psi(p)=p\otimes P\,.$$

Let $z=\begin{bmatrix}(1-p)\otimes I+px^*yp\otimes S^* & 0\\ p\otimes P & (1-p)\otimes I+py^*xp\otimes S\end{bmatrix}$ in $T\otimes M_{2n}$. Then $\pi(z)=\phi(w)\oplus\phi(w)^*$ where π is the quotient map of T onto T/J. Hence

$$d([\phi(w)])=\left[z\begin{bmatrix}1\otimes I & 0\\ 0 & 0\end{bmatrix}z^*\right]-\left[\begin{bmatrix}1\otimes I & 0\\ 0 & 0\end{bmatrix}\right].$$

Since $z\begin{bmatrix}1\otimes I & 0\\ 0 & 0\end{bmatrix}z^*=\begin{bmatrix}1\otimes I & 0\\ 0 & p\otimes P\end{bmatrix}$, we obtain that $d([\phi(w)])=[p\otimes P]$. Q.E.D.

§2. The case of $\alpha(u)=fu$ and $\alpha(v)=v$.

In this section we will show that if $\alpha\in\mathrm{Aut}(A_\theta)$ with $\alpha(u)=fu$ and $\alpha(v)=v$ where f is a unitary element in $C^*(v)$, there are an AF-algebra B, a monomorphism ρ and a unitary element $w\in B$ such that $\rho(\alpha(x))=w\rho(x)w^*$ for any $x\in A_\theta$. Now we consider the crossed product $A_\theta\times_\alpha \boldsymbol{Z}$ of A_θ by α. Then there is a unitary element $z\in A_\theta\times_\alpha \boldsymbol{Z}$ such that $\alpha(x)=zxz^*$ for any $x\in A_\theta$ and A_θ and z generate $A_\theta\times_\alpha \boldsymbol{Z}$. Hence we have the following relations;

$$zuz^*=fu\,,$$

$$zvz^* = v\ ,$$
$$vu = e^{2\pi i\theta} uv\ .$$

Let $C^*(v, z)$ be the C^*-subalgebra of $A_\theta \times_\alpha \boldsymbol{Z}$ generated by v and z and let β be the automorphism of $C^*(v, z)$ defined by $\beta(v) = uvu^* = e^{-2\pi i\theta}$ and $\beta(z) = uzu^* = f^*z$.

LEMMA 2. *With the above assumptions* $\mathrm{Sp}(z) = \boldsymbol{T}$.

PROOF. Suppose that $\mathrm{Sp}(z) \subsetneqq \boldsymbol{T}$. Then we can find a selfadjoint element $a \in A_\theta \times_\alpha \boldsymbol{Z}$ such that $z = e^{ia}$. Hence $[z] = 0$ in $K_1(A_\theta \times_\alpha \boldsymbol{Z})$. On the other hand by the Pimsner-Voiculescu six terms exact sequence we have the following sequence:

$$0 \longrightarrow \mathrm{Im}(\mathrm{id} - \alpha_*) \longrightarrow K_1(A_\theta \times_\alpha) \boldsymbol{Z} \xrightarrow{\partial} K_0(A_\theta) \longrightarrow 0\ .$$

Then by Lemma 1, $\partial([z]) = [1]$. Thus $[z] \neq 0$ in $K_1(A_\theta \times_\alpha \boldsymbol{Z})$. This is a contradiction. Q.E.D.

By Lemma 2, $C^*(v, z)$ is isomorphic to $C(\boldsymbol{T}^2)$ and we identify $C^*(v, z)$ with $C(\boldsymbol{T}^2)$ and regard β as a homeomorphism of $\boldsymbol{T}^2$. Then clearly $A_\theta \times_\alpha \boldsymbol{Z}$ is isomorphic to $C(\boldsymbol{T}^2) \times_\beta \boldsymbol{Z}$. Let τ be the unique faithful tracial state of A_θ and $\tilde{\tau}$ be a faithful tracial state of $A_\theta \times_\alpha \boldsymbol{Z}$ defined by $\tilde{\tau}(g) = \tau(g(0))$ for each $g \in l^1(\boldsymbol{Z}, A_\theta)$. Thus $C(\boldsymbol{T}^2) \times_\beta \boldsymbol{Z}$ has a faithful tracial state. Recall that a separable unital C^*-algebra A is *quasidiagonal* if there is a monomorphism π of A into $B(H)$ such that $\pi(A) \cap K(H) = 0$ where $K(H)$ denotes the C^*-algebra of all compact operators on a Hilbert space H and a sequence $\{p_n\}_{n \in \boldsymbol{N}}$ of finite dimensional orthogonal projections in $B(H)$ such that

