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Properties of Topological Dynamical Systems and Corresponding C^*-Algebras

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Abstract. We show the equivalence of a certain property of topological dynamical systems $\Sigma=(X, G)$ and a particular structure of ideals in the corresponding crossed product $A(\Sigma)=C(X)\rtimes_{\alpha} G$, where X is a compact set and G is a discrete group. As an application, we give a complete characterization for $A(\Sigma)$ to be simple.

§1. Introduction.

Throughout this paper, X is a *compact* Hausdorff space, which is not assumed to satisfy the second axiom of countability, and G a *discrete* group acting on X as a group of homeomorphisms. We denote by $A(\Sigma)$ the transformation group C^*-algebra associated with the topological dynamical system $\Sigma=(X, G)$.

In the theory of transformation group C^*-algebras, we are interested in the relationship between the structure of ideals in the C^*-algebra $A(\Sigma)$ and the property of the dynamical system Σ. In this paper we prove the following: if each non-zero ideal in $A(\Sigma)$ has an intersection with the subalgebra $C(X)$ of $A(\Sigma)$, then $\operatorname{Int} X^t$ is empty for all $t\neq e$, and the converse implication holds if G is amenable, where $C(X)$ is the algebra of all continuous functions on X and X^t is the set of fixed points for the action t on X. In the theory of C^*-algebras, Olesen and Pedersen [6: Theorem 2.5, (i)$\leftrightarrow$(iv)] proved a result corresponding to the equivalence mentioned above for the C^*-dynamical systems consisting of general C^*-algebras and actions of locally compact abelian groups. We here note that this equivalence first appeared in O'Donovan [4; Theorem 1.2.1] (cf. [6; Remark 4.8]). In contrast with theirs, the present proof is rather elementary and self-contained

As an application of our equivalence result, we can give a complete

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characterization for $A(\Sigma)$ to be simple; that is, $A(\Sigma)$ is simple if and only if (1) the action of G on X is minimal, (2) $\mathrm{Int}\, X^t=\emptyset$ for all $t\neq e$, and (3) G is amenable. In the case of abelian groups, this characterization for simplicity is derived from the results by Olesen and Pedersen [5: Theorem 6.5], [6: Theorem 2.5]; besides, the sufficiency of these conditions for $A(\Sigma)$ to be simple is essentially derived from the results by Elliott [1: Theorem 3.2] and Kishimoto [3: Theorem 3.1].

§ 2. Transformation group C^*-algebras.

For each t in G, its action on X is denoted by $x\to t(x)$. The orbit of x in X and the isotropy group for x are denoted by $O(x)$ and G_x respectively. Let α_t be the canonical $*$-automorphism of $C(X)$ induced by the action of t; that is, $\alpha_t(f)(x)=f(t^{-1}(x))$ for all x in X. The transformation C^*-algebra ($=C^*$-crossed product) $A(\Sigma)$ contains a dense $*$-algebra $K(G, C(X))$ of those functions of G into $C(X)$ which vanish outside a finite subset of G. The element $f\delta_t$ in $K(G, C(X))$ means the function defined by $(f\delta_t)(t)=f$ and $(f\delta_t)(s)=0$ for $s\neq t$. Using this notation, every function a in $K(G, C(X))$ is written as follows:

$$a=\sum_{t\in F} a_t\delta_t\ , \qquad (a_t\in C(X))\ ,$$

where F is a finite subset of G. The multiplication and $*$-operation in $K(G, C(X))$ are given by $f\delta_t g\delta_s=f\alpha_t(g)\delta_{ts}$ and $(f\delta_t)^*=\alpha_{t^{-1}}(\bar{f})\delta_{t^{-1}}$, where the bar means the complex conjugate. The C^*-algebra $C(X)$ is regarded as a subalgebra of $A(\Sigma)$ by means of the embedding: $f\to f\delta_e$, where e is the identity in G. If G is amenable, $A(\Sigma)$ coincides with the reduced crossed product $A_r(\Sigma)$. In this case, every element a in $A(\Sigma)$ has a formal expansion with Fourier coefficient;

$$a\sim\sum_{t\in G} a_t\delta_t\ , \qquad (a_t\in C(X))\ .$$

With E, we mean the conditional expectation: $a\to a_e$.

§ 3. Irreducible representations of $A(\Sigma)$.

