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Remarks on Pitman Deficiency

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Introduction.

Let the distributions P_θ be indexed by parameter θ in a set Θ, where Θ is a subset of $\boldsymbol{R}^1$. We consider the testing problem

$$H : \theta = \theta_0 \qquad \text{against} \quad K : \theta > \theta_0 \,.$$

In case that the alternative is close to the null hypothesis, we attempt to compare two tests. A method of the comparison of two tests in the local sense was given by Pitman (Noether [5], Pitman [6]). Pitman introduced the concept of asymptotic relative efficiency of two tests by choosing alternative sequences that approach to the null hypothesis. Roughly speaking, his method is as follows. Let $\{T_{1n_1}\}$, $\{T_{2n_2}\}$ be two tests based on n_1, n_2 samples, respectively, and α_{in_i}, $\beta_{in_i}(\theta)$ $(i=1, 2)$ denote the corresponding levels and power functions. For $i=1, 2$ suppose that $\alpha_{in_i} \to \alpha$ $(0<\alpha<1)$ as $n_i \to \infty$, and choose the alternative sequence $\{\theta_{in_i}\}$ approaching to the null hypothesis θ_0 so that $\beta_{in_i}(\theta_{in_i}) \to \beta$ $(0<\beta<1)$ as $n_i \to \infty$. Then Pitman defined the asymptotic relative efficiency (ARE) of $\{T_{2n}\}$ with respect to $\{T_{1n}\}$ as the limit of the ratio n_1/n_2. The superiority or inferiority between $\{T_{1n}\}$ and $\{T_{2n}\}$ in the local sense is decided whether ARE>1 or ARE<1. If ARE$=1$ then we consider the limit of the difference of sample sizes n_2-n_1, what is called Pitman deficiency, as the second measure of comparison of the two tests. In many cases it occurs that $\theta_{in}=\theta_0+k_i/\sqrt{n}$. But this alternative form is not appropriate for the study of deficiency, because approaching to the null hypothesis is coarse. And so we choose the alternative sequence of the form $\theta_{in}=\theta_0+k_i/\sqrt{n}+l_i/n+m_i/(n\sqrt{n})$ $(i=1, 2)$. By expanding the power functions we compare the two tests under these alternative sequences. Here l_i may be related to the case when Pitman deficiency is infinite. In this paper, however, we study the case when Pitman deficiency is finite only.

In section 1 we consider a method of comparison of two tests in such a case that ARE$=1$. In section 2 the method is applied to two examples and in section 3 we refer to the relation between our method and Pitman deficiency.

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§1. Comparison of two tests.

Let $\{P_\theta : \theta \in \Theta\}$ denote a set of probability distributions on $(\boldsymbol{R}^1, \mathscr{B})$, where Θ denotes a parameter space which is an open subset of $\boldsymbol{R}^1$. $\mathscr{B}$ denotes the Borel σ-field on $\boldsymbol{R}^1$. We consider the testing problem

$$H : \theta = \theta_0 \qquad \text{against} \quad K : \theta > \theta_0 .$$

Here θ_0 is a fixed point of Θ. For $i=1, 2$ let $\{T_{in}\}$ be a sequence of test statistics based on n samples and α_{in}, $\beta_{in}(\theta)$ be the corresponding levels and power functions, respectively. Suppose that $\alpha_{in} \to \alpha$ $(0<\alpha<1)$ as $n\to\infty$ and $\beta_{in}(\theta_{in}) \to \beta$ $(0<\beta<1)$ as $n\to\infty$ for the alternative sequence $\theta_{in} = \theta_0 + k_i/\sqrt{n}$ $(i=1, 2)$. Then Pitman's ARE of $\{T_{2n}\}$ with respect to $\{T_{1n}\}$ is given by k_1/k_2 under the appropriate conditions (cf. Noether [5], Pitman [6]). This fact shows the $\{T_{1n}\}$ and $\{T_{2n}\}$ can be compared by comparing k_1 and k_2. If $k_1 < k_2$ $(k_1 > k_2)$ then $\{T_{1n}\}$ $(\{T_{2n}\})$ is superior to $\{T_{2n}\}$ $(\{T_{1n}\})$ in the local sense. This conclusion suggests to us that Pitman's ARE of two tests having the same asymptotic level and the same asymptotic power is measured by the distance from the alternative hypothesis to the null hypothesis. But if $k_1 = k_2$ then we can not compare $\{T_{1n}\}$ with $\{T_{2n}\}$. In this case when we discuss the comparison of two tests by the distance from the alternative hypothesis to the null hypothesis its approach to the null hypothesis is too coarse to compare. Therefore we choose the alternative sequence of the form

$$\theta_n = \theta_0 + \frac{k}{\sqrt{n}} + \frac{l}{n} + \frac{m}{n\sqrt{n}} . \tag{1.1}$$

In many cases, for the alternative θ such that $\sqrt{n}(\theta - \theta_0)$ is bounded the power functions of test statistics $T_n = T_n(X_1, X_2, \cdots, X_n)$ with asymptotic level α are approximated by the normal distribution as follows.

$$\begin{aligned}\beta_n(\theta) = 1 - \Phi(u_\alpha - c_n(\theta)) &+ \phi(u_\alpha - c_n(\theta)) \\ &\times \left\{\frac{1}{\sqrt{n}} s(u_\alpha - c_n(\theta)) + \frac{1}{n} t(u_\alpha - c_n(\theta))\right\} + o(n^{-1}) ,\end{aligned} \tag{1.2}$$

where Φ and ϕ denote the standard normal distribution function and its density function, u_α is upper α-point of Φ, $c_n(\theta) = \sqrt{n}(\theta - \theta_0)c$, c is a constant, and $s(x)$, $t(x)$ are polynomials of x, whose coefficients depend on the third and fourth cumulant of T_n under the alternative θ, multiplied by $n^{1/2}$ and n, respectively. In view of (1.2), and by Taylor expansions, for the alternative sequence $\{\theta_n\}$ given in (1.1) we have

$$\begin{aligned}\beta_n(\theta_n) = 1 - \Phi(u_\alpha - kc) + \phi(u_\alpha - kc) &\left\{\frac{1}{\sqrt{n}} s(u_\alpha, k, l)\right. \\ &\left. + \frac{1}{n} t(u_\alpha, k, l, m)\right\} + o(n^{-1}) ,\end{aligned} \tag{1.3}$$

where s and t are free from n. For example, suppose that the distributions P_θ $(\theta \in \Theta)$ have mean θ and variance one, and let $X_1, X_2, \cdots, X_n$ be independent identically distributed observations from P_θ. Let

$$T_n = \frac{X_1 + X_2 + \cdots + X_n - n\theta_0}{\sqrt{n}}.$$

Then we have the following Edgeworth expansion under the appropriate conditions.

$$\begin{aligned}\beta_n(\theta) = 1 - \Phi(u_\alpha - c_n(\theta)) + \phi(u_\alpha - c_n(\theta)) \Bigg\{ & \frac{1}{\sqrt{n}} \frac{\kappa_3}{6} ((u_\alpha - c_n(\theta))^2 - 1) \\ & + \frac{1}{n} \Bigg(\frac{\kappa_4}{24} ((u_\alpha - c_n(\theta))^3 - 3(u_\alpha - c_n(\theta))) \\ & + \frac{\kappa_3^2}{72} ((u_\alpha - c_n(\theta))^5 - 10(u_\alpha - c_n(\theta))^3 + 15(u_\alpha - c_n(\theta))) \Bigg) \Bigg\} + o(n^{-1}),\end{aligned}$$

where $c_n(\theta) = \sqrt{n}(\theta - \theta_0)$, and κ_3 and κ_4 are respectively the third and fourth cumulant of $X_1 - \theta$ under the alternative θ. By Taylor expansions, for the alternative sequence $\{\theta_n\}$ given in (1.1) we have

$$\begin{aligned}\beta_n(\theta_n) = 1 - \Phi(u_\alpha - k) + \phi(u_\alpha - k) \Bigg\{ & \frac{1}{\sqrt{n}} \left(1 + \frac{\kappa_3}{6} ((u_\alpha - k)^2 - 1) \right) \\ & + \frac{1}{n} \Bigg(m + \frac{1}{2} l^2 (u_\alpha - k) - \frac{\kappa_3 l}{6} ((u_\alpha - k)^2 + 2(u_\alpha - k) - 1) \\ & + \frac{\kappa_4}{24} ((u_\alpha - k)^3 - 3(u_\alpha - k)) \\ & + \frac{\kappa_3^2}{72} ((u_\alpha - k)^5 - 10(u_\alpha - k)^3 + 15(u_\alpha - k)) \Bigg) \Bigg\} + o(n^{-1}).\end{aligned}$$

Therefore

$$s(u_\alpha, k, l) = 1 + \frac{\kappa_3}{6} ((u_\alpha - k)^2 - 1),$$

$$\begin{aligned}t(u_\alpha, k, l, m) = m + \frac{1}{2} l^2 (u_\alpha - k) - \frac{\kappa_3 l}{6} ((u_\alpha - k)^2 + 2(u_\alpha - k) - 1) \\ + \frac{\kappa_4}{24} ((u_\alpha - k)^3 - 3(u_\alpha - k)) + \frac{\kappa_3^2}{72} ((u_\alpha - k)^5 - 10(u_\alpha - k)^3 + 15(u_\alpha - k)).\end{aligned}$$

See Bhattacharya and Rao [3] with respect to the Edgeworth expansions. Albers [1], Albers, Bickel and Zwet [2] give the validity of expansions of power functions for some statistics.

Let T_{1n}, T_{2n} be two test statistics and $\beta_{1n}(\theta_{1n})$, $\beta_{2n}(\theta_{2n})$ be corresponding power functions for the alternative sequence $\{\theta_{in}\}$ given in (1.1) with k_i, l_i, m_i $(i=1,2)$. Suppose that $\beta_{in}(\theta_{in})$ satisfy (1.2) for $i=1,2$.

$$\beta_{in}(\theta_{in})=1-\Phi(u_\alpha-k_ic)+\phi(u_\alpha-k_ic)\left\{\frac{1}{\sqrt{n}}s_i(u_\alpha,k_i,l_i)\right.$$
$$\left.+\frac{1}{n}t_i(u_\alpha,k_i,l_i,m_i)\right\}+o(n^{-1}).$$

If T_{1n} and T_{2n} have the same asymptotic power then $k_i=k$ $(i=1,2)$. Put $s_1(u_\alpha,k,l_1)=s_2(u_\alpha,k,l_2)$ and $t_1(u_\alpha,k,l_1,m_1)=t_2(u_\alpha,k,l_2,m_2)$. By these relations we will obtain the relations between k, l and m. We assert that if $l_1<l_2$ or $l_1>l_2$ then $\{T_{1n}\}$ and $\{T_{2n}\}$ are distinguishable in the sense of approaching order $1/n$, if $l_1=l_2$ and $m_1\neq m_2$ then $\{T_{1n}\}$ and $\{T_{2n}\}$ are distinguishable in the sense of approaching order $1/n(\sqrt{n})$.

This method shows that the comparison of two tests in such a case that ARE$=1$ can be done more plainly by taking measurements with the distance from the alternative hypothesis to the null hypothesis.

§2. Examples.

In this section we give two examples. In the first example we compare the envelop power with the power of the locally most powerful test.

Example 2.1. Let $X_1, X_2, \cdots, X_n$ be i.i.d random variables with distribution function $F(x-\theta)$, $\theta\in \boldsymbol{R}^1$. Let $f(x)$ be the density function of $F(x)$, and be symmetric about zero and positive on $\boldsymbol{R}^1$, and five times differentiable. We consider the testing problem

$$H:\theta=0 \qquad \text{against} \quad K:\theta>0.$$

Let T_{1n} be the test based on

$$\sum_{i=1}^{n}\log\{f(X_i-\theta_{1n})/f(X_i)\},$$

where $\{\theta_{1n}\}$ is the sequence of alternatives satisfying (1.1), and let T_{2n} be the test based on

$$\sum_{i=1}^{n}f'(X_i)/f(X_i).$$

Albers [1] gives the Edgeworth expansions of the power functions for the tests T_{1n} and

T_{2n} in details, provided that the density function $f(x)$ satisfies additional appropriate regularity conditions and $\sqrt{n}\,\theta$ is bounded. The expansions are as follows.

$$\text{(2.1)} \qquad \beta_{in}(\theta_{in}) = 1 - \Phi(u_\alpha - a_i) + \frac{a_i}{n}(b_{i1}u_\alpha^2 + b_{i2}u_\alpha a_i + b_{i3} + b_{i4}a_i^2)\phi(u_\alpha - a_i) + O(n^{-3/2}),$$

where

$a_i = \theta_{in}(nE_0(\psi_1^2(X_1)))^{1/2} \qquad (i=1, 2)$,

$a_3 = E_0(\psi_1^4(X_1))/\{E_0(\psi_1^2(X_1))\}^2$,

$a_4 = E_0(\psi_2^2(X_1))/\{E_0(\psi_1^2(X_1))\}^2$,

$\psi_j(X_1) = f^{(j)}(X_1)/f(X_1) \qquad (j=1, 2)$,

E_0 denotes the expectation under the null hypothesis,

$f^{(j)}$ denotes j-th derivative of f,

$b_{11} = b_{21} = -(a_3 - 3)/24$,

$b_{12} = b_{22} = -(a_3 - 3)/24$,

$b_{13} = b_{23} = -(a_3 - 3)/24$,

$b_{14} = (2a_3 - 3a_4)/72$,

$b_{24} = (5a_3 - 12a_4 + 9)/72$.

In view of (2.1), we obtain that for $i=1, 2$,

$$\begin{aligned}
\beta_{in}(\theta_{in}) &= 1 - \Phi\left(u_\alpha - \left(k_i + \frac{l_i}{\sqrt{n}} + \frac{m_i}{n}\right)c\right) + \frac{c}{n}\left(k_i + \frac{l_i}{\sqrt{n}} + \frac{m_i}{n}\right) \\
&\quad \times \phi\left(u_\alpha - \left(k_i + \frac{l_i}{\sqrt{n}} + \frac{m_i}{n}\right)c\right)\left\{b_{i1}u_\alpha^2 + b_{i2}u_\alpha\left(k_i + \frac{l_i}{\sqrt{n}} + \frac{m_i}{n}\right)c\right. \\
&\quad \left. + b_{i3} + \left(k_i + \frac{l_i}{\sqrt{n}} + \frac{m_i}{n}\right)^2 c^2 b_{i4}\right\} + O(n^{-3/2}) \\
&= 1 - \Phi(u_\alpha - k_i c) + \phi(u_\alpha - k_i c)\left\{\frac{l_i c}{\sqrt{n}} + \frac{1}{n}\left(m_i c + \frac{1}{2}l_i^2 c^2(u_\alpha - k_i c)\right.\right. \\
&\quad \left.\left. + k_i c(b_{i1}u_\alpha^2 + b_{i2}u_\alpha k_i c + b_{i3} + k_i^2 c^2 b_{i4})\right)\right\} + O(n^{-3/2}),
\end{aligned}$$

where $c = \{E_0(\psi_1^2(X_1))\}^{1/2}$. Since $\{T_{1n}\}$ and $\{T_{2n}\}$ have the same asymptotic power we obtain that $k_1=k_2$. Put $k_1=k_2=k$. Let $1/\sqrt{n}$-terms in $\beta_{1n}(\theta_{1n})$ coincide with $1/\sqrt{n}$-terms in $\beta_{2n}(\theta_{2n})$, and similarly we do also about $1/n$-terms. We obtain that

$$\begin{aligned} &l_1=l_2\,,\\ &m_1-m_2=k\{(b_{21}-b_{11})u_\alpha^2+(b_{22}-b_{12})ku_\alpha c+(b_{23}-b_{13})+(b_{24}-b_{14})k^2c^2\}\\ &\qquad\qquad=(a_3-3a_4+3)k^3c^2/24\,. \end{aligned}$$

REMARK. By easy calculations we observe that

$$a_3-3a_4+3=-3V_0(\psi_1'(X_1))\,,$$

where V_0 denotes the variance under the null hypothesis. Therefore we obtain that $m_1\leqq m_2$.

EXAMPLE 2.2. Let $X_1, X_2, \cdots, X_n$ be i.i.d random variables with normal distribution with mean θ and variance one. We consider the testing problem

$$H\,;\,\theta=0 \qquad \text{against} \quad K:\theta>0\,.$$

We consider two tests as follows.

$$T_{1n}=\sqrt{n}\,\bar{X}_n\,, \qquad T_{2n}=\sqrt{n}\,\bar{X}_n/s_n\,,$$

where

$$\bar{X}_n=\frac{1}{n}\sum_{i=1}^n X_i\,, \qquad s_n^2=\sum_{i=1}^n (X_i-\bar{X}_n)^2/(n-1)\,.$$

The critical region of T_{1n} with level α is given by

$$T_{1n}\geqq u_\alpha\,,$$

where $u_\alpha=\Phi^{-1}(1-\alpha)$. Its power function is as follows.

$$\begin{aligned} \beta_{1n}(\theta_{1n})&=P_{\theta_{1n}}(T_{1n}\geqq u_\alpha)\\ &=1-\Phi(u_\alpha-\sqrt{n}\,\theta_{1n})\\ &=1-\Phi(u_\alpha-k_1)+\frac{l_1}{\sqrt{n}}\phi(u_\alpha-k_1)\\ &\quad+\frac{1}{n}\left\{m_1-\frac{(u_\alpha-k_1)l_1^2}{2}\right\}\phi(u_\alpha-k_1)+O(n^{-3/2})\,. \end{aligned} \tag{2.2}$$

Next, the critical region of T_{2n} is given by

$$T_{2n}\geqq c_n\,,$$

where c_n satisfy that $P_0(T_{2n}\geqq c_n)=\alpha$. The power function of t-test is calculated by

Hodges and Lehmann [4]. They give the normal approximation of the power function as follows.

$$\begin{aligned}(2.3)\qquad \beta_{2n}(\theta_{2n}) &= 1-E(\Phi(c_n s_n-\sqrt{n}\,\theta_{2n}))\\ &= 1-\Phi\left(u_\alpha-\sqrt{n}\,\theta_{2n}\left(1-\frac{u_\alpha^2}{4n}\right)\right)+O(n^{-2})\\ &= 1-\Phi(u_\alpha-k_2)+\frac{l_2}{\sqrt{n}}\phi(u_\alpha-k_2)-\frac{1}{n}\left(\frac{k_2u_\alpha^2-4m_2}{4}\right.\\ &\quad \left.+\frac{l_2^2(u_\alpha-k_2)}{2}\right)\phi(u_\alpha-k_2)+O(n^{-3/2})\,.\end{aligned}$$

In view of (2.2) and (2.3) it must be $k_1=k_2$, because $\beta_{in}(\theta_{in})\to\beta$ as $n\to\infty$ for $i=1, 2$. Put $k_1=k_2=k$. We compare (2.2) with (2.3) in the same way as Example 2.1, and we obtain that

$$\begin{aligned}l_1&=l_2\,,\\ m_1-m_2&=-\frac{ku_\alpha^2}{4}\,.\end{aligned}$$

§3. Relation to Pitman deficiency.

We consider the following testing problem.

$$H:\theta=\theta_0 \qquad \text{against} \quad K:\theta>\theta_0\,.$$

Suppose that for $i=1, 2$ the sequence $\{T_{in}\}$ of test statistics and the sequence $\{c_{in}\}$ of real numbers satisfy that

$$(3.1)\qquad \alpha_{in}=P_{\theta_0}(T_{in}\geqq c_{in})\longrightarrow\alpha \qquad (0<\alpha<1)$$

as $n\to\infty$ and following Edgeworth type expansions are permitted with the alternative θ such as $\sqrt{n}\,(\theta-\theta_0)$ is bounded.

$$\begin{aligned}(3.2)\qquad \beta_{in}(\theta)&=P_\theta(T_{in}\geqq c_{in})\\ &=1-\Phi(u_\alpha-c_n(\theta))+\left\{\frac{1}{\sqrt{n}}s_i(u_\alpha, c_n(\theta))\right.\\ &\quad\left.+\frac{1}{n}t_i(u_\alpha, c_n(\theta))\right\}\phi(u_\alpha-c_n(\theta))+O(n^{-3/2})\,,\end{aligned}$$

where $c_n(\theta)=\sqrt{n}\,(\theta-\theta_0)c$, c is a constant, u_α is upper α-point of Φ, and $s(x, y)$ and $t(x, y)$ are polynomials of x, y.

THEOREM 3.1. *Suppose that for* $i=1, 2$, $\{T_{in}\}$ *satisfy* (3.1) *and* (3.2), *and* $\beta_{in}(\theta_{in})\to\beta$ $(0<\beta<1)$ *as* $n\to\infty$ *for* $\theta_{in}=\theta_0+k_i/\sqrt{n}$. *If* $s_1=s_2$ *then Pitman deficiency, denoting it as* d, *of* $\{T_{2n}\}$ *with respect to* $\{T_{1n}\}$ *is finite, and*

$$d=\frac{2(t_2(u_\alpha, kc)-t_1(u_\alpha, kc))}{kc},$$

where k *satisfies that* $1-\Phi(u_\alpha-kc)=\beta$.

PROOF. Let $\theta_{in_i}=\theta_0+k_i/\sqrt{n_i}$ $(i=1, 2)$. For $i=1, 2$ we obtain that

$$\begin{aligned}(3.3)\qquad \beta_{in_i}(\theta_{in_i}) &= 1-\Phi(u_\alpha-\sqrt{n_i}(\theta_{in_i}-\theta_0)c)\\ &\quad+\left\{\frac{1}{\sqrt{n_i}}s_i(u_\alpha, \sqrt{n_i}(\theta_{in_i}-\theta_0)c)+\frac{1}{n_i}t_i(u_\alpha, \sqrt{n_i}(\theta_{in_i}-\theta_0)c)\right\}\\ &\quad\times\phi(u_\alpha-\sqrt{n_i}(\theta_{in_i}-\theta_0)c)+O(n_i^{-3/2})\\ &=1-\Phi(u_\alpha-k_ic)+\phi(u_\alpha-k_ic)\left\{\frac{1}{\sqrt{n_i}}s_i(u_\alpha, k_ic)+\frac{1}{n_i}t_i(u_\alpha, k_ic)\right\}\\ &\quad+O(n_i^{-3/2}).\end{aligned}$$

Let n_2^* be the solution of equation $\beta_{2n_2}(\theta_{2n_2})=\beta_{1n_1}(\theta_{1n_1})$ under the condition such as $\theta_{2n_2}=\theta_{1n_1}$, and define d_n as $d_n=n_2^*-n_1$. By equation $\theta_{1n_1}=\theta_{2n_2^*}$, we observe that

$$k_2=\sqrt{\frac{n_1+d_n}{n_1}}k_1.$$

In view of (3.3), we observe that

$$\begin{aligned}(3.4)\qquad \beta_{2n_2^*}(\theta_{2n_2^*}) &= 1-\Phi\left(u_\alpha-\sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)\\ &\quad+\phi\left(u_\alpha-\sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)\left\{\frac{1}{\sqrt{n_1+d_n}}s_2\left(u_\alpha, \sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)\right.\\ &\quad\left.+\frac{1}{n_1+d_n}t_2\left(u_\alpha, \sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)\right\}+O(n_1^{-3/2}).\end{aligned}$$

By using Taylor expansions we observe that

$$\sqrt{\frac{n_1+d_n}{n_1}}=\left(1+\frac{d_n}{n_1}\right)^{1/2}=1+\frac{d_n}{2n_1}+O(n_1^{-2}),$$

$$\frac{1}{\sqrt{n_1+d_n}}=\frac{1}{\sqrt{n_1}}\left(1+\frac{d_n}{n_1}\right)^{-1/2}=\frac{1}{\sqrt{n_1}}+O(n_1^{-3/2}),$$

$$\frac{1}{n_1+d_n}=\frac{1}{n_1}\left(1+\frac{d_n}{n_1}\right)^{-1}=\frac{1}{n_1}+O(n_1^{-2}),$$

$$\begin{aligned}\Phi\left(u_\alpha-\sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)&=\Phi\left(u_\alpha-k_1c-\frac{k_1cd_n}{2n_1}+O(n_1^{-2})\right)\\&=\Phi(u_\alpha-k_1c)-\frac{d_nk_1c}{2n_1}\phi(u_\alpha-k_1c)+O(n_1^{-2}),\end{aligned}$$

$$\begin{aligned}\phi\left(u_\alpha-\sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)&=\phi\left(u_\alpha-k_1c-\frac{d_nk_1c}{2n_1}+O(n_1^{-2})\right)\\&=\phi(u_\alpha-k_1c)+O(n_1^{-1}),\end{aligned}$$

$$\begin{aligned}s_2\left(u_\alpha,\sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)&=s_2\left(u_\alpha,k_1c+\frac{d_nk_1c}{2n_1}+O(n_1^{-2})\right)\\&=s_2(u_\alpha,k_1c)+O(n_1^{-1}),\end{aligned}$$

$$\begin{aligned}t_2\left(u_\alpha,\sqrt{\frac{n_1+d_n}{n_1}}k_1c\right)&=t_2\left(u_\alpha,k_1c+\frac{d_nk_1c}{2n_1}+O(n_1^{-2})\right)\\&=t_2(u_\alpha,k_1c)+O(n_1^{-1}).\end{aligned}$$

In view of (3.4) we observe that

$$\begin{aligned}\beta_{2n_2^*}(\theta_{2n_2^*})=1-\Phi(u_\alpha-k_1c)+\phi(u_\alpha-k_1c)&\left\{\frac{1}{\sqrt{n_1}}s_2(u_\alpha,k_1c)\right.\\&\left.+\frac{1}{n_1}\left(t_2(u_\alpha,k_1c)-\frac{d_nk_1c}{2}\right)\right\}+O(n_1^{-3/2}).\end{aligned}$$

It must be that $k_1=k_2$, because $\{T_{1n}\}$ and $\{T_{2n}\}$ have the same asymptotic power. Put $k_1=k_2=k$. By equation $\beta_{1n_1}(\theta_{1n_1})=\beta_{2n_2^*}(\theta_{2n_2^*})$, we obtain that

$$(3.5)\qquad \frac{1}{\sqrt{n_1}}s_1(u_\alpha, kc)+\frac{1}{n_1}t_1(u_\alpha, kc)$$
$$=\frac{1}{\sqrt{n_1}}s_2(u_\alpha, kc)+\frac{1}{n_1}\left(t_2(u_\alpha, kc)-\frac{d_n kc}{2}\right)+O(n_1^{-3/2}).$$

In view of (3.5) if $s_1=s_2$ then we obtain that

$$d_n=\frac{2(t_2(u_\alpha, kc)-t_1(u_\alpha, kc))}{kc}+O(n_1^{-1/2}).$$

The proof has been completed.

REMARK. In Theorem 3.1, n_2 is not necessarily an integer. But by stochastic interpolation we can avoid the difficult situation. That is to say, we define $\beta_{2n_2^*}(\theta_{2n_2^*})$ as follows.

$$\beta_{2n_2^*}(\theta_{2n_2^*})=(1-n_2^*+[n_2^*])\beta_{2[n_2^*]}(\theta_{2[n_2^*]})+(n_2^*-[n_2^*])\beta_{2[n_2^*]+1}(\theta_{2[n_2^*]+1}),$$

where $[x]$ denotes the integer part of x (cf. Hodges and Lehmann [4]).

Let $\theta_{in}=\theta_0+k_i/\sqrt{n}+l_i/n+m_i/(n\sqrt{n})$ $(i=1, 2)$. If two tests $\{T_{1n}\}$ and $\{T_{2n}\}$ have the same asymptotic power then $k_1=k_2$ by the discussion in the proof of Theorem 3.1.

THEOREM 3.2. *Let* $\theta_{in}=\theta_0+k/\sqrt{n}+l_i/n+m_i/(n\sqrt{n})$ $(i=1, 2)$. *Suppose that, for* $i=1, 2$, $\{T_{in}\}$ *satisfy* (3.1), (3.2), *and* $\beta_{in}(\theta_{in})\to\beta$ $(0<\beta<1)$ *as* $n\to\infty$. *Let d denote Pitman deficiency of* $\{T_{2n}\}$ *with respect to* $\{T_{1n}\}$. *If* $s_1=s_2$ *then*

$$d=\frac{2(m_1-m_2)}{k},$$

where k satisfies that $\beta=1-\Phi(u_\alpha-kc)$.

PROOF. In view of (3.2) we obtain that for $i=1, 2$

$$\beta_{in}(\theta_{in})=1-\Phi(u_\alpha-\sqrt{n}(\theta_{in}-\theta_0)c)+\left\{\frac{1}{\sqrt{n}}s_i(u_\alpha, \sqrt{n}(\theta_{in}-\theta_0)c)\right.$$
$$\left.+\frac{1}{n}t_i(u_\alpha, \sqrt{n}(\theta_{in}-\theta_0)c)\right\}\phi(u_\alpha-\sqrt{n}(\theta_{in}-\theta_0)c)+O(n^{-3/2}).$$

Similarly as the proof of Theorem 3.1, using Taylor expansions we obtain that

$$\beta_{in}(\theta_{in})=1-\Phi(u_\alpha-kc)+\frac{1}{\sqrt{n}}(l_ic+s_i(u_\alpha, kc))\phi(u_\alpha-kc)$$
$$+\frac{1}{n}\left\{m_ic+\frac{(u_\alpha-kc)l_i^2c^2}{2}+l_ic(u_\alpha-kc)s_i(u_\alpha, kc)\right.$$

$$\left.+l_i cs_i'(u_\alpha, kc)+t_i(u_\alpha, kc)\right\}\phi(u_\alpha-kc)+O(n^{-3/2}) .$$

Let $1/\sqrt{n}$-terms ($1/n$-terms) in $\beta_{1n}(\theta_{1n})$ coincide with $1/\sqrt{n}$-terms ($1/n$-terms) in $\beta_{2n}(\theta_{2n})$. It follows that

$$l_1 c+s_1(u_\alpha, kc)=l_2 c+s_2(u_\alpha, kc) ,$$

$$m_1 c+\frac{1}{2}(u_\alpha-kc)l_1^2c^2+l_1c(u_\alpha-kc)s_1(u_\alpha, kc)+l_1cs_1'(u_\alpha, kc)+t_1(u_\alpha, kc)$$

$$=m_2 c+\frac{1}{2}(u_\alpha-kc)l_2^2c^2+l_2c(u_\alpha-kc)s_2(u_\alpha, kc)+l_2cs_2'(u_\alpha, kc)+t_2(u_\alpha, kc) .$$

Since $s_1=s_2$, it follows that $l_1=l_2$, and

$$m_1-m_2=\frac{t_2(u_\alpha, kc)-t_1(u_\alpha, kc)}{c}=\frac{kd}{2} .$$

The proof has been completed.

By applying Theorem 3.2 to Example 2.1, we have

$$d=\frac{2(m_1-m_2)}{k}=\frac{(-a_3+3a_4-3)k^2c}{12} .$$

This value coincides with the asymptotic deficiency given by Albers [1].

For Example 2.2 we obtain that

$$d=\frac{2(m_1-m_2)}{k}=-\frac{u_\alpha^2}{2} .$$

This value coincides with the value given by Hodges and Lehmann [4].

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References

[1] W. ALBERS, *Asymptotic Expansions and the Deficiency Concept in Statistics*, Math. Centrum Tracts, **59** (1974).

[2] W. ALBERS, P. J. BICKEL and W. R. VAN ZWET, Asymptotic expansions for the power of distribution free tests in the one-sample problem, Ann. Statist., **4** (1976), 108–156.

[3] R. N. BHATTACHARYA and R. R. RAO, *Normal Approximation and Asymptotic Expansions*, Wiley (1976).

[4] J. L. HODGES and E. L. LEHMANN, Deficiency, Ann. Math. Statist., **41** (1970), 783–801.

[5] G. E. NOETHER, On a theorem of Pitman, Ann. Math. Statist., **26** (1955), 64–68.

[6] E. J. G. PITMAN, *Some Basic Theory for Statistical Inference*, Chapman and Hall (1979).

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