$$\cdots \leqq p_n \leqq p_{n+1} \leqq \cdots\ , \qquad \Big(\bigcup_{n=1}^{\infty} p_n(H)\Big)^- = H$$

and for every $a \in A$

$$\|p_n \pi(a) - \pi(a) p_n\| \to 0\ .$$

Moreover A is *finite* if no proper projection is algebraically equivalent to 1 and A is *stably finite* if $M_n(A)$ is finite for any $n \in \boldsymbol{N}$. By the above definition we can easily see that $C(\boldsymbol{T}^2) \times_\beta \boldsymbol{Z}$ is finite since it has a faithful tracial state.

LEMMA 3. *Let T be a homeomorphism of a compact metrizable space X and α_T be an automorphism of $C(X)$ induced by T. Then the following conditions for $C(X) \times_{\alpha_T} \boldsymbol{Z}$ are equivalent;*

(1) *quasidiagonal,*
(2) *finite,*
(3) *stably finite.*

PROOF. (1) implies (3); By Pimsner [5, Theorem 9] there exists an embedding of $C(X)\times_{\alpha_T}\boldsymbol{Z}$ into an AF-algebra. Hence $C(X)\times_{\alpha_T}\boldsymbol{Z}$ is stably finite since we can regard it as a C*-subalgebra of the AF-algebra.

(3) implies (2); This is trivial.

(2) implies (1); Suppose that $C(X)\times_{\alpha_T}\boldsymbol{Z}$ is not quasidiagonal. Then it follows from Pimsner [5, Proposition 8 and Theorem 9] that we can find a non unitary isometry in $C(X)_{\alpha_T}\boldsymbol{Z}$. However this contradicts (2). Q.E.D.

PROPOSITION 4. *If $\alpha\in\mathrm{Aut}(A_\theta)$ with $\alpha(u)=fu$ and $\alpha(v)=v$ where f is a unitary element in $C^*(v)$, there are an AF-algebra $B(\alpha)$, and a monomorphism ρ_α of $A_\theta\times_\alpha\boldsymbol{Z}$ into $B(\alpha)$.*

PROOF. By Lemma 3, $C(\boldsymbol{T}^2)\times_\beta\boldsymbol{Z}$ is quasidiagonal and $A_\theta\times_\alpha\boldsymbol{Z}$ is isomorphic to $C(\boldsymbol{T}^2)\times_\beta\boldsymbol{Z}$. Hence by Pimsner [5, Theorem 9] we can find an AF-algebra $B(\alpha)$ and a monomorphism ρ_α of $A_\theta\times_\alpha\boldsymbol{Z}$ into $B(\alpha)$. Q.E.D.

§3. The case of $\alpha(u)=fu$ and $\alpha(v)=e^{2\pi it}v$.

For each $t\in\boldsymbol{R}$ let $\beta_t^{(1)}\in\mathrm{Aut}(A_\theta)$ be defined by $\beta_t^{(1)}(u)=e^{2\pi it}u$ and $\beta_t^{(1)}(v)=v$ and let $\beta_t^{(2)}\in\mathrm{Aut}(A_\theta)$ be defined by $\beta_t^{(2)}(u)=u$ and $\beta_t^{(2)}(v)=e^{2\pi it}v$. And we define $\beta_{(s,t)}=\beta_s^{(1)}\circ\beta_t^{(2)}$. Let $SL(2,\boldsymbol{Z})$ be the group of all 2×2 matrices over $\boldsymbol{Z}$ with determinant 1 and let $G=\left\{g\in SL(2,\boldsymbol{Z});\ g=\begin{bmatrix}1&0\\n&1\end{bmatrix}\right\}$. For each $g\in SL(2,\boldsymbol{Z})$ let $\beta_g\in\mathrm{Aut}(A_\theta)$ be defined by $\beta_g(u)=u^av^c$ and $\beta_g(v)=u^bv^d$ where $g=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $a,b,c,d\in\boldsymbol{Z}$.