Since we shall make use of algebraic induced covariant representations of $A(\Sigma)$ induced by isotropy groups, we briefly sketch their structures. Let u be a unitary representation $s\to u_s$ of a subgroup K of G on a Hilbert space $\mathfrak{H}_u$. We write the left coset space $G/K=\{r_\alpha K: \alpha\in\Gamma\}$ for a set of representatives $R=\{r_\alpha: \alpha\in\Gamma\}$ where $r_0=e$. Let $\mathfrak{H}_0$ be the Hilbert space with $\dim\mathfrak{H}_0=\mathrm{card}(G/K)$. We put $\mathfrak{H}=\mathfrak{H}_0\otimes\mathfrak{H}_u$. Then each vector

ξ in $\mathfrak{H}$ is expanded as $\sum_{\alpha\in\Gamma} e_\alpha \otimes \xi_\alpha$ with respect to a fixed complete orthonormal basis $\{e_\alpha : \alpha \in \Gamma\}$ in $\mathfrak{H}_0$. We define a unitary representation L_u^R of G induced by u in the following way:

$$L_u^R(t)(e_\alpha \otimes \xi) = e_\beta \otimes u_s \xi ,$$

where $tr_\alpha = r_\beta s$ for s in K. It is easy to show that L_u^R is a unitary representation of G. We here remark that the underlying Hilbert space for L_u^R depends only on $\mathfrak{H}_u$ and the cardinal number of G/K, whereas in the case of usual induced representation the underlying Hilbert space depends more heavily on the unitary representation of K.

Now take a point x in X with the isotropy subgroup $G_x = \{t \in G : t(x) = x\}$. Taking G_x as the above subgroup K, we may write $O(x) = \{r_\alpha(x) : \alpha \in \Gamma\}$. Let π_x^R be the representation of $C(X)$ on $\mathfrak{H}$ defined by

$$\pi_x^R(f)(e_\alpha \otimes \xi) = f(r_\alpha(x)) e_\alpha \otimes \xi , \qquad (f \in C(X)) .$$

Let L_u^R be the representation of G defined above for a unitary representation u of G_x on $\mathfrak{H}_u$. Then we can see that the pair (π_x^R, L_u^R) is a covariant representation of the C^*-dynamical system $\{C(X), G, \alpha\}$. We denote by $\rho_{x,u} (= \pi_x^R \times L_u^R)$ the representation of $A(\Sigma)$ defined by π_x^R and L_u^R. It can be shown that the representation $\pi_{x,u}^R$ as well as L_u^R does not depend on the choice of the representatives $R = \{r_\alpha : \alpha \in \Gamma\}$ within unitary equivalence (cf. [7: Proposition 4.1.2]).

A representation $(\rho, \mathfrak{H})$ of $A(\Sigma)$ is said to be discrete if there exists a common eigenvector in $\mathfrak{H}$ for all $\rho(f)$ $(f \in C(X))$. In [7: Proposition 4.1.6], it was proved that an irreducible representation ρ of $A(\Sigma)$ is discrete if and only if ρ is unitarily equivalent to $\rho_{x,u}$ for a point x in X and an irreducible representation u of G_x. For x in X, a representation u of G_x and a unit vector ξ in $\mathfrak{H}_u$, we put

$$\psi_{x,u,\xi}(a) = (\rho_{x,u}(a)(e_0 \otimes \xi), e_0 \otimes \xi) , \qquad (a \in A(\Sigma)) .$$

Since the representation of $A(\Sigma)$ associated with a state extension of the evaluation state μ_x of $C(X)$ for a point x is discrete, one can prove the following proposition similar to [7: Proposition 4.1.6].

PROPOSITION 3.1. *A state ψ of $A(\Sigma)$ is an extension of a pure state μ_x of $C(X)$ if and only if ψ is of the form $\psi = \psi_{x,u,\xi}$.*

§ 4. Ideal structure and orbit structure.

In this section, we prove the main result stated before.

