TOKYO J. MATH.
VOL. 15, NO. 2, 1992

Moves for Flow-Spines and Topological Invariants of 3-Manifolds

Ippei ISHII

Keio University

Introduction.

A spine P for a closed 3-manifold M is a 2-dimensional polyhedron in M such that the complement of the regular neighborhood of P is homeomorphic to the 3-ball. Cutting off a closed 3-manifold M along its spine P, we get a 3-ball B^3 with an identification on its boundary. This is a polyhedral representation of M, which is first considered by M. Dehn in the case of closed surfaces, and introduced by H. Seifert in the 3-dimensional case.

A DS-diagram is a polyhedral representation of a special class, which was first introduced in [3]. A spine corresponding to a DS-diagram forms a closed fake surface (cf. [1], [3]). A spine which forms a closed fake surface is called a standard or a simple spine. As is pointed out in [12], a standard spine is the dual of a singular triangulation.

A flow-spine introduced in [7] is a standard spine of a more special class, which is generated by a pair of a non-singular flow and its local section. It was shown in [4] and [7] that a DS-diagram for a flow-spine has an E-cycle. An E-cycle is a cycle of the graph of a DS-diagram which represents a kind of symmetry of a polyhedral representation. (See §1 for precise.)

A closed 3-manifold admits infinitely many flow-spines. In this paper, we shall give conditions for two flow-spines to represent the same manifold, that is, it will be shown that any two flow-spines of a 3-manifold can be transformed from one to another by a finite sequence of operations of three types which we call "moves". A flow-spine is completely determined by a data on the E-cycle, which will be called an E-data (cf. §1). An E-data is the one called a singularity-data in [7]. Our moves of flow-spines are described in terms of E-data. For an easy description of moves of E-data, we introduce the graphic representation of an E-data in §1.

An E-data determines not only a 3-manifold M but also a class of non-singular flows on M (see §1). Moves of E-data are divided into two types, moves which do not change the class of non-singular flows and those which change the class. Moves of the first type are called regular moves and discussed in §2. The second type consists of only

Received July 2, 1991

one move, called a surgery move, and is exhibited in §3.

As an application, in §§4, 5 we shall attempt to give topological invariants of 3-manifolds. The invariants obtained in this paper are similar to those in [12] which are called state sum invariants.

In this paper, we consider only orientable cases. Nonorientable case can be treated in a similar way, but E-data for nonorientable manifolds are somewhat complicated (cf. [7]).

§1. DS-diagrams and E-data.

First we shall recall the notion of *fake surfaces*, *DS-diagrams*, and *DS-diagrams with E-cycle*. These concepts were introduced by H. Ikeda in [1]–[4]. For the precise definition, refer to these papers. Let P be a closed fake surface, and $\mathfrak{S}_j(P)$ be the closure of the j-th singularities of P ($j=1, 2, 3$). A continuous map f from the 2-sphere S^2 onto a closed fake surface P is said to be an *identification map*, if there exists a connected 3-regular graph G embedded in S^2 and satisfying the following conditions:

(i) For any connected component X of S^2-G or $G-V(G)$ ($V(G)$ is the set of vertices of G), $f|X: X\to f(X)$ is a homeomorphism.

(ii) $f^{-1}(f(\mathfrak{S}_3(P)))=V(G)$, and, for each $v\in V(G)$, $f^{-1}(f(v))$ consists of exactly four points.

(iii) $f^{-1}(f(\mathfrak{S}_2(P)))=G$, and $f^{-1}(f(E))$ has exactly three connected components for any component E of $G-V(G)$.

A triple (S^2, G, f) as above is called a *DS-diagram*. Considering S^2 to be the boundary ∂B^3 of the 3-ball B^3 and identifying B^3 by the map f, we get a closed 3-manifold B^3/f which has $\partial B^3/f$ as its standard spine.

A cycle $e=\{E_1, E_2, \cdots, E_{2v}\}$ of the graph G is said to be an *E-cycle* of a DS-diagram (S^2, G, f) if it satisfies that

(i) the underlying space of e, which we denote by the same letter e, is a simple closed curve on S^2,

(ii) $f(E_i)\neq f(E_j)$ for $i\neq j$, and

(iii) for a component H^0 of S^2-e (the other component is denoted by H^1), $f|H^0: H^0\to f(H^0)$ is bijective.

Let (S^2, G, f) be a DS-diagram with an E-cycle e. Without loss of generality, we may assume that

$$B^3=\{(x, y, z)\in \boldsymbol{R}^3;\ x^2+y^2+z^2\leqq 1\}, \qquad S^2=\partial B^3,$$

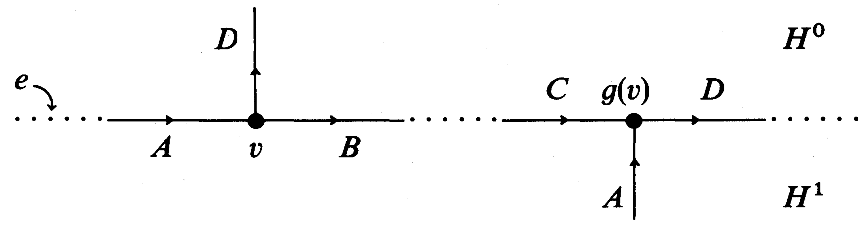
$$e=S^2\cap\{z=0\} \quad \text{(the equator)},$$

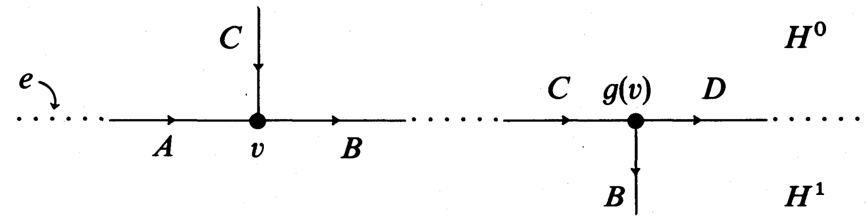
$$H^0=S^2\cap\{z>0\}, \qquad H^1=S^2\cap\{z<0\}.$$

A point v on $V(G)\cap e$ satisfies one of the following (0) or (1) (cf. [4]).

(j) $U\cap(G-e)\subset H^j$, for sufficiently small neighborhood U of v ($j=0$ or 1).