In this section we will show that if $\alpha=\beta_g\circ\beta_{(s,t)}$ with $g\in G$ and $s,t\in\boldsymbol{R}$, there are an AF-algebra B, a monomorphism ρ of A_θ into B and a unitary element $w\in B$ such that $\rho(\alpha(x))=w\rho(x)w^*$ for any $x\in A_\theta$. For each $n\in\boldsymbol{N}$ let $U_n\in M_n$ be defined by

$$U_n=\begin{bmatrix}0&0&\cdots&0&(-1)^{n-1}\\1&0&\cdots&0&0\\0&1&\cdots&0&\cdot\\\cdot&\cdot&\cdots&\cdot&\cdot\\\cdot&\cdot&\cdots&\cdot&\cdot\\\cdot&\cdot&\cdots&0&0\\0&0&\cdots&1&0\end{bmatrix}$$

and let I_n be the unit element of M_n.

LEMMA 5. *Let $\alpha \in \mathrm{Aut}(A_\theta)$. If there exist an $n \in \boldsymbol{N}$, a monomorphism ρ_{α^n} of A_θ into an AF-algebra $B(\alpha^n)$ and a unitary element w_{α^n} such that $\rho_{\alpha^n}(\alpha^n(x)) = w_{\alpha^n}\rho_{\alpha^n}(x)w_{\alpha^n}^*$ for any $x \in A_\theta$, there are a monomorphism ρ_α of A_θ into an AF-algebra $B(\alpha)$ and a unitary element w_α such that $\rho_\alpha(\alpha(x)) = w_\alpha\rho_\alpha(x)w_\alpha^*$ for any $x \in A_\theta$.*

PROOF. Let $B(\alpha) = B(\alpha^n) \otimes M_n$ and ρ_α be a monomorphism of A_θ into $B(\alpha)$ defined by $\rho_\alpha(x) = \bigoplus_{j=0}^{n-1} \rho_{\alpha^n}(\alpha^j(x))$ for each $x \in A_\theta$. Then for any $x \in A_\theta$

$$\begin{aligned}
\rho_\alpha(\alpha(x)) &= \bigoplus_{j=0}^{n-1} \rho_{\alpha^n}(\alpha^{j+1}(x)) \\
&= (I_{n-1} \oplus w_{\alpha^n}) \begin{bmatrix} \rho_{\alpha^n}(\alpha(x)) & 0 & \cdots & 0 & 0 \\ 0 & \rho_{\alpha^n}(\alpha^2(x)) & \cdots & \cdot & \cdot \\ \cdot & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & 0 & \cdot \\ \cdot & \cdot & \cdots & \rho_{\alpha^n}(\alpha^{n-1}(x)) & 0 \\ 0 & 0 & \cdots & 0 & \rho_{\alpha^n}(x) \end{bmatrix} (I_{n-1} \oplus w_{\alpha^n})^* \\
&= \mathrm{Ad}((I_{n-1} \oplus w_{\alpha^n}) U_n^*)\Big(\bigoplus_{j=0}^{n-1} \rho_{\alpha^n}(\alpha^j(x))\Big)
\end{aligned}$$

since $\rho_{\alpha^n}(\alpha^n(x)) = w_{\alpha^n}\rho_{\alpha^n}(x)w_{\alpha^n}^*$. Q.E.D.