THEOREM 4.1. *The following condition* (A) *implies the condition* (B). *If G is amenable, the converse implication holds.*

(A) $I \cap C(X) \neq \{0\}$ *for each non-zero ideal I in $A(\Sigma)$,*

(B) $\operatorname{Int} X^t = \varnothing$ *for all $t \neq e$, where* $\operatorname{Int} X^t$ *means the interior of X^t.*

PROOF. ((A)→(B)). Suppose that $\operatorname{Int} X^t \neq \varnothing$ for some $t \neq e$. Then there exists a continuous function f with $\operatorname{supp}(f)$ contained in X^t. Let I be the ideal generated by the element $f - f\delta_t$ in $A(\Sigma)$. We shall show that $I \cap C(X) = \{0\}$. For each x in X, we take a state extension φ_x to $A(\Sigma)$ of the point measure μ_x such that $\varphi_x(\delta_s) = 1$ if s belongs to G_x and $\varphi_x(\delta_s) = 0$ if s does not belong to G_x. Let h and g be continuous functions on X. Since $\varphi_x(f\delta_t) = \varphi_x(f)\varphi_x(\delta_t)$ for f in $C(X)$, we have

$$\begin{aligned}
\varphi_x(h\delta_r(f - f\delta_t)g\delta_s) &= \varphi_x(h\delta_r f g\delta_s) - \varphi_x(h\delta_r f\delta_t g\delta_s)\\
&= \varphi_x(h\alpha_r(fg)\delta_{rs}) - \varphi_x(h\alpha_r(f)\alpha_{rt}(g)\delta_{rts})\\
&= \varphi_x(h\alpha_r(fg))\varphi_x(\delta_{rs}) - \varphi_x(h\alpha_r(f)\alpha_{rt}(g))\varphi_x(\delta_{rts})\\
&= h(x)f(r^{-1}(x))g(r^{-1}(x))\varphi_x(\delta_{rs}) - h(x)f(r^{-1}(x))g((rt)^{-1}(x))\varphi_x(\delta_{rts})\,. \qquad \cdots\cdots(*)
\end{aligned}$$

If $r^{-1}(x)$ does not belong to X^t, then $f(r^{-1}(x)) = 0$. Otherwise we have $(rt)^{-1}(x) = r^{-1}(x)$. Thus we have $g(r^{-1}(x)) = g((rt)^{-1}(x))$, and $(rts)^{-1}(x) = (rs)^{-1}(x)$, namely $(rs)(rts)^{-1}$ belongs to G_x. Hence $\varphi_x(\delta_{rs}) = \varphi_x(\delta_{rts})$. Therefore it follows that $(*) = 0$. Since each element a in I is approximated by the finite combinations of the elements mentioned above, $\varphi_x(a) = 0$ for all x in X. Thus, if the element a in I belongs to $C(X)$, then $a = 0$.

((B)→(A)). Suppose that $\operatorname{Int} X^t = \varnothing$ for all $t \neq e$ and G is amenable. Let I be an ideal in $A(\Sigma)$ such that $I \cap C(X) = \{0\}$. Let a be an element in I. Since G is amenable, a has the expansion $a \sim \sum_{t \in G} a_t \delta_t$ in $A(\Sigma)$. First we shall show that $a_e = E(a) = 0$, that is, $E(a)(x) = 0$ for all $x \in X$. Let q be the quotient map of $A(\Sigma)$ onto the quotient C^*-algebra $B = A(\Sigma)/I$. The restriction of q to $C(X)$ is then a $*$-isomorphism of $C(X)$ onto $q(C(X))$. Let ν_y be the pure state on $q(C(X))$ associated with a point y in X, that is, $\nu_y(q(f)) = f(y)$ for f in $C(X)$. Let Ψ_y be a pure state extension to B of ν_y. Put $\psi_y = \Psi_y \cdot q$. Then ψ_y is a pure state of $A(\Sigma)$ with $\psi_y(f) = f(y)$ for f in $C(X)$ and

$$\psi_y(a) = (\Psi_y \cdot q)(a) = 0\,. \qquad \cdots\cdots(\mathrm{i})$$

Let $x\ (\in X)$ and $\varepsilon\ (>0)$ be given. Then there exist a neighbourhood U of x such that

$$|E(a)(x) - E(a)(y)| < \varepsilon \qquad \text{for all } y \text{ in } U \qquad \cdots\cdots(\mathrm{ii})$$

and an element $b=\sum_{t\in F} b_t\delta_t$ in $K(G, C(X))$ such that

$$\|a-b\|<\varepsilon \qquad \cdots\cdots \text{(iii)}$$

where F is a finite subset in G. For the subset F and the neighbourhood U, there exists a point y in U such that $t(y)\neq y$ for all t in $F-\{e\}$. In fact, if U were contained in $\cup_{t\in F-\{e\}} X^t$, one of the sets X^t's should have an interior point, and this contradicts the assumption. Next let us consider the following inequality:

$$\begin{aligned}|E(a)(x)| \leqq |a_e(x)-a_e(y)| + |a_e(y)-b_e(y)| \\ + |b_e(y)-\psi_y(b)| + |\psi_y(b)-\psi_y(a)| + |\psi_y(a)| .\end{aligned}$$