By V^j we denote the set of points on $V(G)\cap e$ which satisfy the above condition (j). Moreover for each $v\in V^0$ there exists a point v' on V^1 such that $f(v')=f(v)$, which we denote by $g(v)$. In the case where the represented manifold $M=B^3/f$ is orientable, the points on $V(G)\cap e$ are classified into the following two cases (L) or (R) (cf. [7], and see [3] for the method for indicating the identification map f on a DS-diagram).

(L) 

(R) 

In what follows, we consider only the case where B^3/f is orientable. Hence $V^j=V_l^j\cup V_r^j$ ($j=0, 1$), $g(V_l^0)=V_l^1$ and $g(V_r^0)=V_r^1$, where V_l^j is the set of points on V^j with the above condition (L) and V_r^j is the set of those satisfying (R). Considering e as an oriented circle, we call the 6-tuple $(S^1\,;\,V_l^0, V_r^0, V_l^1, V_r^1\,;\,g)$ an E-data, which we called a singularity-data in [7]. Notice that the notation here is slightly different from the one in [7]. Conversely, given an E-data, we can reconstruct a DS-diagram with the given E-data if there is such a DS-diagram (cf. [7]). Furthermore, fixing the orientations on the 3-ball B^3 and on the equator e, we can regard B^3/f as an oriented manifold. In this way we regard an E-data as an oriented 3-manifold, if the E-data corresponds to a DS-diagram.

Two E-data $\Delta_j=(S^1\,;\,V_{lj}^0, V_{rj}^0, V_{lj}^1, V_{rj}^1\,;\,g_j)$ ($j=1, 2$) are identified with each other if there is an orientation preserving homeomorphism h of S^1 such that $h(V_{l1}^k)=V_{l2}^k$, $h(V_{r1}^k)=V_{r2}^k$ ($k=0, 1$) and $h\circ g_1=g_2\circ h$.

Remark. Recently it is shown by H. Ikeda and M. Kouno that, even if an E-data does not correspond to a DS-diagram, it naturally determines a compact 3-manifold ([6]).

For convenience, we represent an E-data by an oriented and coded graph. Let $\Delta=(S^1\,;\,V_l^0, V_r^0, V_l^1, V_r^1\,;\,g)$ be an E-data, where the orientation on the circles S^1 is fixed. The oriented and coded graph $G^*=G^*(\Delta)$ defined as follows represents the given E-data Δ.

(i) The vertices of G^* consist of V_l^j and V_r^j $(j=0, 1)$,

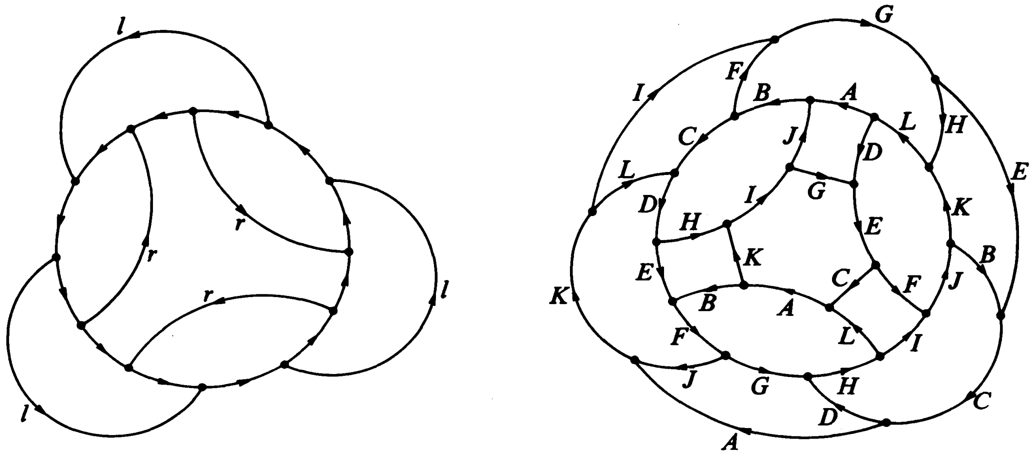
(ii) The oriented edges $E(G^*)$ consist of three classes of edges E^l, E^r and E^x, the edges coded by l, r and x respectively, where these classes are defined by

(l) $[u, v] \in E^l$ iff $u \in V_l^0$ and $v = g(u)$,

(r) $[u, v] \in E^r$ iff $u \in V_r^0$ and $v = g(u)$, and

(x) $[u, v] \in E^x$ iff there is no vertex on the subarc of S^1 going from u to v in the given orientation.

For example, the graph in Fig. 1 (a) represents an E-data (the code x is not written), and this E-data corresponds to the DS-diagram in Fig. 1 (b).



(a) (b)

FIGURE 1

As is shown in [7] and [8], a DS-diagram with E-cycle is closely related to a non-singular flow on the manifold represented by the given DS-diagram. For a DS-diagram with an E-cycle e (we always assume that the E-cycle coincides with the equator of B^3), a non-singular flow $\psi: \boldsymbol{R} \times M \to M$ $(M = B^3/f)$ is defined by

$$\psi(t\,; f(x, y, z)) = f(x, y, z+t)$$
$$\text{if } (x, y, z+s) \in B^3 \quad \text{for } 0 \leqq s \leqq t \ (\text{or } t \leqq s \leqq 0)\,.$$

This definition of ψ is slightly different from the one in [8], but they are essentially the same. The local section Σ is given by

$$\Sigma = \{(x, y, z) \in B^3/f \mid z = 0, x^2 + y^2 \leqq 1 - \delta\}\,,$$

where δ is a sufficiently small positive number. Any orbit of ψ intersects with the interior of Σ, and the graph G of the DS-diagram is given as the image of $\partial\Sigma$ under the Poincaré-map for Σ (cf. [7]). Let Δ be an E-data which determines a closed 3-manifold $M = M(\Delta)$. For this E-data, there is a non-singular flow ψ on M defined as above. This non-singular flow is not unique, but a class of non-singular flows on M is uniquely determined by Δ in some sense. In order to explain this, we make a definition.

DEFINITION 1.1. Two non-singular flows ψ_1 and ψ_2 on an oriented closed manifold M are said to be *equivalent*, written by $\psi_1 \sim \psi_2$, if there are non-singular flows $\hat{\psi}_1$ and $\hat{\psi}_2$, and an orientation preserving homeomorphism $h: M \to M$ such that $h \circ \hat{\psi}_1 = \hat{\psi}_2 \circ h$ and $\hat{\psi}_j$ can be continuously deformed into ψ_j within the set of all non-singular flows on M.