COROLLARY 6. *Let $\alpha \in \mathrm{Aut}(A_\theta)$ with $\alpha(u) = fu$ and $\alpha(v) = e^{2\pi i t}v$ where f is a unitary element in $C^*(v)$ and $t \in \boldsymbol{R}$. If $t \in \boldsymbol{Q}$, there are an AF-algebra $B(\alpha)$, a monomorphism ρ_α of A_θ into $B(\alpha)$ and a unitary element $w_\alpha \in B(\alpha)$ such that $\rho_\alpha(\alpha(x)) = w_\alpha\rho_\alpha(x)w_\alpha^*$ for any $x \in A_\theta$.*

PROOF. Since $t \in \boldsymbol{Q}$, there is an $n \in \boldsymbol{N}$ such that $\alpha^n(u) = gu$ and $\alpha^n(v) = v$ where g is a unitary element in $C^*(v)$. By Proposition 4, α satisfies the assumptions of Lemma 5. Therefore we obtain the conclusion. Q.E.D.

For any automorphism α of a C^*-algebra we denote the Connes spectrum by $\Gamma(\alpha)$.

COROLLARY 7. *Let $\alpha \in \mathrm{Aut}(A_\theta)$ with $\Gamma(\alpha) \subsetneqq \boldsymbol{T}$. Then there are an AF-algebra $B(\alpha)$, a monomorphism ρ_α of A_θ into $B(\alpha)$ and a unitary element $w_\alpha \in B(\alpha)$ such that $\rho_\alpha(\alpha(x)) = w_\alpha\rho_\alpha(x)w_\alpha^*$ for any $x \in A_\theta$.*

PROOF. By Pimsner and Voiculescu [7] we have a monomorphism ρ of A_θ into an AF-algebra B_θ. And since $\Gamma(\alpha) \subsetneqq \boldsymbol{T}$, there are an $n \in \boldsymbol{N}$ and a unitary element $z \in A_\theta$ such that $\alpha^n = \mathrm{Ad}(z)$ and $\alpha(z) = z$. Hence $\rho(\alpha^n(x)) = \rho(z)\rho(x)\rho(z)^*$ for any $x \in A_\theta$. Thus α satisfies the assumptions of Lemma 5. Therefore we obtain the conclusion. Q.E.D.

Let $\tilde{u}$ and $\tilde{v} \in C(\boldsymbol{T}^2)$ be defined by $\tilde{u}(\xi, \zeta)=\xi$ and $\tilde{v}(\xi, \zeta)=\zeta$ for any ξ, $\zeta \in \boldsymbol{T}$. Then $\tilde{u}$ and $\tilde{v}$ are generators of $C(\boldsymbol{T}^2)$. For any $g \in SL(2, \boldsymbol{Z})$ let $\tilde{\beta}_g \in \mathrm{Aut}(C(\boldsymbol{T}^2))$ be defined by $\tilde{\beta}_g(\tilde{u})=\tilde{u}^a\tilde{v}^c$ and $\tilde{\beta}_g(\tilde{v})=\tilde{u}^b\tilde{v}^d$ where $g=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ and $a, b, c, d \in \boldsymbol{Z}$. We note that $\tilde{\beta}_g$ is induced by a toral automorphism of $\boldsymbol{T}^2$. For any s, $t \in \boldsymbol{R}$ let $\tilde{\beta}_{(s,t)} \in \mathrm{Aut}(C(\boldsymbol{T}^2))$ be defined by $\tilde{\beta}_{(s,t)}(\tilde{u})=e^{2\pi i s}\tilde{u}$ and $\tilde{\beta}_{(s,t)}(\tilde{v})=e^{2\pi i t}\tilde{v}$. Then we have the following lemma;

LEMMA 8. *With the above notations the crossed product $C(\boldsymbol{T}^2)\times_{\tilde{\alpha}}\boldsymbol{Z}$ is quasidiagonal where $\tilde{\alpha}=\tilde{\beta}_g\circ\tilde{\beta}_{(s,t)}$.*