We need to estimate only the third term of right hand side. For the pure state ψ_y, by Proposition 3.1 there exists an irreducible representation of $A(\Sigma)$ of the form $\rho_{y,u}=\pi_y^R\times L_u^R$ on $\mathfrak{H}_0\otimes\mathfrak{H}_u$ such that

$$\psi_y(c)=(\rho_{y,u}(c)\xi_e, \xi_e) , \qquad (c\in A(\Sigma)) ,$$

where ξ_e is a unit vector in $e_0\otimes\mathfrak{H}_u$. Since $L_u^R(t)\xi_e$ and ξ_e are orthogonal for $t\in G_y$, we have the following:

$$\begin{aligned}\psi_y(b)&=(\rho_{y,u}(b)\xi_e, \xi_e)=((\sum_{t\in F}\pi_y(b_t))L_u(t)\xi_e, \xi_e)\\ &=(\sum_{t\in G} b_t(t(y))L_u(t)\xi_e, \xi_e)=(b_e(y)\xi_e, \xi_e)=b_e(y) .\end{aligned}$$

Therefore, combining this with (i), (ii) and (iii), we have

$$|E(a)(x)|\leqq 3\varepsilon .$$

Since ε is arbitrary, it follows that $E(a)(x)=0$, thus $E(a)=0$. For each t in G, since $a\delta_{t^{-1}}$ belongs to the ideal I, we have $a_t=E(a\delta_{t^{-1}})=0$. Hence $a=0$. □

REMARK 4.2. By virtue of [8: Proposition 4.1.4], we have that, when G is amenable, the condition (A) is equivalent to the fact that $C(X)$ is a maximal abelian subalgebra of $A(\Sigma)$.

REMARK 4.3. Suppose that G is amenable and the action of G is topologically transitive. Then, in some cases, e.g., when the set $\{t^n: n\in \boldsymbol{Z}\}$ is infinite, the ideal I generated by $f-f\delta_t$ in the proof of the implication (A)→(B) turns out to be an essential ideal.

By Theorem 4.1, we get a complete characterization for $A(\Sigma)$ to be simple.

THEOREM 4.4. *The C^*-algebra $A(\Sigma)$ is simple if and only if* (1) *the action of G on X is minimal,* (2) $\operatorname{Int} X^t = \varnothing$ *for all $t \neq e$, and* (3) *G is amenable.*

As for the reduced transformation group C^*-algebra $A_r(\Sigma)$ for a non-amenable group, it is to be noticed that in some cases the minimality of the dynamical system is enough to imply its simplicity (cf. [2: Theorem 1]).

REMARK 4.5. In the case of abelian group, by Theorem 4.1 we can get a complete characterization for $A(\Sigma)$ to be prime. Namely it follows that $A(\Sigma)$ is prime if and only if (1) the action of G is topologically transitive, and (2) $\operatorname{Int} X^t = \varnothing$ for all $t \neq e$. This characterization is of course well known in the general theory of C^*-crossed product (cf. [5: Theorem 5.8]). Now applying the above characterization, we can get a prime group C^*-algebra $C^*(G)$ associated with an amenable group G, in contrast with the fact that no group C^*-algebra of an amenable group is simple. In fact, let G be the semi-direct product $\boldsymbol{Z}^2 \times \boldsymbol{Z}$ associated with an action $\alpha = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \in \operatorname{Aut}(\boldsymbol{Z}^2)$; i.e., $(a, m)(b, n) = (a + \alpha^m(b), n + m)$ for $(a, m), (b, n) \in \boldsymbol{Z}^2 \times \boldsymbol{Z}$. Let $\tilde{\alpha} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ be the dual action on $\boldsymbol{T}^2 = \hat{\boldsymbol{Z}}^2$ and Σ the topological dynamical system $(\boldsymbol{T}^2, G = \{\tilde{\alpha}^n : n \in \boldsymbol{Z}\})$. The group C^*-algebra $C^*(G)$ is then $*$-isomorphic to the transformation group C^*-algebra $A(\Sigma)$. Since the action $\tilde{\alpha}$ on $\boldsymbol{T}^2$ is topologically transitive and the interior of the set of fixed points for $\tilde{\alpha}^n$ $(n \neq 0)$ is empty, $A(\Sigma)$ is prime, thus $C^*(G)$ is prime, too.

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