Let Δ be an E-data which represents a closed 3-manifold M, and (S^2, G, f) be a DS-diagram given by Δ. Since the graph G is determined only up to isotopy, the corresponding non-singular flow ψ which depends on the choice of G is not unique. However the above defined equivalence class of ψ is uniquely determined. We denote by $[\Delta]$ this equivalence class.

§2. Regular moves of E-data.

For an oriented closed 3-manifold M there are infinitely many E-data Δ such that $M(\Delta)$ is homeomorphic to M. In this section, we consider when two E-data give the same manifold. As is stated in the previous section, an E-data, if it represents a closed 3-manifold M, corresponds to a pair of a non-singular flow on M and its local section. In [7] we called this pair a *normal pair*. Hence, in the case of $[\Delta_1] = [\Delta_2]$, the change from Δ_1 into Δ_2 can be described as a continuous deformation of normal pairs, which is quite analogous to the regular Reidemeister moves of knot projections. Refer to [9] (Chap. 4) for the notion of the Reidemeister move of knot projections.

Before giving the precise definition of moves, we shall summarize the relation between an E-data and a non-singular flow. This will give a good explanation for the reason why the moves of E-data is similar to the regular Reidemeister moves. Let M be a closed 3-manifold, and $\psi: \boldsymbol{R} \times M \to M$ be a non-singular flow. Choose a local section Σ so that the pair (ψ, Σ) forms a normal pair (cf. [7]). Namely any orbit intersects with Σ transversely, and moreover Σ satisfies some generic conditions (see [7] for the precise). Then we can define the Poincaré-map $T: M \to \Sigma$; i.e., for $x \in M$, $T(x)$ is the first returning point to Σ along ψ. The set of the discontinuity points of T, denoted by $P_-(\psi, \Sigma)$, forms a spine of M, and the DS-diagram induced by this spine has an E-cycle ([7]).

Now we shall explain how the E-data of the DS-diagram is derived from the spine $P_-(\psi, \Sigma)$. Let M be oriented, and $\boldsymbol{R}^3$ be usually oriented Euclidean space whose coordinate is denoted by (x, y, z). Let U be a neighborhood of Σ, and $h: U \to \boldsymbol{R}^3$ be an orientation preserving embedding such that $h(\Sigma) \subset \{z=0\}$ and $h(x, y, z+t) = \psi(t; h(x, y, z))$. We settle the orientation on $\partial\Sigma$ so that $h(\Sigma)$ is on the left of $h(\partial\Sigma)$. In this way, for a local section Σ of a flow on an oriented manifold, we can regard the boundary $\partial\Sigma$ as the oriented circle S^1. The required E-data $(S^1; V_l^0, V_r^0, V_l^1, V_r^1; g)$ is given as follows (cf. [7]):

$$V^0 = \{x \in \partial\Sigma \mid T(x) \in \partial\Sigma\}, \qquad g = T \mid V^0,$$
$$V_l^0 = \{x \in V^0 \mid T \mid \partial\Sigma \text{ is left continuous at } x\},$$
$$V_r^0 = \{x \in V^0 \mid T \mid \partial\Sigma \text{ is right continuous at } x\},$$
$$V_l^1 = g(V_l^0), \qquad V_r^1 = g(V_r^1).$$

This E-data is called the one generated by a normal pair (ψ, Σ), and denoted by $\Delta(\psi, \Sigma)$.

See Fig. 2 (i), and assume that

(a) the flow runs from the back of the sheet to the front,

(b) the curves drown there are parts of $\partial\Sigma$, and

(c) Σ lies on the left of its boundary.

Then the singularities of the spine correspond to the crossings of the figure, and so the E-data for this part is given by Fig. 2 (ii). Deforming the local section, we get the situation as in Fig. 3 (i). In this figure, the crossing points encircled by the dotted circle seems to produce a 3-rd singularity of the spine. However there is the local section attached to the part of the boundary numbered by 2 between the under and over crossings. Hence this crossing produces no discontinuity point of the Poincaré-map. Consequently the E-data corresponding to Fig. 3 (i) is given by Fig. 3 (ii). Obviously

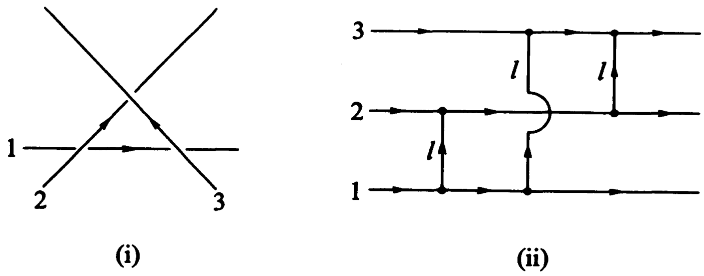


FIGURE 2

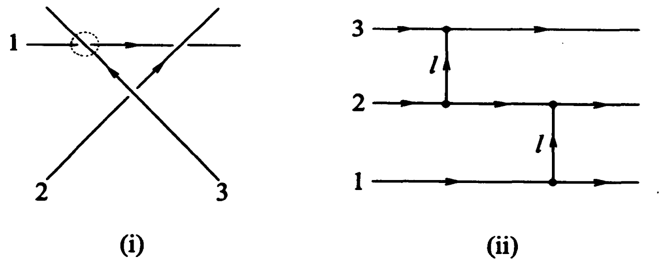


FIGURE 3

the transformation of E-data from Fig. 2 (ii) into Fig. 3 (ii) does not change the represented manifold. This transformation is one of the moves of E-data. The precise definition of regular moves is as follows, and one more move will be introduced in the next section.

DEFINITION 2.1. (Regular moves)

(i) The transformation of E-data from Fig. 2 (ii) into Fig. 3 (ii) is called *the first regular move*, and denoted by R_1.

(ii) Each transformation of E-data in Fig. 4 (1)–(4) will be called *the second regular move*, and denoted by R_2-(x) ($x=1, 2, 3$ or 4), or simply by R_2.

The inverse of the regular move R_j is denoted by R_j^{-1}.