PROOF. Let μ be the Haar measure of $\boldsymbol{T}^2$ with $\mu(\boldsymbol{T}^2)=1$ and let tr be a faithful finite trace of $C(\boldsymbol{T}^2)$ defined by $\mathrm{tr}(x)=\int_{\boldsymbol{T}^2} x\, d\mu$ for any $x \in C(\boldsymbol{T}^2)$. Since μ is two sided invariant and $\tilde{\beta}_g$ is induced by a toral automorphism of $\boldsymbol{T}^2$ leaving μ fixed, $\mathrm{tr}(\tilde{\alpha}(x))=\mathrm{tr}(x)$ for any $x \in C(\boldsymbol{T}^2)$. Hence if $\widetilde{\mathrm{tr}}$ is defined by $\widetilde{\mathrm{tr}}(y)=\mathrm{tr}(y(0))$ for $y \in l^1(\boldsymbol{Z}, C(\boldsymbol{T}^2))$, $\widetilde{\mathrm{tr}}$ is a faithful finite trace of $C(\boldsymbol{T}^2)\times_{\tilde{\alpha}}\boldsymbol{Z}$. Thus $C(\boldsymbol{T}^2)\times_{\tilde{\alpha}}\boldsymbol{Z}$ is quasidiagonal by Lemma 3. Q.E.D.

PROPOSITION 9. *With the above notations let $\alpha=\beta_g\circ\beta_{(s,t)} \in \mathrm{Aut}(A_\theta)$ where s, $t \in \boldsymbol{R}$ and $g \in G$. Then there are an AF-algebra $B(\alpha)$, a monomorphism ρ_α of A_θ into $B(\alpha)$ and a unitary element $w_\alpha \in B(\alpha)$ such that $\rho_\alpha(\alpha(x))=w_\alpha\rho_\alpha(x)w_\alpha^*$ for any $x \in A_\theta$.*

PROOF. By Corollaries 6 and 7 we can assume that $t \notin \boldsymbol{Q}$ and $\Gamma(\alpha)=\boldsymbol{T}$. Since $g \in G$, there is an $n \in \boldsymbol{Z}$ such that $g=\begin{bmatrix} 1 & 0 \\ n & 1\end{bmatrix}$. Let $\gamma \in \mathrm{Aut}(A_\theta)$ be defined by $\gamma(u)=e^{2\pi i s}uv^n=e^{2\pi i(s-n\theta)}v^n u$ and $\gamma(v)=v$. Then there are an AF-algebra $B(\gamma)$ and a monomorphism ρ_γ of $A_\theta\times_\gamma\boldsymbol{Z}$ into $B(\gamma)$ by Proposition 4. Let u, v and w be generators of $A_\theta\times_\gamma\boldsymbol{Z}$ with $vu=e^{2\pi i\theta}uv$ and $\gamma=\mathrm{Ad}(w)$. Let $\tilde{\gamma} \in \mathrm{Aut}(C(\boldsymbol{T}^2))$ be defined by $\tilde{\gamma}=\tilde{\beta}_g\circ\tilde{\beta}_{(0,t)}$, i.e., $\tilde{\gamma}(\tilde{u})=\tilde{u}\tilde{v}^n$ and $\tilde{\gamma}(\tilde{v})=e^{2\pi i t}\tilde{v}$. Then by Lemma 8 and Pimsner [5, Theorem 9] there are an AF-algebra $B(\tilde{\gamma})$ and a monomorphism $\rho_{\tilde{\gamma}}$ of $C(\boldsymbol{T}^2)\times_{\tilde{\gamma}}\boldsymbol{Z}$ into $B(\tilde{\gamma})$. Let $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ be generators of $C(\boldsymbol{T}^2)\times_{\tilde{\gamma}}\boldsymbol{Z}$ with $\tilde{u}\tilde{v}=\tilde{v}\tilde{u}$ and $\tilde{\gamma}=\mathrm{Ad}(\tilde{w})$, and let u_α, v_α and w_α be generators of $A_\theta\times_\alpha\boldsymbol{Z}$ with $v_\alpha u_\alpha=e^{2\pi i\theta}u_\alpha v_\alpha$ and $\alpha=\mathrm{Ad}(w_\alpha)$. We define a homomorphism ρ_α of $A_\theta\times_\alpha\boldsymbol{Z}$ into $B(\gamma)\otimes B(\tilde{\gamma})$ as follows;