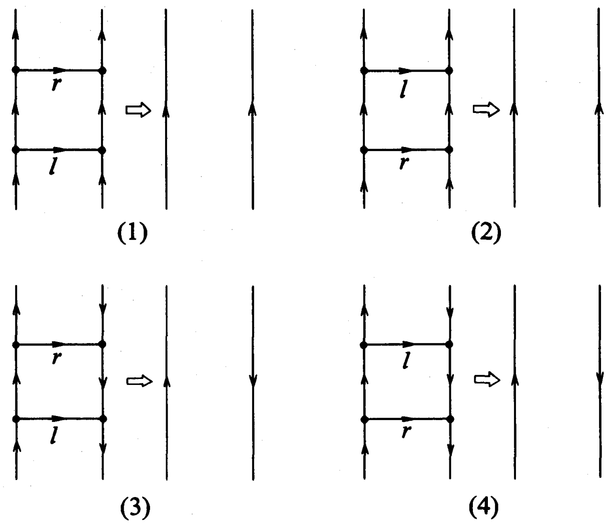


FIGURE 4

If an E-data Δ' is obtained from Δ by applying the move $R_j^{\pm 1}$, then we write $\Delta' = R_j^{\pm 1}(\Delta)$. It is easy to see that if an E-data Δ is realized by a normal pair on some closed 3-manifold M, then each one of $R_1^{\pm 1}(\Delta)$ and $R_2(\Delta)$ is also realized by a normal pair on M. However for $R_2^{-1}(\Delta)$ there might be no closed 3-manifold corresponding to it. By these regular moves we can define equivalence relations.

DEFINITION 2.2. (Regular eqivalence and strongly regular equivalence)

(i) Two E-data Δ_a and Δ_b are said to be *regular equivalent* to each other iff there is a sequence of E-data $\Delta_a = \Delta_1, \Delta_2, \cdots, \Delta_n = \Delta_b$ such that $\Delta_{k+1} = R_j^{\pm 1}(\Delta_k)$ ($j=1$ or 2, $k=1, \cdots, n-1$). We denote this equivalence by $\Delta_a \overset{R}{\sim} \Delta_b$.

(ii) Moreover if any Δ_k ($k=1, 2, \cdots, n$) represents a closed 3-manifold, then we say these E-data are *strongly regular equivalent* to each other, and write $\Delta_a \overset{sR}{\sim} \Delta_b$.

It is not known whether there are two E-data of closed manifolds which are regular equivalent but not strongly regular equivalent.

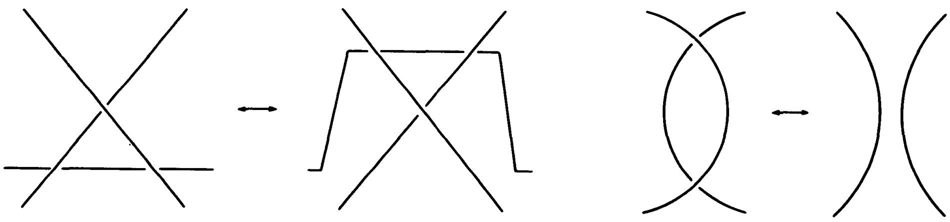
The purpose of this section is to prove the next theorem.

THEOREM 2.3. *Let Δ_1 and Δ_2 be E-data which correspond to oriented closed 3-manifolds $M(\Delta_1)$ and $M(\Delta_2)$ respectively. Then we have that*

$$M(\Delta_1)\simeq M(\Delta_2) \text{ and } [\Delta_1]=[\Delta_2] \text{ if and only if } \Delta_1 \overset{sR}{\sim} \Delta_2\,,$$

where $M(\Delta_1)\simeq M(\Delta_2)$ means that there is an orientation preserving homeomorphism $h: M(\Delta_1)\to M(\Delta_2)$.

PROOF. First we shall consider the case where two E-data Δ_1 and Δ_2 generated by normal pairs (ψ, Σ_1) and (ψ, Σ_2) for the same flow ψ and its disjoint local sections Σ_1 and Σ_2. In this case, connecting Σ_1 and Σ_2 by a band U, we get a normal pair (ψ, Σ_*) for a local section $\Sigma_*=\Sigma_1\cup\Sigma_2\cup U$ which yields an E-data Δ_*. Consider a deformation Σ^t $(-1\leqq t\leqq 1)$ of local sections such that $\Sigma^{-1}=\Sigma_1$, $\Sigma^t\supset\Sigma_1$ for $-1\leqq t\leqq 0$, $\Sigma^0=\Sigma_*$, $\Sigma^t\supset\Sigma_2$ for $0\leqq t\leqq 1$, and $\Sigma^1=\Sigma_2$. Then, for any t, Σ^t intersects with all orbits of ψ. We may assume that the DS-diagram generated by (ψ, Σ^t) changes its isotopy type at finite t's, where it happens the cases in Fig. 5 (i) or (ii) (in these figures the arcs are the boundary of Σ^t, and the flow ψ runs from the back to the front).



(i) (ii)

FIGURE 5

The change in Fig. 5 (ii) yields one of the second regular moves of E-data. For the change in Fig. 5 (i), there are several cases about the orientations of the boundary in the figure. However we can easily check that, in any case, the corresponding transformation of E-data is represented as a composition of the moves R_1 and R_2 (see Fig. 6 for example). This shows that $\Delta(\psi, \Sigma_1)$ and $\Delta(\psi, \Sigma_2)$ are strongly regular equivalent if $\Sigma_1\cap\Sigma_2=\varnothing$.

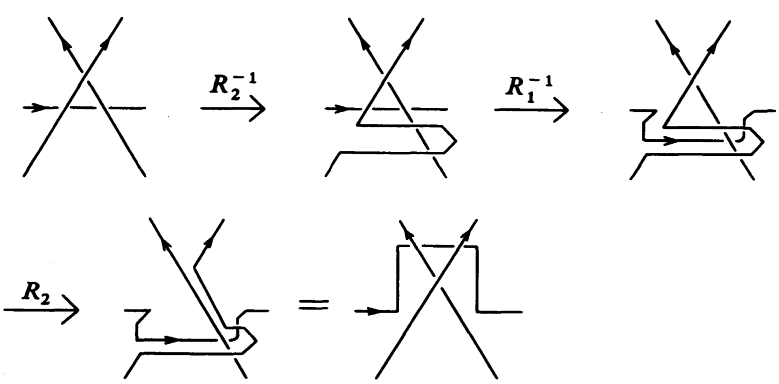


FIGURE 6

In the case of $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, taking another local section Σ_3 such that $\Sigma_3 \cap \Sigma_j = \emptyset$ $(j=1, 2)$ and (ψ, Σ_3) is also a normal pair, we can see that $\Delta(\psi, \Sigma_1) \overset{sR}{\sim} \Delta(\psi, \Sigma_3) \overset{sR}{\sim} \Delta(\psi, \Sigma_2)$.

Now consider the general cases. Suppose a flow ψ_1 can be continuously deformed into ψ_2 within the set of non-singular flows on a 3-manifold M. Then we can take a sequence of non-singular flows $\psi_1 = \psi^1, \psi^2, \cdots, \psi^n = \psi_2$ and local sections $\Sigma^1, \Sigma^2, \cdots, \Sigma^{n-1}$ such that (ψ^k, Σ^k) and (ψ^{k+1}, Σ^k) are normal pairs having the same E-data for each $k=1, 2, \cdots, \nu$. Consequently we get

$$\begin{aligned}\Delta(\psi_1, \Sigma_1) &= \Delta(\psi^1, \Sigma^1) = \Delta(\psi^2, \Sigma^1) \overset{sR}{\sim} \Delta(\psi^2, \Sigma^2) = \Delta(\psi^3, \Sigma^2) \overset{sR}{\sim} \cdots \\ &= \Delta(\psi^{n-1}, \Sigma^{n-2}) \overset{sR}{\sim} \Delta(\psi^{n-1}, \Sigma^{n-1}) = \Delta(\psi^n, \Sigma^{n-1}) = \Delta(\psi_2, \Sigma_2)\,.\end{aligned}$$

This proves that $[\Delta_1] = [\Delta_2]$ implies the strongly regular equivalence of Δ_1 and Δ_2.

Conversely, recalling the way for constructing a non-singular flow from an E-data, we can easily see that if two E-data are strongly regular equivalent, then the corresponding flows are equivalent.

This completes the proof. □

§3. Surgery move of E-data.

In this section, we consider the move of E-data for the case where $M(\Delta_1) \simeq M(\Delta_2)$ and $[\Delta_1] \neq [\Delta_2]$. In order to describe this move, we need the notion of a *surgery of a non-singular flow* given below. Let ψ be a non-singular flow on a closed 3-manifold M. We denote by X the vector field generating ψ. Suppose that ψ has a periodic orbit C with a regular neighborhood U which is homeomorphic to $D^2 \times S^1$ and invariant under ψ. Let (r, θ) be a polar coordinate on $D^2 = \{r \leqq 1\}$ and t be a coordinate on $S^1 = \{\exp(2\pi\sqrt{-1}\,t) \mid t \in \boldsymbol{R}\}$. Moreover assume that $X = \partial/\partial t$ on U. For such an X, we define a vector field Y so that

(i) $Y = X$ on $M - U$,

(ii) $Y = a(r)\partial/\partial t + b(r)\partial/\partial r$ on U,

where $a(r)$ and $b(r)$ are smooth functions such that

(iii) $a(r)$ is increasing, $a(1)=1$ and $a(0)=-1$,

(iv) $b(r)$ is non negative, $b(0)=b(1)=0$, and

(v) $a^2(r)+b^2(r)>0$ for $0 \leqq r \leqq 1$.

We say that Y (or the flow generated by Y) is obtained by a surgery of X (or ψ respectively) along the periodic orbit C.

The next lemma is the key for getting one more move of E-data.

LEMMA 3.1. *Let M be a closed 3-manifold, and ψ_a and ψ_b be non-singular flows on M. Then there is a sequence of non-singular flows $\psi_1, \psi_2, \cdots, \psi_n$ such that*

(i) *$\psi_1 \sim \psi_a$ and $\psi_n \sim \psi_b$, and*

(ii) *ψ_k is obtained by a surgery of ψ_{k-1} along a periodic orbit.*

PROOF. Let TM be the tangent bundle of M. It is known that TM is a trivial bundle. Fix a trivialization $\tau_0 : TM \to M \times \boldsymbol{R}^3$. Then for a vector field X on M there is a function $\tilde{X}: M \to \boldsymbol{R}^3$ such that $\tau_0(X(p)) = (p, \tilde{X}(p))$ $(p \in M)$.

Let X_0 be the vector field given by $\tilde{X}_0(p) \equiv (0, 0, 1)$, and X be an arbitrary non-singular vector field with $\| X(p) \| \equiv 1$. Deforming X slightly if necessary, we may assume that $s = (0, 0, -1)$ is a regular value of $\tilde{X}: M \to S^2 = \{x \in \boldsymbol{R}^3 \mid \| x \| = 1\}$. Hence $\tilde{X}^{-1}(s)$ is a finite union of simple closed curves $C_1, \cdots, C_n$. Since $S^2 - \{s\}$ is contractible, we can deform X into X_1 so that $\tilde{X}_1^{-1}(s) = C_1 \cup C_2 \cup \cdots \cup C_n$ and $X_1 = X_0$ on the outside of a regular neighborhood of $C_1 \cup C_2 \cup \cdots \cup C_n$. Furthermore we can take a continuous function $A(t, p) \in \mathrm{SO}(3, \boldsymbol{R})$ $(0 \leqq t \leqq 1, p \in M)$ such that $A(0, p) \equiv \mathrm{id}$, and the vector field X'_0 defined by

$$\tilde{X}'_0(p) = (0, 0, 1)A(1, p)$$

has closed curves $C_1, \cdots, C_n$ as its periodic orbits. On the other hand, define a vector field X'_1 by $\tilde{X}'_1(p) = \tilde{X}_1(p)A(1, p)$. Making continuous deformations on X'_0 and X'_1 if necessary, we have that $X_0 \sim X'_0$, $X_1 \sim X'_1$ and X'_1 can be obtained from X'_0 by applying surgeries along periodic orbits $C_1, C_2, \cdots, C_n$. This completes the proof. □

According to this lemma and Theorem 2.3, the move describing a surgery along a periodic orbit together with the regular moves will give the generators of moves of E-data.

DEFINITION 3.2. The transformation of E-data in Fig. 7 is called the *surgery move*, and denoted by S.

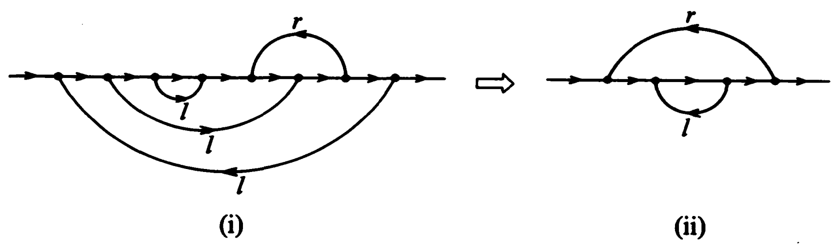


FIGURE 7

Let Δ_a and Δ_b be E-data representing closed manifolds $M(\Delta_a)$ and $M(\Delta_b)$ respectively. We define two more equivalence relations as follows.

DEFINITION 3.3.

(i) Δ_a and Δ_b are said to be *equivalent* to each other, if there is a sequence of E-data $\Delta_a = \Delta_1, \Delta_2, \cdots, \Delta_n = \Delta_b$ such that $\Delta_{k+1} = R_j^{\pm 1}(\Delta_k)$ $(j = 1, 2)$ or $S^{\pm 1}(\Delta_k)$ for $k = 1, \cdots, n-1$. This equivalence is denoted by $\Delta_a \sim \Delta_b$.

(ii) Δ_a and Δ_b are said to be *strongly equivalent* to each other (denoted by $\Delta_a \overset{S}{\sim} \Delta_b$), if any Δ_k in the above definition corresponds to a closed 3-manifold.

Under these definitions, we have the next two theorems.

THEOREM 3.4. *Let Δ_1 and Δ_2 be E-data corresponding to closed 3-manifolds $M(\Delta_1)$ and $M(\Delta_2)$ respectively. Then the fundamental group $\pi_1(M(\Delta_1))$ is isomorphic to $\pi_1(M(\Delta_2))$ if $\Delta_1 \sim \Delta_2$.*

THEOREM 3.5. *Under the same assumption as the above theorem, we have that $M(\Delta_1) \simeq M(\Delta_2)$ if and only if $\Delta_1 \overset{S}{\sim} \Delta_2$.*

Moves for general DS-diagrams (not necessarily with E-cycle) of standard spines (spines which form closed fake surfaces) are proposed in [5], [10] and [11]. Our regular moves are special cases of the moves in those papers and the surgery move can be written as a composition of those moves.

PROOF of THEOREM 3.4. A presentation of $\pi_1(M(\Delta))$ which is given in Theorem 4.1 of [7] is determined only by an E-data Δ. This presentation can be defined for any E-data Δ even if it does not correspond to any closed 3-manifold. We denote by $\Pi(\Delta)$ such a presentation. It can be easily seen that $\Pi(R_j^{\pm 1}(\Delta))$ $(j=1, 2)$ and $\Pi(S^{\pm 1}(\Delta))$ are all obtained by applying the Tietze transformation on $\Pi(\Delta)$. This implies the consequence of Theorem 3.4. □

PROOF of THEOREM 3.5. According to Theorem 2.3 and Lemma 3.1, it is sufficient to show that the surgery move of E-data describes a surgery of a non-singular flow along a periodic orbit.

By (S^2, G_1, f_1) and (S^2, G_2, f_2) we denote the DS-diagrams which are generated by the E-data in Fig. 7 (i) and (ii) respectively. These DS-diagrams are given by Fig. 8 (i) and (ii) respectively. In each diagram, the parts α, β, and γ of the circle C drown by

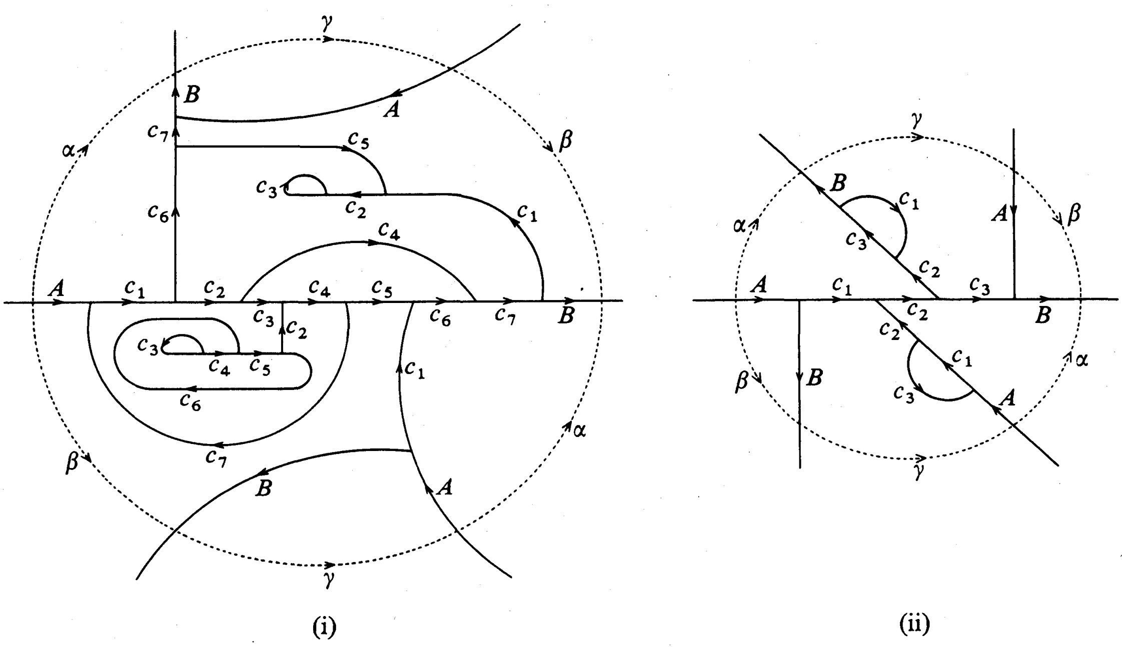


FIGURE 8

broken lines are identified by f_j as indicated in the figure. Let D_0 be a 2-disk properly embedded in the 3-ball B^3 and bounding the circle C, and let D_1 be the disk in ∂B^3 bounded by C. Then $D_0 \cup D_1$ bounds a 3-ball B in B^3. In both cases of Fig. 8 (i) and (ii), B/f_j is a solid torus with a meridian curve homologous to $f_j(x+y)$, where x and y are closed curves on the boundary of the solid torus given by $x=f_j(\alpha+\gamma)$ and $y=f_j(\beta+\gamma)$.

Let ψ_1 and ψ_2 be the non-singular flows for Fig. 8 (i) and fig. 8 (ii) respectively. By a little careful observation upon the construction of the flows, we can see that ψ_j can be taken so that ψ_1 is periodic in B/f_1 and ψ_2 can be continuously deformed into a flow obtained from ψ_1 by a surgery along a periodic orbit which is the core of B/f_1. This proves the theorem. □

§4. State sum invariant.

Recall the graphic representation $G^*(\Delta)$ of an E-data Δ which is introduced in §1. Throughout this and the next section, we will fix the notation for $G^*(\Delta)$ as follows:

NOTATION.
1) $E(G^*(\Delta))=E^x \cup E^l \cup E^r$, and $v=\#(E^l \cup E^r)$.
2) By $v_1, v_2, \cdots, v_v$, we denote the elements of $E^l \cup E^r$.
3) By $E_1, E_2, \cdots, E_{2v}$, we denote the elements of E^x.
4) $c(v_k)$ $(=l$ or $r)$ is the code of v_k.
5) The numbering to the elements of $E^l \cup E^r$ and E^x will be fixed once for all.

For an element v_k of $E^l \cup E^r$, we define four edges $E_{k(1)}$, $E_{k(2)}$, $E_{k(3)}$ and $E_{k(4)}$ of E^x by the following rule.

DEFINITION 4.1. The edges $E_{k(j)}$ $(j=1, \cdots, 4)$ are defined by the first picture in Fig. 9 if $c(v_k)=l$, and by the second if $c(v_k)=r$.

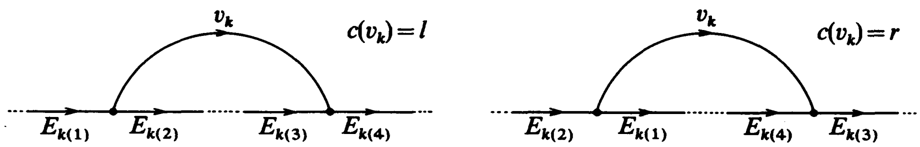


FIGURE 9

Let $J=\{1, 2, \cdots, s\}$ be a finite set, called a set of colors. A coloring of E^x by J is a map $\gamma: E^x \to J$. Let W_l and W_r be complex valued functions on J^4. We define a complex number $\Gamma(\Delta)$ for each E-data Δ by the following formula:

$$\Gamma(\Delta)=\sum_{\gamma} \prod_{k=1}^{v} W_{c(v_k)}(\gamma(E_{k(1)}), \gamma(E_{k(2)}), \gamma(E_{k(3)}), \gamma(E_{k(4)})) ,$$

where the sum is taken all over the colorings. If we could define the functions W_l and W_r so that $\Gamma(\Delta)$ is invariant under the regular moves of E-data, then, according to

Theorem 2.3, $\Gamma(\Delta)$ gives an invariant of the pair of the manifold $M(\Delta)$ and the class $[\Delta]$ of non-singular flows. If it is invariant also under the surgery move, then $\Gamma(\Delta)$ becomes a topological invariant of $M(\Delta)$ by Theorem 3.5. The required conditions on W_l and W_r are as follows:

(4.1) $$\sum_{i,j,k} W_l(a_1, i, b_1, k)W_l(k, b_2, j, c_2)W_l(i, a_2, c_1, j) = \sum_j W_l(b_1, j, c_1, c_2)W_l(a_1, a_2, j, b_2)\,,$$

(4.2.1) $$\sum_{i,j} W_l(a, i, b, j)W_r(c, i, d, j) = \delta_{ac}\delta_{bd}\,,$$

(4.2.2) $$\sum_{i,j} W_l(i, a, j, b)W_r(i, c, j, d) = \delta_{ac}\delta_{bd}\,,$$

(4.2.3) $$\sum_{i,j} W_l(a, i, j, b)W_r(c, i, j, d) = \delta_{ac}\delta_{bd}\,,$$

(4.2.4) $$\sum_{i,j} W_l(i, a, b, j)W_r(i, c, d, j) = \delta_{ac}\delta_{bd}\,,$$

(4.3) $$\sum W_l(j_1, j_2, j_5, j_6)W_l(j_2, j_3, j_3, j_4)W_l(j_7, b, a, j_1)W_r(j_7, j_6, j_5, j_4) = \delta_{ab}$$
(the sum is taken over $j_1, \cdots, j_7$).

PROPOSITION 4.2.
(i) *$\Gamma(\Delta)$ is invariant under the first regular move R_1 if the condition* (4.1) *is satisfied.*
(ii) *$\Gamma(\Delta)$ is invariant under the second regular move R_2-(j) if the condition* (4.2.j) *is satisfied* ($j=1, \cdots, 4$).
(iii) *$\Gamma(\Delta)$ is invariant under the surgery move S if the conditions* (4.2) *and* (4.3) *are satisfied.*

PROOF. For Fig. 2 (ii) and Fig. 3 (ii) which indicate the move R_1, assume that colors are given to x-coded edges as in Fig. 10. Then it can be easily seen that the condition (4.1) implies the invariance of $\Gamma(\Delta)$ under the move R_1.

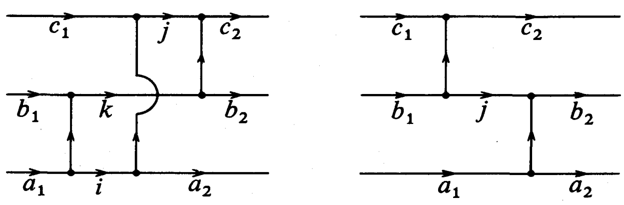


FIGURE 10

Similarly, coloring the figures indicating the move R_2 (Fig. 4) as in Fig. 11, we can see the second statement.

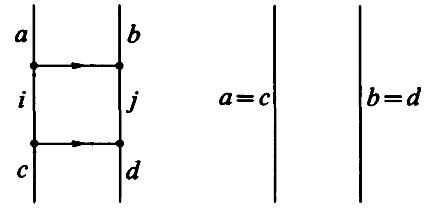


FIGURE 11

Giving colors to x-coded edges in Fig. 7 as shown in Fig. 12, we can see that if the left-hand side of (4.3) is equal to

$$\sum_{j}\sum_{i,k} W_l(j, k, i, j)W_r(b, k, i, a),$$

then $\Gamma(\Delta)$ is invariant under the surgery move. By (4.2.3), this quantity is equal to $\sum_j \delta_{jb}\delta_{ja}=\delta_{ab}$. □

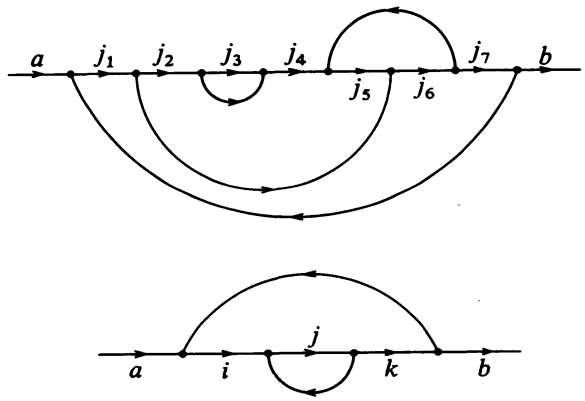


FIGURE 12

§5. Examples of the solutions for (4.1), (4.2.j) and (4.3).

In this section we will give solutions for the equations (4.1) and (4.2.j) in the case of $J=\{1, 2\}$ or $\{1, 2, 3\}$. For convenience, we represent the function W_l by a matrix as follows. We denote by L_{pq} an $s\times s$ matrix ($s=\sharp J$) whose (i, j)-element is $W_l(q, p, i, j)$, and by L an $s^2\times s^2$ matrix defined by

$$L=\begin{vmatrix} L_{11} & L_{12} & \cdots\cdots & L_{1s} \\ L_{21} & L_{22} & \cdots\cdots & L_{2s} \\ \vdots & \vdots & & \vdots \\ L_{s1} & L_{s2} & \cdots\cdots & L_{ss} \end{vmatrix}.$$

Moreover we define an $s^2 \times s^2$ matrix $\hat{L}$ by

$$\hat{L}=\left|\begin{array}{cccc} L_{11} & L_{21} & \cdots\cdots & L_{s1} \\ L_{12} & L_{22} & \cdots\cdots & L_{s2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_{1s} & L_{2s} & \cdots\cdots & L_{ss} \end{array}\right| .$$

Using the function W_r, we define $s \times s$ matrices R_{pq} and $s^2 \times s^2$ matrices R and $\hat{R}$ by the same rule as L_{pq}, L and $\hat{L}$. Then the conditions (4.2.1) and (4.2.2) are equivalent to the condition $R^{-1}=L^T$, and the conditions (4.2.3) and (4.2.4) are equivalent to $\hat{R}^{-1}=\hat{L}^T$, where L^T and $\hat{L}^T$ denote the transposed matrices.

In order to get functions W_l and W_r invariant under the regular moves, we should determine them in the following way. First take an L satisfying the condition (4.1), and put $R^T=L^{-1}$. If this R satisfies also $\hat{R}^T=\hat{L}^{-1}$, then these L and R give a solution for the equations (4.1) and (4.2.j). Therefore it is most important to solve the equation (4.1). In what follows, in the case $J=\{1, 2\}$ or $\{1, 2, 3\}$, we shall solve this equation under a restricted conditions

(5.1) $$W_l(q, p, i, j)=0 \quad \text{if} \quad p>q \text{ or } i>j .$$

The case of $J=\{1, 2\}$.

By the restriction (5.1) we can put

$$L_{11}=\left|\begin{array}{cc} u_1 & x_1 \\ 0 & u_2 \end{array}\right| , \qquad L_{12}=\left|\begin{array}{cc} y_1 & z \\ 0 & y_2 \end{array}\right| ,$$

$$L_{21}= 0 , \qquad L_{22}=\left|\begin{array}{cc} u_3 & x_2 \\ 0 & u_4 \end{array}\right| .$$

Solving the equation (4.1) directly, we get two solutions up to the permutation of J:

(5.2) $u_1=-1$, $u_j=1$ $(j=2, 3, 4)$, $z=1$, $x_1y_1=-2$, $x_2=y_2=0$,

(5.3) $u_j=1$ $(j=1, 3, 4)$, $u_2=-1$, $z=-1$, $x_1y_2=2$, $x_2=y_1=0$.

For both of these solutions, defining the matrix R by $R^T=L^{-1}$, we have also $\hat{R}^T=\hat{L}^{-1}$. Moreover we can check that these solutions satisfy also the equality (4.3). Therefore $\Gamma(\Delta)$ defined by these solutions give topological invariants of $M(\Delta)$.

The case of $J=\{1, 2, 3\}$.

In this case, as one of solutions of (4.1), we get

$$L_{11}=\begin{vmatrix} \omega & -\delta_1 b & \delta_1 b^2 \\ 0 & \omega^2 & -\omega^2 b \\ 0 & 0 & 1 \end{vmatrix}, \quad L_{12}=\begin{vmatrix} -\delta_2 a & 2\omega^2 & -\omega^2 b \\ 0 & \delta_2 a & \delta_2 \\ 0 & 0 & 0 \end{vmatrix},$$

$$L_{13}=\begin{vmatrix} \delta_2 a^2 & -\omega^2 a & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad L_{22}=\begin{vmatrix} \omega^2 & \delta_1 b & 0 \\ 0 & \omega & b \\ 0 & 0 & 1 \end{vmatrix},$$

$$L_{23}=\begin{vmatrix} -\omega^2 a & \delta_1 & 0 \\ 0 & a & 1 \\ 0 & 0 & 0 \end{vmatrix}, \quad L_{33}=1 \ \text{(the identity matrix)},$$

and $L_{pq}=0$ for $p>q$, where $\omega=\exp(2\pi\sqrt{-1}/3)$, and a, b, δ_1 and δ_2 are constants satisfying $ab=\omega-1$ and $\delta_1\delta_2=-1$. We can show that the value $\Gamma(\Delta)$ defined by this solution is invariant under the moves R_1, R_2 and S.

References

[1] H. Ikeda, Acyclic fake surfaces, Topology, **10** (1971), 9–36.

[2] ———, Identification maps on S^2, Kobe J. Math., **2** (1985), 163–167.

[3] H. Ikeda and Y. Inoue, Invitation to DS-diagrams, Kobe J. Math., **2** (1985), 169—186.

[4] H. Ikeda, DS-diagrams with E-cycle, Kobe J. Math., **3** (1986), 103–112.

[5] H. Ikeda, M. Yamashita and K. Yokoyama, Deformations of DS-diagrams, *Topology and Computer Sciences*, ed. by S. Suzuki, Kinokuniya (1987), 81–138.

[6] H. Ikeda and M. Kouno, Like a flow-spine, preprint.

[7] I. Ishii, Flows and spines, Tokyo J. Math., **9** (1986), 505–525.

[8] I. Ishii, Combinatorial construction of a non-singular flow on a 3-manifold, Kobe J. Math., **3** (1986), 201–208.

[9] L. H. Kauffman, *On Knots*, Ann. Math. Studies, **115** (1987), Princeton Univ. press.

[10] S. V. Matveev, Transformations of special spines and the Zeeman conjecture, Math. USSR Izv., **31** (1988), 423–434.

[11] R. Piergallini, Standard moves for standard polyhedra and spines, Rend. Circ. Mat. Palermo (2) Suppl., **18** (1988), 391–414.

[12] V. G. Turaev and O. Y. Viro, State sum invariants of 3-manifolds and quantum 6*j*-symbols, preprint.

Present Address:
Department of Mathematics, Faculty of Science and Technology, Keio University
Hiyoshi, Yokohama 223, Japan