$$\rho_\alpha(u_\alpha)=\rho_\gamma(u)\otimes\rho_{\tilde{\gamma}}(\tilde{u}) \ ,$$
$$\rho_\alpha(v_\alpha)=\rho_\gamma(v)\otimes\rho_{\tilde{\gamma}}(\tilde{v}) \ ,$$
$$\rho_\alpha(w_\alpha)=\rho_\gamma(w)\otimes\rho_{\tilde{\gamma}}(\tilde{w}) \ .$$

Then we can easily see that

$$\rho_\alpha(v_\alpha)\rho_\alpha(u_\alpha)=e^{2\pi i\theta}\rho_\alpha(u_\alpha)\rho_\alpha(v_\alpha)\ ,$$
$$\rho_\alpha(w_\alpha)\rho_\alpha(u_\alpha)\rho_\alpha(w_\alpha)^*=e^{2\pi is}\rho_\alpha(u_\alpha)\rho_\alpha(v_\alpha)^n\ ,$$

and

$$\rho_\alpha(w_\alpha)\rho_\alpha(v_\alpha)\rho_\alpha(w_\alpha)^*=e^{2\pi it}\rho_\alpha(v_\alpha)\ .$$

Hence the above definition of ρ_α is well defined. Since $\Gamma(\alpha)=\boldsymbol{T}$ and A_θ is simple, $A_\theta\times_\alpha\boldsymbol{Z}$ is simple. Thus ρ_α is injective. Q.E.D.

§ 4. The main theorem.

PROPOSITION 10. *Let $\alpha\in\mathrm{Aut}(A_\theta)$ with $\alpha(u)=fu^*$ and $\alpha(v)=e^{2\pi it}v^*$ where f is a unitary element in $C^*(v)$ and $t\in\boldsymbol{R}$. Then there are an AF-algebra $B(\alpha)$, a monomorphism ρ_α and a unitary element $w_\alpha\in B(\alpha)$ such that $\rho_\alpha(\alpha(x))=w_\alpha\rho_\alpha(x)w_\alpha^*$ for any $x\in A_\theta$.*

PROOF. We have that $\alpha^2(u)\in C^*(v)u$ and $\alpha^2(v)=v$. Hence by Proposition 4 and Lemma 5 we obtain the conclusion. Q.E.D.

THEOREM 11. *Let $\alpha\in\mathrm{Aut}(A_\theta)$ be defined by $\alpha(u)=e^{2\pi is}uv^n$ and $\alpha(v)=e^{2\pi it}v$, or $\alpha(u)=e^{2\pi is}u^*v^n$ and $\alpha(v)=e^{2\pi it}v^*$, where $s,t\in\boldsymbol{R}$ and $n\in\boldsymbol{Z}$. Then for any unitary element z in A_θ, $\mathrm{Ad}(z)\circ\alpha$ can be extended to an inner automorphism of an AF-algebra.*

PROOF. By Propositions 9 and 10 this is clear. Q.E.D.

Before we state a corollary, we need some notations. Let A_θ^∞ be the dense $*$-subalgebra of all smooth elements of A_θ with respect to the canonical action of $\boldsymbol{T}^2$ and let A_θ^F be the $*$-subalgebra of finite linear combinations of monomials in u and v.

COROLLARY 12. *Let $\alpha\in\mathrm{Aut}(A_\theta)$ be leaving invariant a canonical subalgebra isomorphic to $C(\boldsymbol{T})$. If θ has the generic Diophantine property and $\alpha(A_\theta^\infty)=A_\theta^\infty$ or if $\alpha(A_\theta^F)=A_\theta^F$, α can be extended to an inner automorphism of an AF-algebra.*

PROOF. By Elliott [3] and Brenken [1] α satisfies the assumptions of Theorem 11. Hence we obtain the conclusion. Q.E.D.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN