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Initial Boundary Value Problem for the Wave Equation in a Domain with a Corner

Masaru TANIGUCHI and Fukuzo SUZUKI

Waseda University and Gunma College of Technology
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Introduction.

In this paper, we consider the mixed problem for the wave equation in the domains $\{(t, x, y, w) \mid t>0, x>0, y>0, w \in \boldsymbol{R}^n\}$ and $\{(t, x, y, z, w) \mid t>0, x>0, y>0, z>0, w \in \boldsymbol{R}^n\}$. On the boundary $x=0$, an oblique boundary condition is given and on the other boundary ($y=0$ or $z=0$), the Dirichlet or Neumann boundary condition is given. Such a problem was considered in [1], [2], [5], [14], [16] and [17].

The aim of this paper is to give the energy inequality for the above problem with non-homogeneous boundary condition, with the aid of which we can prove an existence and uniqueness theorem. Our result is an extension of the result in [1], [16] and [17]. This energy estimate is the same as the one for the mixed problem in the domain with smooth boundary. The similar result is obtained only in [16] and [17].

Our method is to reduce the mixed problem for the wave equation in a domain with a corner to the one for symmetric hyperbolic systems of first order with positive (or non-negative) boundary condition on the boundaries. This method is discovered in [27], and is given in Appendix. Such a method is treated and developed in [18], [7], [19], [20], [22], [24] and [26], and used in [17], [23] and [25]. In [27], we improved the results in [19: §4] and [26: §3], and obtain our results by the use of its reformation in [27].

As for the other results on the mixed problem for hyperbolic equations in a domain with non-smooth boundary, we can refer to [6], [13], [15] and [21].

As for the other symmetrization on the Cauchy problem and the mixed problem for hyperbolic equation, we can refer to [3], [4], [8], [9], [10] and [11].

An outline of this paper is as follows. In §1, we give the notations. In §2, we state the problems and the results. In §3, we discuss the symmetrization of the mixed problems (P.1), (P.2) and (P.3). In §4, we treat the boundary estimate of the solution. In §5, we prove Main Theorem 1. In §6, we prove Main Theorem 2. In §7, we prove Main

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Theorem 3. In Appendix, we state a result in [27].

§1. Notation.

$\boldsymbol{R}^m$ ($\boldsymbol{C}^m$) : m-dimensional real (complex) Euclidean space.

$\boldsymbol{R}^m_+$: the set $\{(x, y) \mid x>0, y\in \boldsymbol{R}^{m-1}\}$.

$[\,,\,]$: the inner product in $\boldsymbol{C}^m$.

$$(u, v)=\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdydx \quad\text{or}\quad \int_0^\infty\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdzdydx .$$

$$\langle u, v\rangle=\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdy \quad\text{or}\quad \int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdzdy .$$

$$\langle\langle u, v\rangle\rangle=\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdx \quad\text{or}\quad \int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdzdx .$$

$$\langle\langle\langle u, v\rangle\rangle\rangle=\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} u\cdot\bar{v}\, dwdydx .$$

$$\|u\|^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+\delta+|\rho|=m}\int_0^T\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} \left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdydxdt$$

$$\text{or}\quad \sum_{\alpha+\beta+\gamma+\delta+\theta+|\rho|=m}\int_0^T\int_0^\infty\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} \left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial z}\right)^\theta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdzdydxdt .$$

$$\langle u\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+|\rho|=m}\int_0^T\int_0^\infty\int_{\boldsymbol{R}^n} \left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial y}\right)^\gamma\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdydt$$

$$\text{or}\quad \sum_{\alpha+\beta+\gamma+\delta+|\rho|=m}\int_0^T\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n} \left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial y}\right)^\gamma\left(\frac{\partial}{\partial z}\right)^\delta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdzdydt .$$

$$\langle\langle u\rangle\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+|\rho|=m}\int_0^T\int_0^\infty\int_{\boldsymbol{R}^n} \left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdxdt$$

or $$\sum_{\alpha+\beta+\gamma+\delta+|\rho|=m}\int_0^T\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n}\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial z}\right)^\delta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdzdxdt\,.$$

$\langle\!\langle u\rangle\!\rangle_m$ $\cdots$ the norm of Sobolev space $H_m(\boldsymbol{R}^1_{x+}\times\boldsymbol{R}^n_w)$

or the norm of Sobolev space $H_m(\boldsymbol{R}^1_{x+}\times\boldsymbol{R}^1_{z+}\times\boldsymbol{R}^n_w)$.

$$\langle\!\langle\!\langle u\rangle\!\rangle\!\rangle_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+\delta+|\rho|=m}\int_0^T\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n}\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdydxdt\,.$$

$\langle\!\langle\!\langle u\rangle\!\rangle\!\rangle_m$ $\cdots$ the norm of Sobolev space $H_m(\boldsymbol{R}^1_{x+}\times\boldsymbol{R}^1_{y+}\times\boldsymbol{R}^n_w)$.

$$|||u(t)|||^2_{m,\mu}=\sum_{\alpha+\beta+\gamma+\delta+|\rho|=m}\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n}\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdydx$$

or $$\sum_{\alpha+\beta+\gamma+\delta+\theta+|\rho|=m}\int_0^\infty\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n}\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial z}\right)^\theta\left(\frac{\partial}{\partial w}\right)^\rho u\right|^2 dwdzdydx\,.$$

$$\mathscr{F}_{(x,w)}u=\int_{-\infty}^\infty\int_{\boldsymbol{R}^n}e^{-(ix\cdot\xi+iw\cdot\zeta)}u(x,w)\,dwdx$$

$$(w=(w_1,\cdots,w_n)\,,\quad \zeta=(\zeta_1,\cdots,\zeta_n)\,,\quad w\cdot\zeta=\textstyle\sum_{j=1}^n w_j\zeta_j)\,.$$

$$\Lambda^{\pm\theta}_{x,w,\mu}=\bar{\mathscr{F}}_{(x,w)}((\xi^2+|\zeta|^2+\mu^2)^{\pm\theta/2})\mathscr{F}_{(x,w)}\quad(|\zeta|=(|\zeta_1|^2+\cdots+|\zeta_n|^2)^{1/2})\,.$$

$\mathscr{H}_{m,\mu}[(\boldsymbol{R}^1_+)^l\times\boldsymbol{R}^n]$: the space of functions which are obtained by the completion of $C_0^\infty[(\overline{\boldsymbol{R}^1_+})^l\times\boldsymbol{R}^n]$ with the norm $\|u\|_{m,\mu,\infty}$ $(l=3, 4)$.

§2. The statement of the problem and the result.

We consider the mixed problems

$$\begin{cases} L_1[u]=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}-\displaystyle\sum_{j=1}^n\dfrac{\partial^2 u}{\partial w_j^2}+m(t,x,y,w)u=f(t,x,y,w) \\ u(0,x,y,w)=u_0(x,y,w)\,,\qquad u_t(0,x,y,w)=u_1(x,y,w) \end{cases}$$

$$
\text{(P.1)}\quad \begin{cases} B_1[u]|_{x=0} = \dfrac{\partial u}{\partial x} + b(t, y, w)\dfrac{\partial u}{\partial y} + \sum_{j=1}^{n} d_j(t, y, w)\dfrac{\partial u}{\partial w_j} \\ \qquad\qquad -c(t, y, w)\dfrac{\partial u}{\partial t} + \gamma(t, y, w)u|_{x=0} = g(t, y, w) \\ B_2[u]|_{y=0} = u|_{y=0} = h(t, x, w) \\ (t, x, y, w) \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^n , \end{cases}
$$

$$
\text{(P.2)}\quad \begin{cases} L_1[u] = f(t, x, y, w) \\ u(0, x, y, w) = u_0(x, y, w), \qquad u_t(0, x, y, w) = u_1(x, y, w) \\ B_1[u]|_{x=0} = g(t, y, w) \\ B_3[u]|_{y=0} = \left.\dfrac{\partial u}{\partial y}\right|_{y=0} = h(t, x, w) \\ (t, x, y, w) \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^n \end{cases}
$$

and

$$
\text{(P.3)}\quad \begin{cases} L_2[u] = \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial^2 u}{\partial z^2} - \sum_{j=1}^{n} \dfrac{\partial^2 u}{\partial w_j^2} + m(t, x, y, z, w)u = f(t, x, y, z, w) \\ u(0, x, y, z, w) = u_0(x, y, z, w), \qquad u_t(0, x, y, z, w) = u_1(x, y, z, w) \\ B_4[u]|_{x=0} = \dfrac{\partial u}{\partial x} + b_1(t, y, z, w)\dfrac{\partial u}{\partial y} + b_2(t, y, z, w)\dfrac{\partial u}{\partial z} + \sum_{j=1}^{n} d_j(t, y, z, w)\dfrac{\partial u}{\partial w_j} \\ \qquad\qquad -c(t, y, z, w)\dfrac{\partial u}{\partial t} + \gamma(t, y, z, w)u|_{x=0} = g(t, y, z, w) \\ B_5[u]|_{y=0} = u|_{y=0} = h(t, x, z, w) \\ B_6[u]|_{z=0} = u|_{z=0} = k(t, x, y, w) \\ (t, x, y, z, w) \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^n , \end{cases}
$$

where $w = (w_1, w_2, \cdots, w_n)$ $(n \geq 1)$, the coefficient m belongs to $\mathscr{B}^0((\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n)$ $(\mathscr{B}^0((\overline{\boldsymbol{R}^1_+})^4 \times \boldsymbol{R}^n))$ and is constant outside a compact set in $(\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n$ $((\overline{\boldsymbol{R}^1_+})^4 \times \boldsymbol{R}^n)$, the coefficients b, d_j, c and γ in (P.1) and (P.2) belong to $\mathscr{B}^1((\overline{\boldsymbol{R}^1_+})^2 \times \boldsymbol{R}^n)$ and are constant outside a compact set in $(\overline{\boldsymbol{R}^1_+})^2 \times \boldsymbol{R}^n$, the coefficients b_1, b_2, d_j, c and γ in (P.3) belong to $\mathscr{B}^1((\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n)$ and are constant outside a compact set in $(\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n$.

We assume the following condition for the problem (P.1) or (P.2),

$$
\text{(A.I)}\qquad |1+c| - |1-c| > 2\left\{|b|^2 + \sum_{j=1}^{n} |d_j|^2\right\}^{1/2}
$$

for all $(t, y, w) \in \overline{\boldsymbol{R}^1_+} \times \overline{\boldsymbol{R}^1_+} \times \boldsymbol{R}^n$.

We assume the following condition for the problem (P.3), (A.II) $$|1+c|-|1-c|>2\left\{|b_1|^2+|b_2|^2+\sum_{j=1}^{n}|d_j|^2\right\}^{1/2}$$

for all $(t, y, z, w)\in\overline{\boldsymbol{R}_+^1}\times\overline{\boldsymbol{R}_+^1}\times\overline{\boldsymbol{R}_+^1}\times\boldsymbol{R}^n$.

REMARK 1. Assume the condition (A.I) ((A.II)). Then, the boundary operator B_1 (B_4) satisfies the uniform Lopatinski condition.

Now, we state our results.

MAIN THEOREM 1. *Assume the condition* (A.I). *Then, there exist positive constants* C *and* μ_0 *such that, for a solution* u *of* (P.1) *which belongs to* $\mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^3\times\boldsymbol{R}^n]$, *the following energy inequality holds,*

(2.1) $$|||u(t)|||_{1,\mu}^2+\mu\|u\|_{1,\mu,t}^2+\langle u\rangle_{1,\mu,t}^2+\langle\langle u\rangle\rangle_{1,\mu,t}^2 \leq C\left\{|||u(0)|||_{1,\mu}^2+\frac{1}{\mu}\|f\|_{0,\mu,t}^2+\langle g\rangle_{0,\mu,t}^2+\langle\langle h\rangle\rangle_{1,\mu,t}^2\right\}$$

for all $t\in\boldsymbol{R}_+^1$ *and all* $\mu\geq\mu_0$.

REMARK 2. Under the condition

$$|1-c|-|1-c|\geq 2\left\{|b|^2+\sum_{j=1}^{n}|d_j|^2\right\}^{1/2}$$

for all $(t, y, w)\in\overline{\boldsymbol{R}_+^1}\times\overline{\boldsymbol{R}_+^1}\times\boldsymbol{R}^n$, we hope to solve the problem (P.1) in future.

MAIN THEOREM 2. *Assume the condition* (A.I). *Then, there exist positive constants* C *and* μ_0 *such that, for a solution* u *of* (P.2) *which belongs to* $\mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^3\times\boldsymbol{R}^n]$, *the following energy inequality holds,*

(2.2) $$|||u(t)|||_{1,\mu}^2+\mu\|u\|_{1,\mu,t}^2+\langle u\rangle_{1,\mu,t}^2+\mu\sum_{j=0}^{1}\left\langle\left\langle\Lambda_{x,w,\mu}^{-1/2}\left(\frac{\partial}{\partial y}\right)^j u\right\rangle\right\rangle_{1-j,\mu,t}^2 \leq C\left\{|||u(0)|||_{1,\mu}^2+\frac{1}{\mu}\|f\|_{0,\mu,t}^2+\langle g\rangle_{0,\mu,t}^2+\frac{1}{\mu}\langle\langle\Lambda_{x,w,\mu}^{1/2}h\rangle\rangle_{0,\mu,t}^2\right\}$$

for all $t\in\boldsymbol{R}_+^1$ *and all* $\mu\geq\mu_0$.

REMARK 3. For the problem (P.2) with homogeneous boundary conditions, the condition (A.I) is replaced by the following conditions:

$$|1-c|-|1-c|\geq 2\left\{|b|^2+\sum_{j=1}^{n}|d_j|^2\right\}^{1/2}$$

and

$$(|\tilde{b}|, c)\neq(1, 1) \qquad \left(|\tilde{b}|=\left(|b|^2+\sum_{j=1}^{n}|d_j|^2\right)^{1/2}\right)$$

for all $(t, y, w) \in \overline{\boldsymbol{R}_+^1} \times \overline{\boldsymbol{R}_+^1} \times \boldsymbol{R}^n$. Then, we have

$$\text{(2.3)} \qquad |||u(t)|||_{1,\mu}^2 + \mu \| u \|_{1,\mu,t}^2 \leq C\left\{ |||u(0)|||_{1,\mu}^2 + \frac{1}{\mu}\|f\|_{0,\mu,t}^2 \right\}.$$

MAIN THEOREM 3. *Assume the condition* (A.II). *Then, there exist positive constants* C *and* μ_0 *such that, for a solution* u *of* (P.3) *which belongs to* $\mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^4 \times \boldsymbol{R}^n]$, *the following energy inequality holds,*

$$\text{(2.4)} \qquad |||u(t)|||_{1,\mu}^2 + \mu \| u \|_{1,\mu,t}^2 + \langle u \rangle_{1,\mu,t}^2 + \langle\langle u \rangle\rangle_{1,\mu,t}^2 + \langle\langle\langle u \rangle\rangle\rangle_{1,\mu,t}^2$$
$$\leq C\left\{ |||u(0)|||_{1,\mu}^2 + \frac{1}{\mu}\|f\|_{0,\mu,t}^2 + \langle g \rangle_{0,\mu,t}^2 + \langle\langle h \rangle\rangle_{1,\mu,t}^2 + \langle\langle\langle k \rangle\rangle\rangle_{1,\mu,t}^2 \right\}$$

for all $t \in \boldsymbol{R}_+^1$ *and all* $\mu \geq \mu_0$.

REMARK 4. Under the condition

$$|1+c| - |1-c| \geq 2\left\{ |b_1|^2 + |b_2|^2 + \sum_{j=1}^{n} |d_j|^2 \right\}^{1/2}$$

for all $(t, y, z, w) \in \overline{\boldsymbol{R}_+^1} \times \overline{\boldsymbol{R}_+^1} \times \overline{\boldsymbol{R}_+^1} + \boldsymbol{R}^n$, we hope to solve the problem (P.3) in future.

REMARK 5. We will treat the regularity problem of the solution u of the problems (P.1), (P.2) and (P.3) in future.

§3. The symmetrization of the mixed problems (P.1), (P.2) and (P.3).

In this section, we transform the mixed problems (P.1), (P.2) and (P.3) to the ones for symmetric hyperbolic systems of first order which have a positive boundary condition on a boundary $x=0$ and a non-negative boundary condition on the boundary $y=0$ ($z=0$), by using the result in [27].

We set

$$\text{(3.1)} \qquad U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} u_t - u_x + \theta_1(\alpha - i\beta)u_y + \theta_1(p - iq)u_z \\ \theta_1(u_t + u_x) + (\alpha + i\beta)u_y + (p + iq)u_z \\ u_y \\ u_z \end{pmatrix}$$

where $u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0$, θ_1 is a complex constant, α, β, p and q are real constants which satisfy the inequalities $\alpha^2 + \beta^2 \leq 1$ and $p^2 + q^2 \leq 1$.

LEMMA 3.1. *U satisfies the following equation*:

$$\text{(3.2)} \qquad M_1 U_t = A_{11} U_x + A_{12} U_y + A_{13} U_z$$

where

$$M_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\gamma l&-al\\0&0&-al&sl\end{pmatrix},\quad A_{11}=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&\gamma m&-am\\0&0&-am&sm\end{pmatrix},$$

$$A_{12}=\begin{pmatrix}0&\alpha-i\beta&\gamma&-a\\ \alpha+i\beta&0&\gamma\theta_1&-a\theta_1\\ \gamma&\gamma\bar\theta_1&-2\gamma\operatorname{Re}\{\theta_1(\alpha-i\beta)\}&2a\operatorname{Re}\{\theta_1(\alpha-i\beta)\}\\ -a&-a\bar\theta_1&2a\operatorname{Re}\{\theta_1(\alpha-i\beta)\}&-2s\operatorname{Re}\{\theta_1(\alpha-i\beta)\}\end{pmatrix},$$

$$A_{13}=\begin{pmatrix}0&p-iq&-a&s\\ p+iq&0&-a\theta_1&s\theta_1\\ -a&-a\bar\theta_1&-2\gamma\operatorname{Re}\{\theta_1(p-iq)\}&2a\operatorname{Re}\{\theta_1(p-iq)\}\\ s&s\bar\theta_1&2a\operatorname{Re}\{\theta_1(p-iq)\}&-2s\operatorname{Re}\{\theta_1(p-iq)\}\end{pmatrix},$$

$a=\operatorname{Re}\{(\alpha-i\beta)(p+iq)\}$, $l=1+|\theta_1|^2$, $m=1-|\theta_1|^2$,

$\gamma=1-\alpha^2-\beta^2$ and $s=1-p^2-q^2$.

PROOF. By simple calculations, we obtain Lemma 3.1. Q.E.D.

REMARK 6. We have that

$$M_1>0 \quad\text{if and only if}\quad \begin{cases}\gamma>0\\ \gamma s-a^2>0\,.\end{cases} \tag{3.3}$$

COROLLARY. *For $U={}^t(U_1, U_2, U_3, \cdots, U_n, U_{n+1})={}^t(u_t-u_{x_1}, \theta_2(u_t+u_{x_1}), u_{x_2}, \cdots, u_{x_{n-1}}, u_{x_n})$, we have*

$$M_2U_t=A_{21}U_{x_1}+\sum_{j=2}^{n}A_{2j}U_{x_j} \tag{3.4}$$

where $u_{tt}-\sum_{j=1}^{n}u_{x_jx_j}=0$,

$M_2=\operatorname{diag}(1, 1, 1+|\theta_2|^2, \cdots, 1+|\theta_2|^2, 1+|\theta_2|^2)$,

$A_{21}=\operatorname{diag}(-1, 1, 1-|\theta_2|^2, \cdots, 1-|\theta_2|^2, 1-|\theta_2|^2)$,

$$A_{22}=\begin{pmatrix}0&0&1&&0\\0&0&\theta_2&\mathbf{0}&\cdot\\1&\bar\theta_2&0&&\cdot\\&&\mathbf{0}&&\cdot\\&&&&\cdot\\0&\cdot&\cdot&\cdot&0\end{pmatrix},\quad A_{23}=\begin{pmatrix}0&0&0&1&\\0&0&0&\theta_2&\mathbf{0}\\0&0&0&0&\\1&\bar\theta_2&0&0&\\&&\mathbf{0}&&\mathbf{0}\end{pmatrix},$$

$$\cdots\cdots, \quad A_{2n}=\begin{pmatrix} & & & & & 1 \\ & & & & & \theta_2 \\ & & & & & 0 \\ & & 0 & & & \cdot \\ & & & & & \cdot \\ & & & & & \cdot \\ 1 & \bar{\theta}_2 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

From now on, we treat the symmetrization of the problems (P.1), (P.2) and (P.3).

Firstly, we consider the problem (P.1) under the condition (A.I).

The functions $c(t, y, w)$, $b(t, y, w)$ and $d_j(t, y, w)$ belong to $\mathscr{B}^1((\overline{R^1_+})^2 \times R^n)$ and are constant outside a compact set in $(\overline{R^1_+})^2 \times R^n$. Therefore, by the condition (A.I), we obtain

$$\inf_{(t,y,w)\in(\overline{R^1_+})^2\times R^n}\left\{1-\frac{2}{|1+c(t,y,w)|}\left(|b(t,y,w)|^2+\sum_{j=1}^n|d_j(t,y,w)|^2\right)^{1/2}\right\}>0. \tag{3.5}$$

We set

$$\begin{cases} \delta_0=\dfrac{1}{2}\inf\limits_{(t,y,w)\in(\overline{R^1_+})^2\times R^n}\left\{1-\dfrac{2}{|1+c|}\left(|b|^2+\sum\limits_{j=1}^n|d_j|^2\right)^{1/2}\right\} \\ O_1=\{(t,y,w)\in\overline{R^1_+}\times\overline{R^1_+}\times R^n \mid |1-c(t,y,w)|<\delta_0\} \\ O_2=\{(t,y,w)\in\overline{R^1_+}\times\overline{R^1_+}\times R^n \mid |1-c(t,y,w)|>\delta_0/2\}. \end{cases} \tag{3.6}$$

Then, $\{O_j\}_{j=1}^2$ is an open covering of $\overline{R^1_+}\times\overline{R^1_+}\times R^n$. Therefore, we have the functions $\varphi_1(t, y, w)$ and $\varphi_2(t, y, w)$ which belong to $\mathscr{B}((\overline{R^1_+})^2\times R^n)$, $\varphi_1+\varphi_2=1$ on $(\overline{R^1_+})^2\times R^n$ and $\operatorname{supp}[\varphi_j]\subseteqq O_j$ $(j=1, 2)$. Now, we consider the localization problem for (P.1)

$$\begin{cases} L_1[\varphi_j u]=\varphi_j f \\ (\varphi_j u)(0, x, y, w)=\varphi_j(0, y, w)u_0 \\ (\varphi_j u)_t(0, x, y, w)=\varphi_j(0, y, w)u_1+\varphi_{jt}(0, y, w)u_0 \\ B_1[\varphi_j u]|_{x=0}=[B_1, \varphi_j]u|_{x=0}+\varphi_j(t, y, w)g(t, y, w) \\ B_2[\varphi_j u]|_{y=0}=\varphi_j(t, 0, w)h \\ (t, x, y, w)\in R^1_+\times R^1_+\times R^1_+\times R^n \end{cases} \tag{3.7}$$

$(j=1, 2)$.

We set

$$v_j=\varphi_j(t, y, w)u(t, x, y, w) \qquad (j=1, 2), \tag{3.8}$$

$$U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \cdot \\ \cdot \\ \cdot \\ U_{n+3} \\ U_{n+4} \end{pmatrix}=\begin{pmatrix} v_{1t}-(v_{1x}+\gamma v_1) \\ \theta_3(v_{1t}+(v_{1x}+\gamma v_1)) \\ v_{1y} \\ v_{1w_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{1w_n} \\ v_1 \end{pmatrix} \tag{3.9}$$

and

$$V=\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \cdot \\ \cdot \\ \cdot \\ V_{n+3} \\ V_{n+4} \end{pmatrix}=\begin{pmatrix} v_{2t}-(v_{2x}+\gamma v_2) \\ \theta_4(v_{2t}+(v_{2x}+\gamma v_2)) \\ v_{2y} \\ v_{2w_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{2w_n} \\ v_2 \end{pmatrix} \tag{3.10}$$

where

$$\theta_3=\frac{\sqrt{\delta_0}}{1+c(t, y, w)} \tag{3.11}$$

and

$$\theta_4=\sqrt{\frac{1-c(t, y, w)}{1+c(t, y, w)}}\,. \tag{3.12}$$

Then, we have the following theorems by Corollary of Lemma 3.1.

THEOREM 3.2. *For U in* (3.9), *we obtain*

$$\begin{cases} M_3U_t=A_3U_x+A_{30}U_y+\sum_{j=1}^{n} A_{3j}U_{w_j}+E_3U+F_3 \\ U(0, x, y, w)=U_0(x, y, w) \\ P_3U|_{x=0}=G_3(t, y, w) \\ Q_3U|_{y=0}=H_3(t, x, w) \\ (t, x, y, w)\in \boldsymbol{R}^1_+\times \boldsymbol{R}^1_+\times \boldsymbol{R}^1_+\times \boldsymbol{R}^n \end{cases} \tag{3.13}$$

where

$$M_3 = \mathrm{diag}(1, 1, 1+|\theta_3|^2, \cdots, 1+|\theta_3|^2, 1),$$

$$A_3 = \mathrm{diag}(-1, 1, 1-|\theta_3|^2, \cdots, 1-|\theta_3|^2, 1),$$

$$A_{30} = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ 0 & 0 & \theta_3 & \mathbf{0} & \cdot \\ 1 & \bar{\theta}_3 & 0 & & \cdot \\ & \mathbf{0} & & & \cdot \\ & & & & 0 \\ 0 & \cdot & \cdot & \cdot\ 0 & 1 \end{pmatrix}, \quad \cdots, \quad A_{3n} = \begin{pmatrix} & & & & & 1 & 0 \\ & & & & & \theta_3 & \cdot \\ & & \mathbf{0} & & & 0 & \cdot \\ & & & & & \cdot & \cdot \\ & & & & & \cdot & \cdot \\ 1 & \bar{\theta}_3 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$E_3 \cdots$ *an* $(n+4)\times(n+4)$ *matrix which has the property that for* $E_3 = (e_{ij})$, e_{ij} *belongs to* $\mathscr{B}^0((\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n)$ *and is a constant outside a compact set in* $(\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n$,

$$F_3 = {}^t(\tilde{f}, \theta_3\tilde{f}, 0, \cdots, 0) \qquad (\tilde{f} = \varphi_1 f),$$

$$P_3 = \left(1, -\frac{1-c}{\sqrt{\delta_0}}, -\frac{2b}{1+c}, -\frac{2d_1}{1+c}, \cdots, -\frac{2d_n}{1+c}, 0\right),$$

$$Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & & & & 0 \\ 0 & 1 & 0 & 0 & & \mathbf{0} & & \cdot \\ 0 & 0 & 0 & 1 & & & & \cdot \\ & & & & \cdot & & & \cdot \\ & \mathbf{0} & & & & \cdot & & \cdot \\ & & & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$$G_3 = -\frac{2}{1+c}\{[B_1, \varphi_1]u|_{x=0} + \varphi_1(t, y, w)g\},$$

$$H_3 = {}^t(\tilde{h}_t - \tilde{h}_x - \tilde{\gamma}\tilde{h}, \theta_{30}(\tilde{h}_t + \tilde{h}_x + \tilde{\gamma}\tilde{h}), \tilde{h}_{w_1}, \cdots, \tilde{h}_{w_n}, \tilde{h})$$

$$(\tilde{h} = \varphi_1(t, 0, w)h, \ \theta_{30} = \theta_3|_{y=0}, \ \tilde{\gamma} = \gamma|_{y=0}),$$

and there exists a positive constant C such that

$$\begin{cases} [A_3 U, U] \geq C[U, U] & \textit{for all} \quad U \in \mathrm{Ker}\, P_3 \\ [A_{30} U, U] = 0 & \textit{for all} \quad U \in \mathrm{Ker}\, Q_3 . \end{cases} \tag{3.14}$$

REMARK 7. In [27], we obtained the same result for second order hyperbolic equations in $\boldsymbol{R}^1_{t+} \times \boldsymbol{R}^1_{x+} \times \boldsymbol{R}^n$ as (3.13) and (3.14) (see Appendix for a result in [27]).

Here, we give the proof of Theorem 3.2 by the same method as the one in [27].

PROOF. By simple calculations, we have (3.13). Furthermore, we get easily

$$[A_{30}U, U]=0 \qquad \text{for all} \quad U\in \operatorname{Ker} Q_3 .$$

Let $U\in \operatorname{Ker} Q_3$. Then, we obtain

$$U_1=\tilde{\theta}_3 U_2+\frac{2b}{1+c}U_3+\frac{2}{1+c}\sum_{j=1}^{n} d_j U_{j+3} \qquad \left(\tilde{\theta}_3=\frac{1-c}{\sqrt{\delta_0}}\right). \tag{3.15}$$

We set

$$I=[A_3U, U] \qquad \text{for} \quad U\in \operatorname{Ker} P_3 \tag{3.16}$$

and

$$|\tilde{b}|=\left(|b|^2+\sum_{j=1}^{n}|d_j|^2\right)^{1/2}. \tag{3.17}$$

Then, we have

$$I=\left\{-|U_1|^2+|U_2|^2+(1-|\theta_3|^2)|U_3|^2+(1-|\theta_3|^2)\sum_{j=1}^{n}|U_{j+3}|^2+|U_{n+4}|^2\right\}. \tag{3.18}$$

For $|\tilde{b}|>0$, by (3.15), we get

$$\begin{aligned}|U_1|^2&=\left|\tilde{\theta}_3 U_2+\frac{2}{1+c}\left(bU_3+\sum_{j=1}^{n} d_j U_{j+3}\right)\right|^2\\ &\le |\tilde{\theta}_3|^2|U_2|^2+4\frac{|\tilde{b}|^{1/2}}{\sqrt{|1+c|}}|U_2|\cdot\frac{|\tilde{\theta}_3|}{\sqrt{|1+c|}}|\tilde{b}|^{1/2}\\ &\quad\cdot\left|\frac{b}{|\tilde{b}|}U_3+\sum_{j=1}^{n}\frac{d_j}{|\tilde{b}|}U_{j+3}\right|+\frac{4}{|1+c|^2}|\tilde{b}|^2\left\{|U_3|^2+\sum_{j=1}^{n}|U_{j+3}|^2\right\}\\ &\le\left(|\tilde{\theta}_3|^2+\frac{2|\tilde{b}|}{|1+c|}\right)|U_2|^2+\left\{\frac{2|\tilde{\theta}_3|^2|\tilde{b}|}{|1+c|}+\frac{4|\tilde{b}|^2}{|1+c|^2}\right\}\left(|U_3|^2+\sum_{j=1}^{n}|U_{j+3}|^2\right).\end{aligned} \tag{3.19}$$

We notice that (3.19) holds for $|\tilde{b}|=0$. By (3.18) and (3.19), we obtain

$$\begin{aligned}I\ge&\left\{1-|\tilde{\theta}_3|^2-\frac{2|\tilde{b}|}{|1+c|}\right\}|U_2|^2\\ &+\left\{1-|\theta_3|^2-\frac{2|\tilde{\theta}_3|^2|\tilde{b}|}{|1+c|}-\frac{4|\tilde{b}|^2}{|1+c|^2}\right\}\left(|U_3|^2+\sum_{j=1}^{n}|U_{j+3}|^2\right)+|U_{n+4}|^2.\end{aligned} \tag{3.20}$$

We set

$$K_1=1-|\tilde{\theta}_3|^2-\frac{2|\tilde{b}|}{|1+c|}, \qquad K_2=1-|\theta_3|^2-\frac{2|\tilde{\theta}_3|^2|\tilde{b}|}{|1+c|}-\frac{4|\tilde{b}|^2}{|1+c|^2}. \tag{3.21}$$

By $\mathrm{Re}\, c>0$, $\mathrm{supp}[\varphi_1]\subseteq O_1$, (3.11) and $\tilde{\theta}_3=(1-c)/\sqrt{\delta_0}$, we get

$$|\tilde{\theta}_3|^2<\delta_0, \qquad |\theta_3|^2<\delta_0. \tag{3.22}$$

Then, by (3.6) and (3.22), we have

$$K_1>1-\frac{2|\tilde{b}|}{|1+c|}-\delta_0\geq 2\delta_0-\delta_0=\delta_0>0$$

and

$$\begin{aligned} K_2&>1-\frac{4|\tilde{b}|^2}{|1+c|}-\left(1+\frac{2|\tilde{b}|}{|1+c|}\right)\delta_0 \\ &=\left(1+\frac{2|\tilde{b}|}{|1+c|}\right)\left(1-\frac{2|\tilde{b}|}{|1+c|}-\delta_0\right) \\ &\geq\left(1+\frac{2|\tilde{b}|}{|1+c|}\right)(2\delta_0-\delta_0)=\left(1+\frac{2|\tilde{b}|}{|1+c|}\right)\delta_0>0. \end{aligned}$$

Therefore, we have Theorem 3.2. Q.E.D.

THEOREM 3.3. *For V in* (3.10), *we have*

$$\begin{cases} M_4V_t=A_4V_x+A_{40}V_y+\sum\limits_{j=1}^{n}A_{4j}V_{w_j}+E_4V+F_4 \\ V(0, x, y, w)=V_0(x, y, w) \\ P_4V|_{x=0}=G_4(t, y, w) \\ Q_4V|_{y=0}=H_4(t, x, w) \\ (t, x, y, w)\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^n \end{cases} \tag{3.23}$$

where

$$M_4=\mathrm{diag}(1, 1, 1+|\theta_4|^2, \cdots, 1+|\theta_4|^2, 1),$$

$$A_4=\mathrm{diag}(-1, 1, 1-|\theta_4|^2, \cdots, 1-|\theta_4|^2, 1),$$

$$A_{40}=\begin{pmatrix} 0 & 0 & 1 & & 0 \\ 0 & 0 & \theta_4 & \mathbf{0} & \cdot \\ 1 & \bar{\theta}_4 & 0 & & \cdot \\ & & & & \cdot \\ & & \mathbf{0} & & 0 \\ 0 & \cdot & \cdot & \cdot\ 0 & 1 \end{pmatrix}, \quad \cdots, \quad A_{4n}=\begin{pmatrix} & & & & & 1 & 0 \\ & & & & & \theta_4 & \cdot \\ & & & \mathbf{0} & & 0 & \cdot \\ & & & & & \cdot & \cdot \\ & & & & & \cdot & \cdot \\ 1 & \bar{\theta}_4 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$E_4 \cdots$ *an* $(n+4)\times(n+4)$ *matrix with the same property as* E_3,

$$F_4={}^t(\tilde{\tilde{f}}, \theta_4\tilde{\tilde{f}}, 0, \cdots, 0) \qquad (\tilde{\tilde{f}}=\varphi_2 f),$$

$$P_4=\left(1, -\theta_4, -\frac{2b}{1+c}, -\frac{2d_1}{1+c}, \cdots, -\frac{2d_n}{1+c}, 0\right),$$

$$Q_4=Q_3,$$

$$G_4=-\frac{2}{1+c}\{[B_1, \varphi_2]u|_{x=0}+\varphi_2(t, y, w)g\},$$

$$H_4={}^t(\tilde{\tilde{h}}_t-\tilde{\tilde{h}}_x-\tilde{\gamma}\tilde{\tilde{h}}, \theta_{40}(\tilde{\tilde{h}}_t+\tilde{\tilde{h}}_x+\tilde{\gamma}\tilde{\tilde{h}}), \tilde{\tilde{h}}_{w_1}, \cdots, \tilde{\tilde{h}}_{w_n}, \tilde{\tilde{h}})$$
$$(\tilde{\tilde{h}}=\varphi_2(t, 0, w)h, \theta_{40}=\theta_4|_{y=0}, \tilde{\gamma}=\gamma|_{y=0}),$$

and there exists a positive constant C such that

$$\begin{cases}[A_4V, V]\geq C[V, V] & \textit{for all} \quad V\in \operatorname{Ker} P_4 \\ [A_{40}V, V]=0 & \textit{for all} \quad V\in \operatorname{Ker} Q_4 .\end{cases} \tag{3.24}$$

REMARK 8. In [27], we obtained the same result for second order hyperbolic equations in $\boldsymbol{R}^1_{t+}\times\boldsymbol{R}^1_{x+}\times\boldsymbol{R}^n$ as (3.23) and (3.24) (see Appendix for a result in [27]). Here, we give the proof of Theorem 3.3 by the same method as the one in [27].

PROOF. By simple calculations, we obtain (3.23). Furthermore, we get easily

$$[A_{40}V, V]=0 \qquad \text{for all} \quad V\in \operatorname{Ker} Q_4 .$$

Let $V\in \operatorname{Ker} P_4$. Then, we have

$$V_1=\theta_4V_2+\frac{2b}{1+c}V_3+\frac{2}{1+c}\sum_{j=1}^{n} d_jV_{j+3} . \tag{3.25}$$

We set

$$J=[A_4V, V] \qquad \text{for} \quad V\in \operatorname{Ker} P_4 . \tag{3.26}$$

Then, by the same method as the one in Theorem 3.2, we obtain

$$J\geq\left\{1-|\theta_4|^2-\frac{2|\tilde{b}|}{|1+c|}\right\}|V_2|^2 \tag{3.27}$$
$$+\left\{1-|\theta_4|^2-\frac{2|\theta_4|^2|\tilde{b}|}{|1+c|}-\frac{4|\tilde{b}|^2}{|1+c|^2}\right\}\left(|V_3|^2+\sum_{j=1}^{n}|V_{j+3}|^2\right)+|V_{n+4}|^2$$

where $|\tilde{b}|=(|b|^2+\sum_{j=1}^n|d_j|^2)^{1/2}$.

We set

$$K_3=1-|\theta_4|^2-\frac{2|\tilde{b}|}{|1+c|}, \qquad K_4=1-|\theta_4|^2-\frac{2|\theta_4|^2|\tilde{b}|}{|1+c|}-\frac{4|\tilde{b}|^2}{|1+c|^2} . \tag{3.28}$$

By (A.I) and (3.12), we get

$$K_3=1-\frac{|1-c|}{|1+c|}-\frac{2|\tilde{b}|}{|1+c|}=\frac{|1+c|-|1-c|-2|\tilde{b}|}{|1+c|}>0 \tag{3.29}$$

and

$$\begin{aligned} K_4&=1-\frac{|1-c|}{|1+c|}-\frac{2|1-c||\tilde{b}|}{|1+c|^2}-\frac{4|\tilde{b}|^2}{|1+c|^2} \\ &=\left(1+\frac{2|\tilde{b}|}{|1+c|}\right)\frac{1}{|1+c|}(|1+c|-|1-c|-2|\tilde{b}|)>0\,. \end{aligned} \tag{3.30}$$

Therefore, we have Theorem 3.3. Q.E.D.

Secondly, we consider the problem (P.2) under the condition (A.I).

THEOREM 3.4. *For U in* (3.9), *we obtain*

$$\begin{cases} M_3U_t=A_3U_x+A_{30}U_y+\sum_{j=1}^{n}A_{3j}U_{w_j}+E_3U+F_3 \\ U(0,x,y,w)=U_0(x,y,w) \\ P_3U|_{x=0}=G_3(t,y,w) \\ Q_5U|_{y=0}=H_5(t,x,w) \\ (t,x,y,w)\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^n \end{cases} \tag{3.31}$$

where

$$Q_5=(0,0,1,0,\cdots,0)\,,\qquad H_5=\varphi_1(t,0,w)h+\varphi_{1y}(t,0,w)u(t,x,0,w)\,,$$

and there exists a positive constant C such that

$$\begin{cases} [A_3U,U]\geq C[U,U] & \textit{for all}\quad U\in \operatorname{Ker}P_3 \\ [A_{30}U,U]\geq 0 & \textit{for all}\quad U\in\operatorname{Ker}Q_5\,. \end{cases} \tag{3.32}$$

PROOF. By simple calculations, we have (3.31) and

$$[A_{30}U,U]\geq 0\qquad \text{for all}\quad U\in\operatorname{Ker}Q_5\,.$$

Also, we obtain

$$[A_3U,U]\geq C[U,U]\qquad \text{for all}\quad U\in\operatorname{Ker}P_3$$

by the proof of Theorem 3.2. Q.E.D.

THEOREM 3.5. *For V in* (3.10), *we have*

(3.33)
$$\begin{cases} M_4V_t = A_4V_x + A_{40}V_y + \sum_{j=1}^{n} A_{4j}V_{w_j} + E_4V + F_4 \\ V(0, x, y, w) = V_0(x, y, w) \\ P_4V|_{x=0} = G_4(t, y, w) \\ Q_6V|_{y=0} = H_6(t, x, w) \\ (t, x, y, w) \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^n \end{cases}$$

where

$$Q_6 = Q_5 = (0, 0, 1, 0, \cdots, 0)\,, \qquad H_6 = \varphi_2(t, 0, w)h + \varphi_{2y}(t, 0, w)u(t, x, 0, w)$$

and there exists a positive constant C such that

(3.34)
$$\begin{cases} [A_4V, V] \geq C[V, V] & \textit{for all} \quad V \in \operatorname{Ker} P_4 \\ [A_{40}V, V] \geq 0 & \textit{for all} \quad V \in \operatorname{Ker} Q_6\,. \end{cases}$$

PROOF. By simple calculations, we have (3.33) and

$$[A_{40}V, V] \geq 0 \qquad \text{for all} \quad V \in \operatorname{Ker} Q_6\,.$$

Also, we get

$$[A_4V, V] \geq C[V, V] \qquad \text{for all} \quad V \in \operatorname{Ker} P_4$$

by the proof of Theorem 3.3. Q.E.D.

Thirdly, we treat the problem (P.3) under the condition (A.II).

We use the same method as the one in treating the problem (P.1) under the condition (A.I).

We set

(3.35)
$$\begin{cases} \delta_1 = \dfrac{1}{2} \inf\limits_{(t,y,z,w) \in (\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n} \left\{ 1 - \dfrac{2}{|1+c|} \left(|b_1|^2 + |b_2|^2 + \sum_{j=1}^{n} |d_j|^2 \right)^{1/2} \right\} \\ \tilde{O}_1 = \{(t, y, z, w) \in (\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n \mid |1 - c(t, y, z, w)| < \delta_1\} \\ \tilde{O}_2 = \{(t, y, z, w) \in (\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n \mid |1 - c(t, y, z, w)| > \delta_1/2\}\,. \end{cases}$$

Then, $\delta_1 > 0$ and $\{\tilde{O}_j\}_{j=1}^{2}$ is an open covering of $(\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n$. Therefore, we have the functions $\psi_1(t, y, z, w)$ and $\psi_2(t, y, z, w)$ which belong to $\mathscr{B}((\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n)$, $\psi_1 + \psi_2 = 1$ on $(\overline{\boldsymbol{R}^1_+})^3 \times \boldsymbol{R}^n$ and $\operatorname{supp}[\psi_j] \subseteqq \tilde{O}_j$ $(j = 1, 2)$. Now, we treat the localization problem for (P.3)

(3.36)
$$\begin{cases} L_2[\psi_j u] = \psi_j f \\ (\psi_j u)(0, x, y, z, w) = \psi_j(0, y, z, w)u_0 \\ (\psi_j u)_t(0, x, y, z, w) = \psi_j(0, y, z, w)u_1 + \psi_{jt}(0, y, z, w)u_0 \\ B_4[\psi_j u]|_{x=0} = [B_4, \psi_j]u|_{x=0} + \psi_j(t, y, z, w)g \end{cases}$$

$$\begin{cases} B_5[\psi_j u]|_{y=0} = \psi_j(t, 0, z, w)h \\ B_6[\psi_j u]|_{z=0} = \psi_j(t, y, 0, w)k \\ (t, x, y, z, w) \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^n \end{cases}$$

$$(j=1, 2).$$

We set

$$v_j = \psi_j(t, y, z, w)u(t, x, y, z, w) \qquad (j=1, 2), \tag{3.37}$$

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ \cdot \\ \cdot \\ \cdot \\ U_{n+4} \\ U_{n+5} \end{pmatrix} = \begin{pmatrix} v_{1t} - (v_{1x} + \gamma v_1) \\ \theta_5(v_{1t} + (v_{1x} + \gamma v_1)) \\ v_{1y} \\ v_{1z} \\ v_{1w_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{1w_n} \\ v_1 \end{pmatrix} \tag{3.38}$$

and

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ \cdot \\ \cdot \\ \cdot \\ V_{n+4} \\ V_{n+5} \end{pmatrix} = \begin{pmatrix} v_{2t} - (v_{2x} + \gamma v_2) \\ \theta_6(v_{2t} + (v_{2x} + \gamma v_2)) \\ v_{2y} \\ v_{2z} \\ v_{2w_1} \\ \cdot \\ \cdot \\ \cdot \\ v_{2w_n} \\ v_2 \end{pmatrix} \tag{3.39}$$

where

$$\theta_5 = \frac{\sqrt{\delta_1}}{1 + c(t, y, z, w)} \tag{3.40}$$

and

$$\theta_6 = \sqrt{\frac{1 - c(t, y, z, w)}{1 + c(t, y, z, w)}}\,. \tag{3.41}$$

Then, by the same method as the one in the proof of Theorem 3.2 and Theorem 3.3, we have the following theorems.

THEOREM 3.6. *For U in* (3.38), *we obtain*

$$
(3.42)\quad \begin{cases} M_7 U_t = A_7 U_x + A_{70} U_y + A_{700} U_z + \sum_{j=1}^{n} A_{7j} U_{w_j} + E_7 U + F_7 \\ U(0, x, y, z, w) = U_0(x, y, z, w) \\ P_7 U|_{x=0} = G_7(t, y, z, w) \\ Q_7 U|_{y=0} = H_7(t, x, z, w) \\ R_7 U|_{z=0} = K_7(t, x, y, w) \\ (t, x, y, z, w) \in (\boldsymbol{R}_+^1)^4 \times \boldsymbol{R}^n \end{cases}
$$

where

$$M_7 = \mathrm{diag}(1, 1, 1 + |\theta_5|^2, \cdots, 1 + |\theta_5|^2, 1)\,,$$

$$A_7 = \mathrm{diag}(-1, 1, 1 - |\theta_5|^2, \cdots, 1 - |\theta_5|^2, 1)\,,$$

$$
A_{70} = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ 0 & 0 & \theta_5 & \mathbf{0} & \cdot \\ 1 & \bar{\theta}_5 & 0 & & \cdot \\ & & & & \cdot \\ & & \mathbf{0} & & \\ & & & & 0 \\ 0 & \cdot & \cdot & \cdot\ 0 & 1 \end{pmatrix}, \qquad
A_{700} = \begin{pmatrix} 0 & 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & \theta_5 & \mathbf{0} & \cdot \\ 0 & 0 & 0 & 0 & & \cdot \\ 1 & \bar{\theta}_5 & 0 & 0 & & \cdot \\ & & \mathbf{0} & & & \cdot \\ & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot\ 0 & 1 \end{pmatrix},
$$

$$
\cdots\cdots, \qquad A_{7n} = \begin{pmatrix} & & & & & 1 & 0 \\ & & & & & \theta_5 & \cdot \\ & & \mathbf{0} & & & 0 & \cdot \\ & & & & & \cdot & \cdot \\ & & & & & \cdot & \cdot \\ 1 & \bar{\theta}_5 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},
$$

$E_7 \cdots$ *an* $(n+5) \times (n+5)$ *matrix with the same property as* E_3,

$F_7 = {}^t(f_1, \theta_5 f_1, 0, \cdots, 0) \qquad (f_1 = \psi_1 f)$,

$$P_7 = \left(1, -\frac{1-c}{\sqrt{\delta_1}}, -\frac{2b_1}{1+c}, -\frac{2b_2}{1+c}, -\frac{2d_1}{1+c}, \cdots, -\frac{2d_n}{1+c}, 0\right),$$

$$Q_7=\begin{pmatrix} 1 & 0 & 0 & 0 & & & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathbf{0} & & \cdot & \cdot \\ 0 & 0 & 0 & 1 & & & \cdot & \cdot \\ & & \mathbf{0} & & & & \cdot & \cdot \\ & & & & & & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$$R_7=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & & & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \mathbf{0} & & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & & & \cdot & \cdot \\ & & \mathbf{0} & & & & & \cdot & \cdot \\ & & & & & & & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$$G_7=-\frac{2}{1+c}\{[B_4, \psi_1]u|_{x=0}+\psi_1(t, y, z, w)g\},$$

$$H_7={}^t(h_{1t}-h_{1x}-\gamma_1 h_1, \theta_{51}(h_{1t}+h_{1x}+\gamma_1 h_1), h_{1z}, h_{1w_1}, \cdots, h_{1w_n}, h_1)$$
$$(h_1=\psi_1(t, 0, z, w)h, \theta_{51}=\theta_5|_{y=0}, \gamma_1=\gamma|_{y=0}),$$

$$K_7={}^t(k_{1t}-k_{1x}-\gamma_2 k_1, \theta_{52}(k_{1t}+k_{1x}+\gamma_2 k_1), k_{1y}, k_{1w_1}, \cdots, k_{1w_n}, k_1)$$
$$(k_1=\psi_1(t, y, 0, w)k, \theta_{52}=\theta_5|_{z=0}, \gamma_2=\gamma|_{z=0}),$$

and there exists a positive constant C such that

$$(3.43)\qquad \begin{cases} [A_7 U, U]\ge C[U, U] & \textit{for all} \quad U\in \operatorname{Ker} P_7 \\ [A_{70}U, U]=0 & \textit{for all} \quad U\in \operatorname{Ker} Q_7 \\ [A_{700}U, U]=0 & \textit{for all} \quad U\in \operatorname{Ker} R_7 . \end{cases}$$

THEOREM 3.7. *For V in* (3.39), *we have*

$$(3.44)\qquad \begin{cases} M_8 V_t=A_8 V_x+A_{80}V_y+A_{800}V_z+\sum_{j=1}^{n} A_{8j}V_{w_j}+E_8 U+F_8 \\ V(0, x, y, z, w)=V_0(x, y, z, w) \\ P_8 V|_{x=0}=G_8(t, y, z, w) \\ Q_8 V|_{y=0}=H_8(t, x, z, w) \\ R_8 V|_{z=0}=K_8(t, x, y, w) \\ (t, x, y, z, w)\in(\boldsymbol{R}^1_+)^4\times \boldsymbol{R}^n \end{cases}$$

where

$$M_8=\operatorname{diag}(1,1,1+|\theta_6|^2,\cdots,1+|\theta_6|^2,1)\,,$$

$$A_8=\operatorname{diag}(-1,1,1-|\theta_6|^2,\cdots,1-|\theta_6|^2,1)\,,$$

$$A_{80}=\begin{pmatrix} 0 & 0 & 1 & & & 0\\ 0 & 0 & \theta_6 & & \mathbf{0} & \cdot\\ 1 & \bar{\theta}_6 & 0 & & & \cdot\\ & & & & & \cdot\\ & & \mathbf{0} & & & 0\\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},\qquad A_{800}=\begin{pmatrix} 0 & 0 & 0 & 1 & & & 0\\ 0 & 0 & 0 & \theta_6 & & \mathbf{0} & \cdot\\ 0 & 0 & 0 & 0 & & & \cdot\\ 1 & \bar{\theta}_6 & 0 & 0 & & & \cdot\\ & & & \mathbf{0} & & & \cdot\\ & & & & & & 0\\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$$\cdots\cdots,\qquad A_{8n}=\begin{pmatrix} & & & & & 1 & 0\\ & & & & & \theta_6 & \cdot\\ & & \mathbf{0} & & & 0 & \cdot\\ & & & & & \cdot & \cdot\\ & & & & & \cdot & \cdot\\ 1 & \bar{\theta}_6 & 0 & \cdot & \cdot & 0 & 0\\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix},$$

$E_8\cdots$ *an* $(n+5)\times(n+5)$ *matrix with the same property as* E_3,

$$F_8={}^t(f_2,\theta_6 f_2,0,\cdots,0)\qquad (f_2=\psi_2 f)\,,$$

$$P_8=\left(1,-\theta_6,-\frac{2b_1}{1+c},-\frac{2b_2}{1+c},-\frac{2d_1}{1+c},\cdots,-\frac{2d_n}{1+c},0\right),$$

$$Q_8=Q_7\,,\quad R_8=R_7\,,$$

$$G_8=-\frac{2}{1+c}\{[B_4,\psi_2]u|_{x=0}+\psi_2(t,y,z,w)g\}\,,$$

$$H_8={}^t(h_{2t}-h_{2x}-\gamma_1 h_2,\theta_{61}(h_{2t}+h_{2x}+\gamma_1 h_2),h_{2z},h_{2w_1},\cdots,h_{2w_n},h_2)$$
$$(h_2=\psi_2(t,0,z,w)h,\ \theta_{61}=\theta_6|_{y=0},\ \gamma_1=\gamma|_{y=0})\,,$$

$$K_8={}^t(k_{2t}-k_{2x}-\gamma_2 k_2,\theta_{62}(k_{2t}+k_{2x}+\gamma_2 k_2),k_{2y},k_{2w_1},\cdots,k_{2w_n},k_2)$$
$$(k_2=\psi_2(t,y,0,w)k,\ \theta_{62}=\theta_6|_{z=0},\ \gamma_2=\gamma|_{z=0})\,,$$

and there exists a positive constant C such that

$$\text{(3.45)}\qquad \begin{cases} [A_8V,V]\ge C[V,V] & \textit{for all}\quad V\in\operatorname{Ker}P_8\\ [A_{80}V,V]=0 & \textit{for all}\quad V\in\operatorname{Ker}Q_8\\ [A_{800}V,V]=0 & \textit{for all}\quad V\in\operatorname{Ker}R_8\,. \end{cases}$$

§4. Boundary estimates.

In this section, we prepare several propositions to estimate the boundary value of u on the boundaries $y=0$ and $z=0$.

Firstly, we consider the problem (P.1) under the condition (A.I).

We set

$$U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \cdot \\ \cdot \\ \cdot \\ U_{n+3} \\ U_{n+4} \end{pmatrix}=\begin{pmatrix} u_t-u_x+\sqrt{2}^{-1}u_y \\ u_t+u_x+\sqrt{2}^{-1}u_y \\ u_y \\ u_{w_1} \\ \cdot \\ \cdot \\ \cdot \\ u_{w_n} \\ u \end{pmatrix} \tag{4.1}$$

for the solution u of the problem (P.1) which belongs to $\mathscr{H}_{2,\mu}[(R^1_+)^3\times R^n]$.

PROPOSITION 4.1. *For U in* (4.1), *we have*

$$\begin{cases} \tilde{M}_1U_t=\tilde{A}_1U_x+\tilde{A}_{10}U_y+\sum_{j=1}^{n}\tilde{A}_{1j}U_{w_j}+\tilde{E}_1U+\tilde{F}_1 \\ (t, x, y, w)\in R^1_+\times R^1_+\times R^1_+\times R^n \end{cases} \tag{4.2}$$

and

$$U|_{y=0}={}^t\left(h_t-h_x+\frac{u_y}{\sqrt{2}}, h_1+h_x+\frac{u_y}{\sqrt{2}}, u_y, h_{w_1}, \cdots, h_{w_n}, h\right) \tag{4.3}$$

where

$$\tilde{M}_1=\mathrm{diag}(1, 1, 1, 2, \cdots, 2, 1)\,, \qquad \tilde{A}_1=\mathrm{diag}(-1, 1, 0, 0, \cdots, 0, 1)\,,$$

$$\tilde{A}_{10}=\begin{pmatrix} 0 & 1/\sqrt{2} & 1/2 & & & & \\ 1/\sqrt{2} & 0 & 1/2 & & & 0 & \\ 1/2 & 1/2 & -1/\sqrt{2} & & & & \\ & & & -\sqrt{2} & & & \\ & & & & \ddots & & \\ & 0 & & & & -\sqrt{2} & \\ & & & & & & 1 \end{pmatrix},$$

$$
\tilde{A}_{1j}=
\begin{array}{r c}
 & \qquad\qquad\quad (j+3)\ \text{column} \\
\begin{array}{r} \\ \\ \\ \\ \\ (j+3)\ \text{row} \\ \\ \\ \\ \end{array} &
\left(\begin{array}{ccccccc}
 & & & & 1 & & 0 \\
 & & & & 1 & & \cdot \\
 & 0 & & & 0 & & \cdot \\
 & & & & \cdot & 0 & \cdot \\
 & & & & \cdot & & \cdot \\
1 & 1 & 0 & \cdots & 0 & & \cdot \\
 & & & & & & \cdot \\
 & & 0 & & & & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{array}\right)
\end{array}
\qquad (j=1, \cdots, n)\,,
$$

$\tilde{E}_1 \cdots$ *an* $(n+4)\times(n+4)$ *matrix with the same property as* E_3 *in* §3,

and

$$\tilde{F}_1 = {}^t(f, f, 0, \cdots, 0)\,.$$

PROOF. By simple calculations, we have Proposition 4.1. Q.E.D.

PROPOSITION 4.2. *For* U *in* (4.1), *the following inequality holds,*

$$
\frac{d}{dt}(\tilde{M}_1 e^{-\mu t}U, e^{-\mu t}U) \le C_1\langle e^{-\mu t}U, e^{-\mu t}U\rangle - C_2\langle\!\langle e^{-\mu t}u_y, e^{-\mu t}u_y\rangle\!\rangle + C_3\langle\!\langle e^{-\mu t}h\rangle\!\rangle_1^2 + \frac{C_4}{\mu}(e^{-\mu t}f, e^{-\mu t}f) \tag{4.4}
$$

for all $\mu \ge \mu_1$ *where* μ_1, C_1, C_2, C_3 *and* C_4 *are positive constants.*

PROOF. By Proposition 4.1, we have

$$
\begin{aligned}
\frac{d}{dt}(\tilde{M}_1 e^{-\mu t}U, e^{-\mu t}U) &= -2\mu(\tilde{M}_1 e^{-\mu t}U, e^{-\mu t}U) \\
&\quad + \left(e^{-\mu t}\left(\tilde{A}_1 U_x + \tilde{A}_{10}U_y + \sum_{j=1}^{n} \tilde{A}_{1j}U_{w_j} + \tilde{E}_1 U + \tilde{F}_1\right), e^{-\mu t}U\right) \\
&\quad + \left(e^{-\mu t}U, e^{-\mu t}\left(\tilde{A}_1 U_x + \tilde{A}_{10}U_y + \sum_{j=1}^{n} \tilde{A}_{1j}U_{w_j} + \tilde{E}_1 U + \tilde{F}_1\right)\right) \\
&\le C_1\langle e^{-\mu t}U, e^{-\mu t}U\rangle - C_2\langle\!\langle e^{-\mu t}u_y, e^{-\mu t}u_y\rangle\!\rangle + C_3\langle\!\langle e^{-\mu t}h\rangle\!\rangle_1^2 + \frac{C_4}{\mu}(e^{-\mu t}f, e^{-\mu t}f)
\end{aligned}
$$

for all $\mu \ge \mu_1$ where μ_1, C_1, C_2, C_3 and C_4 are positive constants. Q.E.D.

Secondly, we use the same vector U in (4.1) for the solution u of the problem (P.2) under the condition (A.I) which belongs to $\mathscr{H}_{2,\mu}[(\boldsymbol{R}^1_+)^3 \times \boldsymbol{R}^n]$.

PROPOSITION 4.3. *For U in* (4.1), *the following inequality holds,*

$$\langle\langle \Lambda_{x,w,\mu}^{-1/2} U \rangle\rangle_{0,\mu,t}^2 \leq C\left\{\frac{1}{\mu}|||U(t)|||_{0,\mu}^2 + \frac{1}{\mu}|||U(0)|||_{0,\mu}^2 + \|U\|_{0,\mu,t}^2 + \frac{1}{\mu}\langle U\rangle_{0,\mu,t}^2 + \frac{1}{\mu^2}\|\tilde{F}_1\|_{0,\mu,t}^2\right\} \tag{4.5}$$

for all $t \in \boldsymbol{R}_+^1$ and all $\mu \geq \mu_2$ where C and μ_2 are positive constants.

PROOF. U satisfies the equation (4.2). We set $U(t, x, y, w) = 0$ $(x<0)$. Then, by the Fourier transform of (4.2) with respect to (x, w), we get

$$\tilde{M}_1 \hat{U}_t = i\xi \tilde{A}_1 \hat{U} - \tilde{A}_1 \hat{\hat{U}}(t, 0, y, \zeta) + \tilde{A}_{10} \hat{U}_y + i\left(\sum_{j=1}^n \tilde{A}_{1j}\zeta_j\right)\hat{U} + \widehat{\tilde{E}_1 U} + \hat{\tilde{F}}_1, \tag{4.6}$$

where

ξ is the dual variable of x, $\zeta = (\zeta_1, \cdots, \zeta_n)$ is the dual vector of $w = (w_1, \cdots, w_n)$,

$$\hat{p} = \int_{\boldsymbol{R}^{n+1}} e^{-i(x\cdot\xi + w\cdot\zeta)} p(x, w)\,dxdw \quad \text{and} \quad \hat{\hat{U}} = \int_{\boldsymbol{R}^n} e^{-iw\cdot\zeta} U(t, 0, y, w)\,dw.$$

We set

$$V = e^{-\mu t}(\xi^2 + |\zeta|^2 + \mu^2)^{-1/4}\hat{U}, \tag{4.7}$$

where $|\zeta|^2 = |\zeta_1|^2 + \cdots + |\zeta_n|^2$. Then, by (4.6) and (4.7), we obtain

$$\begin{aligned} V_y = {} & \tilde{A}_{10}^{-1}\tilde{M}_1 V_t + \mu\tilde{A}_{10}^{-1}\tilde{M}_1 V - \tilde{A}_{10}^{-1} i\left(\tilde{A}_1\xi + \sum_{j=1}^n \tilde{A}_{1j}\zeta_j\right)V \\ & + e^{-\mu t}\tilde{A}_{10}^{-1}\tilde{A}_1(\xi^2 + |\zeta|^2 + \mu^2)^{-1/4}\hat{\hat{U}}(t, 0, y, \zeta) \\ & - \tilde{A}_{10}^{-1} e^{-\mu t}(\xi^2 + |\zeta|^2 + \mu^2)^{-1/4}\{\widehat{\tilde{E}_1 U} + \hat{\tilde{F}}_1\} \end{aligned} \tag{4.8}$$

and $\tilde{A}_{10}^{-1}\tilde{M}_1$ is an Hermite matrix. Therefore, we have

$$\begin{aligned} -\frac{d}{dy}[V, V] = {} & -[V_y, V] - [V, V_y] \\ = {} & -\Big[\tilde{A}_{10}^{-1}\tilde{M}_1 V_t + \mu\tilde{A}_{10}^{-1}\tilde{M}_1 V - \tilde{A}_{10}^{-1} i\left(\tilde{A}_1\xi + \sum_{j=1}^n \tilde{A}_{1j}\zeta_j\right)V \\ & + e^{-\mu t}\tilde{A}_{10}^{-1}\tilde{A}_1(\xi^2 + |\zeta|^2 + \mu^2)^{-1/4}\hat{\hat{U}}(t, 0, y, \zeta) \\ & - \tilde{A}_{10}^{-1} e^{-\mu t}(\xi^2 + |\zeta|^2 + \mu^2)^{-1/4}\{\widehat{\tilde{E}_1 U} + \hat{\tilde{F}}_1\}, V\Big] \\ & -\Big[V, \tilde{A}_{10}^{-1}\tilde{M}_1 V_t + \mu\tilde{A}_{10}^{-1}\tilde{M}_1 V - \tilde{A}_{10}^{-1} i\left(\tilde{A}_1\xi + \sum_{j=1}^n \tilde{A}_{1j}\zeta_j\right)V \end{aligned}$$

$$+e^{-\mu t}\tilde{A}_{10}^{-1}\tilde{A}_1(\xi^2+|\zeta|^2+\mu^2)^{-1/4}\hat{\hat{U}}(t,0,y,\zeta)$$
$$\left.-\tilde{A}_{10}^{-1}e^{-\mu t}(\xi^2+|\zeta|^2+\mu^2)^{-1/4}\{\widehat{\tilde{E}_1 V}+\hat{\tilde{F}}_1\}\right]$$

$$=-\frac{d}{dt}[\tilde{A}_{10}^{-1}\tilde{M}_1 V, V]-2\mu[\tilde{A}_{10}^{-1}\tilde{M}_1 V, V]$$
$$+\left\{\left[e^{-\mu t}\tilde{A}_{10}^{-1}i\left(\tilde{A}_1\xi+\sum_{j=1}^{n}\tilde{A}_{1j}\zeta_j\right)(\xi^2+|\zeta|^2+\mu^2)^{-1/2}\hat{U}, e^{-\mu t}\hat{U}\right]\right.$$
$$\left.+\left[e^{-\mu t}\hat{U}, e^{-\mu t}\tilde{A}_{10}^{-1}i\left(\tilde{A}_1\xi+\sum_{j=1}^{n}\tilde{A}_{1j}\zeta_j\right)(\xi^2+|\zeta|^2+\mu^2)^{-1/2}\hat{U}\right]\right\}$$
$$-[e^{-\mu t}\tilde{A}_{10}^{-1}\tilde{A}_1(\xi^2+|\zeta|^2+\mu^2)^{-1/2}\hat{\hat{U}}(t,0,y,\zeta), e^{-\mu t}\hat{U}]$$
$$-[e^{-\mu t}\hat{U}, e^{-\mu t}\tilde{A}_{10}^{-1}\tilde{A}_1(\xi^2+|\zeta|^2+\mu^2)^{-1/2}\hat{\hat{U}}(t,0,y,\zeta)]$$
$$+[e^{-\mu t}\tilde{A}_{10}^{-1}(\xi^2+|\zeta|^2+\mu^2)^{-1/4}(\widehat{\tilde{E}_1 U}+\hat{\tilde{F}}_1), V]$$
$$+[V, e^{-\mu t}\tilde{A}_{10}^{-1}(\xi^2+|\zeta|^2+\mu^2)^{-1/4}(\widehat{\tilde{E}_1 U}+\hat{\tilde{F}}_1)]\,.$$

Therefore, we obtain

$$\langle\!\langle A_{x,w,\mu}^{-1/2}U\rangle\!\rangle_{0,\mu,t}^2=\int_0^t\int_{-\infty}^{\infty}\int_{\mathbf{R}^n}\int_{-\infty}^{\infty}\left\{-\frac{d}{dy}[V,V]\right\}dyd\zeta d\xi dt \tag{4.9}$$
$$\leq C\left\{\frac{1}{\mu}|||U(t)|||_{0,\mu}^2+\frac{1}{\mu}|||U(0)|||_{0,\mu}^2+\|U\|_{0,\mu,t}^2+\frac{1}{\mu}\langle U\rangle_{0,\mu,t}^2+\frac{1}{\mu^2}\|\tilde{F}_1\|_{0,\mu,t}^2\right\}$$

for all $t\in\mathbf{R}_+^1$ and all $\mu\geq\mu_2$ where C and μ_2 are positive constants. Q.E.D.

Thirdly, we consider the problem (P.3) under the condition (A.II).

We set

$$U=\begin{pmatrix} U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ \cdot\\ \cdot\\ \cdot\\ U_{n+4}\\ U_{n+5}\end{pmatrix}=\begin{pmatrix} u_t-u_x+\sqrt{2}^{-1}u_y+\sqrt{2}^{-1}u_z\\ u_t+u_x+\sqrt{2}^{-1}u_y+\sqrt{2}^{-1}u_z\\ u_y\\ u_z\\ u_{w_1}\\ \cdot\\ \cdot\\ \cdot\\ u_{w_n}\\ u\end{pmatrix}, \tag{4.10}$$

for the solution u of the problem (P.3) which belongs to $\mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^4 \times \boldsymbol{R}^n]$.

PROPOSITION 4.4. *For U in* (4.10), *we have*

$$\begin{cases} \tilde{M}_2 U_t = \tilde{A}_2 U_x + \tilde{A}_{20} U_y + \tilde{A}_{200} U_z + \sum_{j=1}^{n} \tilde{A}_{2j} U_{w_j} + \tilde{E}_2 U + \tilde{F}_2 \\ (t, x, y, z, w) \in (\boldsymbol{R}_+^1)^4 \times \boldsymbol{R}^n \end{cases} \tag{4.11}$$

and

$$\begin{cases} U|_{y=0} = {}^t\!\left(h_t - h_x + \dfrac{u_y}{\sqrt{2}} + \dfrac{h_z}{\sqrt{2}}, h_t + h_x + \dfrac{u_y}{\sqrt{2}} + \dfrac{h_z}{\sqrt{2}}, u_y, h_z, h_{w_1}, \cdots, h_{w_n}, h\right) \\ U|_{z=0} = {}^t\!\left(k_t - k_x + \dfrac{k_y}{\sqrt{2}} + \dfrac{u_z}{\sqrt{2}}, k_t + k_x + \dfrac{k_y}{\sqrt{2}} + \dfrac{u_z}{\sqrt{2}}, k_y, u_z, k_{w_1}, \cdots, k_{w_n}, k\right), \end{cases} \tag{4.12}$$

where

$$\tilde{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & \boldsymbol{0} & \\ 0 & 0 & 1 & -1 & & & & \\ 0 & 0 & -1 & 1 & & & & \\ & & & & 2 & & & \\ & & & & & \ddots & & \\ & \boldsymbol{0} & & & & & 2 & \\ & & & & & & & 1 \end{pmatrix},$$

$\tilde{A}_2 = \operatorname{diag}(-1, 1, 0, 0, \cdots, 0, 1)$,

$$\tilde{A}_{20} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/2 & -1/2 & & & & \\ 1/\sqrt{2} & 0 & 1/2 & -1/2 & & & & \\ 1/2 & 1/2 & -1/\sqrt{2} & 1/\sqrt{2} & & & \boldsymbol{0} & \\ -1/2 & -1/2 & 1/\sqrt{2} & -1/\sqrt{2} & & & & \\ & & & & -\sqrt{2} & & & \\ & & & & & \ddots & & \\ & & \boldsymbol{0} & & & & -\sqrt{2} & \\ & & & & & & & 1 \end{pmatrix},$$

$$\tilde{A}_{200}=\begin{pmatrix}
0 & 1/\sqrt{2} & -1/2 & 1/2 & & & & \\
1/\sqrt{2} & 0 & -1/2 & 1/2 & & & & \\
-1/2 & -1/2 & -1\sqrt{2} & 1/\sqrt{2} & & \mathbf{0} & & \\
1/2 & 1/2 & 1/\sqrt{2} & -1/\sqrt{2} & & & & \\
 & & & & -\sqrt{2} & & & \\
 & & \mathbf{0} & & & \ddots & & \\
 & & & & & & -\sqrt{2} & \\
 & & & & & & & 1
\end{pmatrix},$$

$$\tilde{A}_{2j}=\begin{array}{r c}
 & \hphantom{1\ 1\ 0\ \cdots\ }(j+4)\ \text{column} \\
 \begin{array}{r} \\ \\ \\ \\ \\ (j+4)\ \text{row} \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
 & & & & 1 & & 0 \\
 & & & & 1 & & \cdot \\
 & & \mathbf{0} & & 0 & & \cdot \\
 & & & & \cdot & \mathbf{0} & \cdot \\
 & & & & \cdot & & \cdot \\
1 & 1 & 0 & \cdots & 0 & & \cdot \\
 & & & & & & \cdot \\
 & & \mathbf{0} & & & & \cdot \\
 & & & & & & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}
\end{array} \qquad (j=1,\cdots,n),$$

$\tilde{E}_2 \cdots$ *an* $(n+5)\times(n+5)$ *matrix with the same property as* E_3 *in* §3,

and

$\tilde{F}_2={}^t(f, f, 0, \cdots, 0)$.

PROOF. By simple calculations we obtain Proposition 4.4. Q.E.D.

PROPOSITION 4.5. *For U in* (4.10), *the following inequality holds,*

$$\begin{aligned}
(4.13)\qquad & \frac{d}{dt}(\tilde{M}_2 e^{-\mu t}U, e^{-\mu t}U) \\
& \leq C_1\langle e^{-\mu t}U, e^{-\mu t}U\rangle - C_2\langle\langle e^{-\mu t}u_y, e^{-\mu t}u_y\rangle\rangle - C_3\langle\langle\langle e^{-\mu t}u_z, e^{-\mu t}u_z\rangle\rangle\rangle \\
& \quad + C_4\langle\langle e^{-\mu t}h\rangle\rangle_1^2 + C_5\langle\langle\langle e^{-\mu t}k\rangle\rangle\rangle_1^2 + \frac{C_6}{\mu}(e^{-\mu t}f, e^{-\mu t}f)
\end{aligned}$$

for all $\mu \geq \mu_3$ *where* μ_3, C_1, C_2, C_3, C_4, C_5 *and* C_6 *are positive constants.*

PROOF. By the same method as the one in the proof of Proposition 4.2, and Proposition 4.4, we have Proposition 4.5. Q.E.D.

§5. The proof of Main Theorem 1.

In this section, we shall prove Main Theorem 1.

Firstly, we consider U which is defined by (3.9) for the solution $u \in \mathscr{H}_{2,\mu}[(\boldsymbol{R}^1_+)^3 \times \boldsymbol{R}^n]$ of the problem (P.1) under the condition (A.I). By Theorem 3.2, we obtain

$$
\begin{aligned}
(5.1)\qquad & \frac{d}{dt}(M_3 e^{-\mu t}U, e^{-\mu t}U) \\
&= -2\mu(M_3 e^{-\mu t}U, e^{-\mu t}U) + (e^{-\mu t}M_3 U_t, e^{-\mu t}U) + (e^{-\mu t}U, e^{-\mu t}M_3 U_t) \\
&= -2\mu(M_3 e^{-\mu t}U, e^{-\mu t}U) \\
&\quad + \left(e^{-\mu t}\left(A_3 U_x + A_{30}U_y + \sum_{j=1}^{n} A_{3j}U_{w_j} + E_3 U + F_3\right), e^{-\mu t}U\right) \\
&\quad + \left(e^{-\mu t}U, e^{-\mu t}\left(A_3 U_x + A_{30}U_y + \sum_{j=1}^{n} A_{3j}U_{w_j} + E_3 U + F_3\right)\right) \\
&\leq -C_1\mu(e^{-\mu t}U, e^{-\mu t}U) + \frac{C_2}{\mu}(e^{-\mu t}F_3, e^{-\mu t}F_3) \\
&\quad - \langle A_3 e^{-\mu t}U, e^{-\mu t}U\rangle - \langle\!\langle A_{30}e^{-\mu t}U, e^{-\mu t}U\rangle\!\rangle
\end{aligned}
$$

where C_1 and C_2 are positive constants.

Now, we set

$$
(5.2)\qquad P_{31} = \begin{pmatrix} 1 & -\dfrac{1-c}{\sqrt{\delta_0}} & -\dfrac{2b}{1+c} & -\dfrac{2d_1}{1+c} & \cdots & -\dfrac{2d_n}{1+c} & 0 \\ & & & \mathbf{0} & & & \end{pmatrix}
$$

and

$$
(5.3)\qquad Q_{31} = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & \mathbf{0} & & \\ 0 & 0 & 0 & & & & \\ \hline & & & 1 & & & 0 \\ & \mathbf{0} & & & \ddots & & \\ & & & 0 & & \ddots & \\ & & & & & & 1 \end{array}\right)
$$

where P_{31} and Q_{31} are $(n+4)\times(n+4)$ matrices. Then, we have

$$
P_3(U - P_{31}U)|_{x=0} = 0 \quad \text{and} \quad Q_3(U - Q_{31}U)|_{y=0} = 0\,.
$$

Using G_3 and H_3 which are defined in (3.13), we get

$$
(5.4) \qquad \begin{cases} U-P_{31}U|_{x=0}\in \operatorname{Ker} P_3 \\ P_{31}U|_{x=0}={}^t(G_3, 0, \cdots, 0)=G_{31} \end{cases}
$$

and

$$
(5.5) \qquad \begin{cases} U-Q_{31}U|_{y=0}\in \operatorname{Ker} Q_3 \\ Q_{31}U|_{y=0}={}^t(\tilde{h}_t-\tilde{h}_x-\tilde{\gamma}\tilde{h}, \theta_{30}(\tilde{h}_t+\tilde{h}_x+\tilde{\gamma}\tilde{h}), 0, \tilde{h}_{w_1}, \cdots, \tilde{h}_{w_n}, \tilde{h})=H_{31}\,. \end{cases}
$$

Then, by (3.14) and (5.4), we obtain

$$
\begin{aligned}
(5.6) \qquad \langle A_3 e^{-\mu t}U, e^{-\mu t}U\rangle &= \langle A_3 e^{-\mu t}(I-P_{31})U, e^{-\mu t}(I-P_{31})U\rangle \\
&\quad +2\operatorname{Re}\langle A_3 e^{-\mu t}U, e^{-\mu t}G_{31}\rangle-\langle A_3 e^{-\mu t}G_{31}, e^{-\mu t}G_{31}\rangle \\
&\geq C_3\langle e^{-\mu t}U, e^{-\mu t}U\rangle - C_4\langle e^{-\mu t}G_{31}, e^{-\mu t}G_{31}\rangle
\end{aligned}
$$

where C_3 and C_4 are positive constants. Similarly, by (3.14) and (5.5), we have

$$
\begin{aligned}
(5.7) \qquad \langle\!\langle A_{30} e^{-\mu t}U, e^{-\mu t}U\rangle\!\rangle &= \langle\!\langle A_{30} e^{-\mu t}(I-Q_{31})U, e^{-\mu t}(I-Q_{31})U\rangle\!\rangle \\
&\quad +2\operatorname{Re}\langle\!\langle A_{30} e^{-\mu t}U, e^{-\mu t}H_{31}\rangle\!\rangle-\langle\!\langle A_{30} e^{-\mu t}H_{31}, e^{-\mu t}H_{31}\rangle\!\rangle \\
&\geq -\delta_1\langle\!\langle e^{-\mu t}v_{1y}, e^{-\mu t}v_{1y}\rangle\!\rangle - C_5\langle\!\langle e^{-\mu t}H_{31}, e^{-\mu t}H_{31}\rangle\!\rangle
\end{aligned}
$$

where δ_1 is a sufficiently small positive constant, and C_5 is a positive constant.

Thus, by (5.1), (5.6) and (5.7), we get

$$
\begin{aligned}
(5.8) \qquad \frac{d}{dt}(M_3 e^{-\mu t}U, e^{-\mu t}U) &\leq -C_1\mu(e^{-\mu t}U, e^{-\mu t}U)+\frac{C_2}{\mu}(e^{-\mu t}F_3, e^{-\mu t}F_3) \\
&\quad -C_3\langle e^{-\mu t}U, e^{-\mu t}U\rangle+\delta_1\langle\!\langle e^{-\mu t}v_{1y}, e^{-\mu t}v_{1y}\rangle\!\rangle \\
&\quad +C_4\langle e^{-\mu t}G_{31}, e^{-\mu t}G_{31}\rangle+C_5\langle\!\langle e^{-\mu t}H_{31}, e^{-\mu t}H_{31}\rangle\!\rangle\,.
\end{aligned}
$$

Nextly, for V in (3.10), we have the following inequality by Theorem 3.3 and the same method as the above one,

$$
\begin{aligned}
(5.9) \qquad \frac{d}{dt}(M_4 e^{-\mu t}V, e^{-\mu t}V) &\leq -C_1'\mu(e^{-\mu t}V, e^{-\mu t}V)+\frac{C_2'}{\mu}(e^{-\mu t}F_4, e^{-\mu t}F_4) \\
&\quad -C_3'\langle e^{-\mu t}V, e^{-\mu t}V\rangle+\delta_2\langle\!\langle e^{-\mu t}v_{2y}, e^{-\mu t}v_{2y}\rangle\!\rangle \\
&\quad +C_4'\langle e^{-\mu t}G_{41}, e^{-\mu t}G_{41}\rangle+C_5'\langle\!\langle e^{-\mu t}H_{41}, e^{-\mu t}H_{41}\rangle\!\rangle
\end{aligned}
$$

where

$$
(5.10) \qquad \begin{cases} G_{41}={}^t(G_4, 0, \cdots, 0) \\ H_{41}={}^t(\tilde{\tilde{h}}_t-\tilde{\tilde{h}}_x-\tilde{\gamma}\tilde{\tilde{h}}, \theta_{40}(\tilde{\tilde{h}}_t+\tilde{\tilde{h}}_x+\tilde{\gamma}\tilde{\tilde{h}}), 0, \tilde{\tilde{h}}_{w_1}, \cdots, \tilde{\tilde{h}}_{w_n}, \tilde{\tilde{h}})\,, \end{cases}
$$

C_j' $(j=1, \cdots, 5)$ is a positive constant and δ_2 is a sufficiently small positive constant.

LEMMA 5.1. *Let u be the solution of the problem* (P.1) *which belongs to $\mathscr{H}_{2,\mu}[(\boldsymbol{R}^1_+)^3\times\boldsymbol{R}^n]$. Then, there exist positive constants C and $\tilde{\mu}_1$ such that the following inequality holds for all $t\in\boldsymbol{R}^1_+$ and all $\mu\geq\tilde{\mu}_1$,*

$$\begin{aligned}(5.11)\qquad &|||\mu u(t)|||^2_{0,\mu}+\mu\|\mu u\|^2_{0,\mu,t}+\langle\mu u\rangle^2_{0,\mu,t}+\langle\!\langle\mu u\rangle\!\rangle^2_{0,\mu,t}\\ &\leq C\{|||u(0)|||^2_{1,\mu}+\mu\|u_t\|^2_{0,\mu,t}+\mu\|u_x\|^2_{0,\mu,t}+\mu\|u_y\|^2_{0,\mu,t}\}\,.\end{aligned}$$

PROOF. See [7]. Q.E.D.

PROOF OF MAIN THEOREM 1. By Proposition 4.2, (5.8), (5.9) and Lemma 5.1, we have Main Theorem 1. Q.E.D.

§6. The proof of Main Theorem 2.

In this section, we shall prove Main Theorem 2.

Firstly, we consider U which is defined by (3.9) for the solution $u\in\mathscr{H}_{2,\mu}[(\boldsymbol{R}^1_+)^3\times\boldsymbol{R}^n]$ of the problem (P.2) under the condition (A.I). By Theorem 3.4 and the same method as the one in §5, we obtain

$$\begin{aligned}(6.1)\qquad &\frac{d}{dt}(M_3e^{-\mu t}U, e^{-\mu t}U)\leq -C_1\mu(e^{-\mu t}U, e^{-\mu t}U)+\frac{C_2}{\mu}(e^{-\mu t}F_3, e^{-\mu t}F_3)\\ &-\langle A_3e^{-\mu t}U, e^{-\mu t}U\rangle-\langle\!\langle A_{30}e^{-\mu t}U, e^{-\mu t}U\rangle\!\rangle\end{aligned}$$

where C_1 and C_2 are positive constants. Now, we set

$$(6.2)\qquad P_{51}=P_{31}\qquad(\text{for } P_{31},\text{ see } (5.2))$$

and

$$(6.3)\qquad Q_{51}=\begin{pmatrix} & 0 & 0 & & \\ & & 0 & 0 & \\ 0 & 0 & 1 & & \\ & 0 & & & \end{pmatrix},$$

where P_{51} and Q_{51} are $(n+4)\times(n+4)$ matrices. Then, we obtain

$$P_3(U-P_{51}U)|_{x=0}=0\quad\text{and}\quad Q_5(U-Q_{51}U)|_{y=0}=0\,.$$

By G_3 and H_5 which are defined in (3.31), we have

$$(6.4)\qquad\begin{cases}(U-P_{51}U)|_{x=0}\in\operatorname{Ker}P_3\,,\quad (U-Q_{51}U)|_{y=0}\in\operatorname{Ker}Q_5\\ P_{51}U|_{x=0}={}^t(G_3,0,\cdots,0)=G_{51}\\ Q_{51}U|_{y=0}={}^t(0,0,H_5,0,\cdots,0)=H_{51}\,.\end{cases}$$

Then, by (3.32) and (6.4), we get

(6.5) $$\langle A_3 e^{-\mu t}U, e^{-\mu t}U\rangle \geq C_3\langle e^{-\mu t}U, e^{-\mu t}U\rangle - C_4\langle e^{-\mu t}G_{51}, e^{-\mu t}G_{51}\rangle$$

where C_3 and C_4 are positive constants. Similarly, we have

(6.6) $$\begin{aligned}\langle\!\langle A_{30}e^{-\mu t}U, e^{-\mu t}U\rangle\!\rangle &= \langle\!\langle A_{30}e^{-\mu t}(I-Q_{51})U, e^{-\mu t}(I-Q_{51})U\rangle\!\rangle \\ &\quad + 2\mathrm{Re}\langle\!\langle A_{30}e^{-\mu t}U, e^{-\mu t}H_{51}\rangle\!\rangle - \langle\!\langle A_{30}e^{-\mu t}H_{51}, e^{-\mu t}H_{51}\rangle\!\rangle \\ &\geq -\delta_3\mu\langle\!\langle \Lambda_{x,w,\mu}^{-1/2}e^{-\mu t}U, \Lambda_{x,w,\mu}^{-1/2}e^{-\mu t}U\rangle\!\rangle \\ &\quad -\frac{C_5}{\mu}\langle\!\langle \Lambda_{x,w,\mu}^{1/2}e^{-\mu t}H_{51}, \Lambda_{x,w,\mu}^{1/2}e^{-\mu t}H_{51}\rangle\!\rangle\end{aligned}$$

where δ_3 is a sufficiently small positive constant and C_5 is a positive constant. Thus, by (6.1), (6.5) and (6.6), we obtain

(6.7) $$\begin{aligned}\frac{d}{dt}(M_3e^{-\mu t}U, e^{-\mu t}U) &\leq -C_1\mu(e^{-\mu t}U, e^{-\mu t}U) + \frac{C_2}{\mu}(e^{-\mu t}F_3, e^{-\mu t}F_3) \\ &\quad - C_3\langle e^{-\mu t}U, e^{-\mu t}U\rangle + \delta_3\mu\langle\!\langle \Lambda_{x,w,\mu}^{-1/2}e^{-\mu t}U, \Lambda_{x,w,\mu}^{-1/2}e^{-\mu t}U\rangle\!\rangle \\ &\quad + C_4\langle e^{-\mu t}G_{51}, e^{-\mu t}G_{51}\rangle + \frac{C_5}{\mu}\langle\!\langle \Lambda_{x,w,\mu}^{1/2}e^{-\mu t}H_{51}, \Lambda_{x,w,\mu}^{1/2}e^{-\mu t}H_{51}\rangle\!\rangle\,.\end{aligned}$$

Nextly, for V in (3.10), we have the following inequality by Theorem 3.5 and the same method as the above one,

(6.8) $$\begin{aligned}\frac{d}{dt}(M_4e^{-\mu t}V, e^{-\mu t}V) &\leq -C_1'\mu(e^{-\mu t}V, e^{-\mu t}V) + \frac{C_2'}{\mu}(e^{-\mu t}F_4, e^{-\mu t}F_4) \\ &\quad - C_3'\langle e^{-\mu t}V, e^{-\mu t}V\rangle + \delta_4\mu\langle\!\langle \Lambda_{x,w,\mu}^{-1/2}e^{-\mu t}V, \Lambda_{x,w,\mu}^{-1/2}e^{-\mu t}V\rangle\!\rangle \\ &\quad + C_4'\langle e^{-\mu t}G_{61}, e^{-\mu t}G_{61}\rangle + \frac{C_5'}{\mu}\langle\!\langle \Lambda_{x,w,\mu}^{1/2}e^{-\mu t}H_{61}, \Lambda_{x,w,\mu}^{1/2}e^{-\mu t}H_{61}\rangle\!\rangle\end{aligned}$$

where

(6.9) $$G_{61} = {}^t(G_4, \cdots, 0)\,, \qquad H_{61} = {}^t(0, 0, H_6, 0, \cdots, 0)\,,$$

C_j' $(j=1, \cdots, 5)$ is a positive constant and δ_4 is a sufficiently small positive constant.

Lemma 6.1. *Let $u \in \mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^3 \times \boldsymbol{R}^n]$ be the solution of the problem* (P.2). *Then, there exist positive constants C and $\tilde{\mu}_2$ such that the following inequality holds for all $t \in \boldsymbol{R}_+^1$ and all $\mu \geq \tilde{\mu}_2$,*

(6.10) $$\begin{aligned}&|||\mu u(t)|||_{0,\mu}^2 + \mu\|\mu u\|_{0,\mu,t}^2 + \langle \mu u\rangle_{0,\mu,t}^2 + \mu\langle\!\langle \Lambda_{x,w,\mu}^{-1/2}\mu u\rangle\!\rangle_{0,\mu,t}^2 \\ &\leq C\{|||u(0)|||_{1,\mu}^2 + \mu\|u_t\|_{0,\mu,t}^2 + \mu\|u_x\|_{0,\mu,t}^2 + \mu\|u_y\|_{0,\mu,t}^2\}\,.\end{aligned}$$

Proof. See [16]. Q.E.D.

Proof of Main Theorem 2. By (6.7), (6.8), Proposition 4.3 and Lemma 6.1, we have Main Theorem 2. Q.E.D.

§7. The proof of Main Theorem 3.

In this section, we shall prove Main Theorem 3.

LEMMA 7.1. *Let u be the solution of the problem* (P.3) *which belongs to* $\mathscr{H}_{2,\mu}[(R^1_+)^4\times R^n]$. *Then, there exist positive constants C and $\tilde{\mu}_3$ such that the following inequality holds for all $t\in R^1_+$ and all $\mu\geq\tilde{\mu}_3$:*

$$\begin{aligned}(7.1)\qquad & |||\mu u(t)|||^2_{0,\mu}+\mu\|\mu u\|^2_{0,\mu,t}+\langle\mu u\rangle^2_{0,\mu,t}+\langle\!\langle\mu u\rangle\!\rangle^2_{0,\mu,t}+\langle\!\langle\!\langle\mu u\rangle\!\rangle\!\rangle^2_{0,\mu,t}\\ &\leq C\{|||u(0)|||^2_{1,\mu}+\mu\|u_t\|^2_{0,\mu,t}+\mu\|u_x\|^2_{0,\mu,t}+\mu\|u_y\|^2_{0,\mu,t}+\mu\|u_z\|^2_{0,\mu,t}\}\,.\end{aligned}$$

PROOF. By the same method as the one in [7], we have Lemma 7.1. Q.E.D.

PROOF OF MAIN THEOREM 3. By the same method as the one in §5, Theorem 3.7, Theorem 3.8, Proposition 4.4 and Lemma 7.1, we obtain Main Theorem 3. Q.E.D.

Appendix. On a result in [27].

We state a result in [27] which is used in §3.

We treat the mixed problem

$$(\mathrm{P})\quad\begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-2\sum\limits_{j=1}^n h_j(t,x)\dfrac{\partial^2 u}{\partial t\partial x_j}-\sum\limits_{i,j=1}^n a_{ij}(t,x)\dfrac{\partial^2 u}{\partial x_i\partial x_j}\\ \qquad\qquad +a_0(t,x)\dfrac{\partial u}{\partial t}+\sum\limits_{j=1}^n a_j(t,x)\dfrac{\partial u}{\partial x_j}+d(t,x)u=f(t,x)\\ u(0,x)=u_0(x)\,,\qquad u_t(0,x)=u_1(x)\\ B[u]|_{x_1=0}=a_{11}(t,0,x')^{-1/2}\left\{a_{11}(t,0,x')\dfrac{\partial u}{\partial x_1}+\sum\limits_{j=2}^n a_{1j}(t,0,x')\dfrac{\partial u}{\partial x_j}\right.\\ \quad\left.+h_1(t,0,x')\dfrac{\partial u}{\partial t}\right\}+\sum\limits_{j=2}^n b_j(t,x')\dfrac{\partial u}{\partial x_j}-c(t,x')\left(1+\dfrac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{1/2}\\ \quad\cdot\left\{\dfrac{\partial u}{\partial t}-\left(1+\dfrac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{-1}\sum\limits_{j=2}^n\left(h_j(t,0,x')-\dfrac{h_1(t,0,x')}{a_{11}(t,0,x')}a_{1j}(t,0,x')\right)\dfrac{\partial u}{\partial x_j}\right\}\\ \quad+\gamma(t,x')u|_{x_1=0}=g(t,x')\\ (t,x)=(t,x_1,x')\in R^1_+\times R^1_+\times R^{n-1}\end{cases}$$

where $x=(x_1,x_2,\cdots,x_n)$, $x'=(x_2,\cdots,x_n)$, $n\geq 2$, the coefficients h_j and a_{ij} (a_0, a_j and d) belong to $\mathscr{B}^1(R\times\overline{R^n_+})$ ($\mathscr{B}^0(R\times\overline{R^n_+})$) and are constant outside a compact set in $R\times\overline{R^n_+}$, the coefficients b_j, c and γ belong to $\mathscr{B}^1(\boldsymbol{R}\times\boldsymbol{R}^{n-1})$ and are constant outside a compact set in $\boldsymbol{R}\times\boldsymbol{R}^{n-1}$.

We assume the following conditions for the problem (P):

(C.1) The operator L is regularly hyperbolic on $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$ and $a_{11}(t, x)>0$ on $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$.

(C.2)
$$|1+c(t, x')|-|1-c(t, x')| > 2\left\{\sup_{\eta'}|\operatorname{Re} b(t, x', \eta')|^2+\sup_{\eta'}|\operatorname{Im} b(t, x', \eta')|^2\right\}^{1/2}$$

for all $(t, x', \eta')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}\times(\boldsymbol{R}^{n-1}-\{0\})$ where

$$(1)\quad\begin{cases} b(t, x', \eta')=\sum_{j=2}^{n} b_j(t, x')\eta'/d(t, x', \eta') \\ d(t, x', \eta')=\Bigg[\sum_{i,j=2}^{n} a_{ij}(t, 0, x')\eta_i\eta_j-\dfrac{1}{a_{11}(t, 0, x')}\left(\sum_{j=2}^{n} a_{1j}(t, 0, x')\eta_j\right)^2 \\ \qquad+\left(1+\dfrac{h_1(t, 0, x')^2}{a_{11}(t, 0, x')}\right)^{-1}\left\{\sum_{j=2}^{n}\left(h_j(t, 0, x')-\dfrac{h_1(t, 0, x')}{a_{11}(t, 0, x')}a_{1j}(t, 0, x')\right)\eta_j\right\}^2\Bigg]^{1/2} \\ \eta'=(\eta_2, \cdots, \eta_n)\in\boldsymbol{R}^{n-1}-\{0\}\,. \end{cases}$$

REMARK. In (C.1), the condition $a_{11}(t, x)>0$ on $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$ is replaced by the condition $a_{11}(t, 0, x')>0$ on $\overline{\boldsymbol{R}^1_+}\times\boldsymbol{R}^{n-1}$.

By the treatment in the framework of L^2-theory, we have only to consider the following two cases respectively:

$$\text{(I)}\quad |1-c(t, x')|<\frac{1}{2}\inf_{(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}}\left\{1-\frac{2}{|1+c(t, x')|}\left(\sum_{j=2}^{n}|\tilde{b}_j(t, x')|^2\right)^{1/2}\right\},$$

$$\text{(II)}\quad |1-c(t, x')|>\frac{1}{4}\inf_{(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}}\left\{1-\frac{2}{|1+c(t, x')|}\left(\sum_{j=2}^{n}|\tilde{b}_j(t, x')|^2\right)^{1/2}\right\}$$

for all $(t, x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}$ where

$$\sigma(L)=\tilde{\xi}^2+\tilde{d}(\eta')^2-\tilde{\tau}^2\,,$$

$$\tilde{\xi}=\frac{1}{\sqrt{a_{11}(t, x)}}\left\{a_{11}(t, x)\xi+\sum_{j=2}^{n} a_{1j}(t, x)\eta_j+h_1(t, x)\tau\right\},$$

$$\tilde{\tau}=\left(1+\frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2}\left\{\tau-\left(1+\frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1}\sum_{j=2}^{n}\left(h_j(t, x)-\frac{h_1(t, x)}{a_{11}(t, x)}a_{1j}(t, x)\right)\eta_j\right\},$$

$$\tilde{d}(\eta')=\Bigg[\sum_{i,j=2}^{n} a_{ij}(t,x)\eta_i\eta_j-\frac{1}{a_{11}(t,x)}\Big(\sum_{j=2}^{n} a_{1j}(t,x)\eta_j\Big)^2 \\ +\Big(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\Big)^{-1}\Big\{\sum_{j=2}^{n}\Big(h_j(t,x)-\frac{h_1(t,x)}{a_{11}(t,x)}a_{1j}(t,x)\Big)\eta_j\Big\}^2\Bigg]^{1/2},$$

$$\tilde{d}(\eta')^2={}^t\eta' M\eta'={}^t\zeta'\cdot\zeta' \quad (\eta'={}^t(\eta_2,\cdots,\eta_n)\,,\ M>0\,,\ \zeta'={}^t(\zeta_2,\cdots,\zeta_n))\,,$$

$${}^tNMN=\begin{pmatrix}\alpha_2(t,x) & & 0\\ & \ddots & \\ 0 & & \alpha_n(t,x)\end{pmatrix}>0\,,$$

$$\zeta'=\begin{pmatrix}\zeta_2\\ \vdots\\ \zeta_n\end{pmatrix}=\begin{pmatrix}\sqrt{\alpha_2} & & 0\\ & \ddots & \\ 0 & & \sqrt{\alpha_n}\end{pmatrix}N^{-1}\cdot\eta'\,,$$

$$\tilde{\zeta}'=\begin{pmatrix}\tilde{\zeta}_2\\ \vdots\\ \tilde{\zeta}_n\end{pmatrix}=\begin{pmatrix}\sqrt{\alpha_2} & & 0\\ & \ddots & \\ 0 & & \sqrt{\alpha_n}\end{pmatrix}N^{-1}\Bigg|_{x_1=0}\cdot\eta'\,,$$

$$\sum_{j=2}^{n} b_j(t,x')\eta_j=\sum_{j=2}^{n}\tilde{b}_j(t,x')\tilde{\zeta}_j\,,\qquad b(t,x',\eta')=\Big(\sum_{j=2}^{n}\tilde{b}_j\tilde{\zeta}_j\Big)\Big/\sqrt{\tilde{\zeta}_2^2+\cdots+\tilde{\zeta}_n^2}$$

and

$$|1+c(t,x')|-|1-c(t,x')|>2\Big(\sum_{j=2}^{n}|\tilde{b}_j(t,x')|^2\Big)^{1/2}.$$

We set

$$Q_0=\Big(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\Big)^{1/2}\Big\{\frac{\partial}{\partial t}-\Big(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\Big)^{-1} \\ \cdot\sum_{j=2}^{n}\Big(h_j(t,x)-\frac{h_1(t,x)}{a_{11}(t,x)}a_{1j}(t,x)\Big)\frac{\partial}{\partial x_j}\Big\},\tag{2}$$

$$Q_1=\frac{1}{\sqrt{a_{11}(t,x)}}\left\{a_{11}(t,x)\frac{\partial}{\partial x_1}+\sum_{j=2}^{n}a_{1j}(t,x)\frac{\partial}{\partial x_j}+h_1(t,x)\frac{\partial}{\partial t}\right\}+\gamma(t,x'), \tag{3}$$

$$Q_j=\sum_{l=2}^{n}p_{jl}(t,x)\frac{\partial}{\partial x_l}\qquad (j=2,\cdots,n) \tag{4}$$

where

$$\begin{pmatrix} p_{22}(t,x) & \cdots & p_{2n}(t,x) \\ \vdots & & \vdots \\ p_{n2}(t,x) & \cdots & p_{nn}(t,x) \end{pmatrix}=\begin{pmatrix} \sqrt{\alpha_2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\alpha_n} \end{pmatrix}N^{-1}. \tag{5}$$

Then, we have

$$L=Q_0\cdot Q_0-Q_1\cdot Q_1-\sum_{j=2}^{n}Q_j\cdot Q_j+(\text{lower order operator}). \tag{6}$$

Case (I): $|1-c(t,x')|<\dfrac{1}{2}\inf_{(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}}\left\{1-\dfrac{2}{|1+c(t,x')|}\left(\sum_{j=2}^{n}|\tilde{b}_j(t,x')|^2\right)^{1/2}\right\}$

for all $(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}$.

We set

$$u=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \cdot \\ \cdot \\ \cdot \\ U_{n+1} \\ U_{n+2} \end{pmatrix}=\begin{pmatrix} Q_0u-Q_1u \\ z_1(Q_0u+Q_1u) \\ Q_2u \\ Q_3u \\ \cdot \\ \cdot \\ \cdot \\ Q_nu \\ u \end{pmatrix} \tag{7}$$

where u is the solution of the problem (P),

$$z_1=\frac{\sqrt{\varepsilon_0}}{1+c(t,x')} \tag{8}$$

and

$$\varepsilon_0=\frac{1}{2}\inf_{(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}}\left\{1-\frac{2}{|1+c(t,x')|}\left(\sum_{j=2}^{n}|\tilde{b}_j(t,x')|^2\right)^{1/2}\right\}. \tag{9}$$

Now, we state our result in [27].

THEOREM A. *The problem* (P) *under the conditions* (C.1) *and* (C.2) *is transformed into the system*:

$$
(10)\qquad \begin{cases} \tilde{\tilde{M}}_1 U_t = \tilde{\tilde{A}}_{11} U_{x_1} + \sum_{j=2}^{n} \tilde{\tilde{A}}_{1j} U_{x_j} + \tilde{\tilde{E}}_1 U + \tilde{\tilde{F}}_1(t, x) \\ U(0, x) = U_0(x) \\ \tilde{\tilde{P}}_1 U\big|_{x_1=0} = \tilde{\tilde{G}}_1(t, x') \\ (t, x) = (t, x_1, x') \in \boldsymbol{R}^1_+ \times \boldsymbol{R}^1_+ \times \boldsymbol{R}^{n-1} \end{cases}
$$

where

$$
\begin{aligned}
\tilde{\tilde{M}}_1 &= \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} \operatorname{diag}(1, 1, 1+|z_1|^2, \cdots, 1+|z_1|^2, 1) \\
&\quad - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}} \operatorname{diag}(-1, 1, 1-|z_1|^2, \cdots, 1-|z_1|^2, 1)\,, \\
\tilde{\tilde{A}}_{11} &= \sqrt{a_{11}(t, x)}\, \operatorname{diag}(-1, 1, 1-|z_1|^2, \cdots, 1-|z_1|^2, 1)\,, \\
\tilde{\tilde{A}}_{1j} &= \frac{a_{1j}(t, x)}{a_{11}(t, x)} \tilde{\tilde{A}}_{11} + \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1/2} \left(h_j(t, x) - \frac{h_1(t, x)}{a_{11}(t, x)} a_{1j}(t, x)\right) \\
&\quad \cdot \operatorname{diag}(1, 1, 1+|z_1|^2, \cdots, 1+|z_1|^2, 1)
\end{aligned}
$$

$$
+ \sum_{l=2}^{n} p_{lj}
\begin{array}{r}
\\ \\ \\ \\ \\ \\ \\ (l+1)\ \text{row} \\ \\ \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} & & & & & (l+1)\ \text{column} & & & & \end{array} \\
\begin{pmatrix}
0 & \cdot & \cdot & \cdot & 0 & 1 & & & & 0 \\
0 & \cdot & \cdot & \cdot & 0 & z_1 & & & & \cdot \\
\cdot & & & & \cdot & 0 & & & & \cdot \\
\cdot & & & & \cdot & \cdot & & & & \cdot \\
\cdot & & & & \cdot & \cdot & & 0 & & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & 0 & & & & \cdot \\
1 & \bar{z}_1 & 0 & \cdot & 0 & 0 & & & & \cdot \\
 & & & & & & & & & \cdot \\
 & & & & 0 & & & & & \cdot \\
 & & & & & & & & & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1
\end{pmatrix}
\end{array}
$$

$$(j = 2, \cdots, n)\,,$$

$\tilde{\tilde{E}}_1 \cdots$ *an* $(n+2)\times(n+2)$ *matrix which has the property that for* $\tilde{\tilde{E}}_1=(e_{ij})$, e_{ij} *belongs to* $\mathscr{B}^0(\boldsymbol{R}\times\boldsymbol{R}^n_+)$ *and is a constant outside a compact set in* $\boldsymbol{R}\times\overline{\boldsymbol{R}^n_+}$,

$\tilde{\tilde{F}}_1={}^t(f, z_1 f, 0, \cdots, 0)$,

$$\tilde{\tilde{P}}_1=\left(1,\ -\frac{1-c(t,x')}{\sqrt{\varepsilon_0}},\ -\frac{2}{1+c}\tilde{b}_2, \cdots, -\frac{2}{1+c}\tilde{b}_n, 0\right),$$

$$\tilde{\tilde{G}}_1=-\frac{2g(t,x')}{1+c(t,x')}$$

and there is a positive constant C such that

$$[\tilde{\tilde{A}}_{11}U, U]\geq C[U, U] \qquad \textit{for all} \quad U\in \operatorname{Ker}\tilde{\tilde{P}}_1 . \tag{11}$$

Case (II): $|1-c(t,x')|>\dfrac{1}{4}\inf\limits_{(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}}\left\{1-\dfrac{2}{|1+c(t,x')|}\left(\sum\limits_{j=2}^n|\tilde{b}_j(t,x')|^2\right)^{1/2}\right\}$

for all $(t,x')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}$.

For the solution u of the problem (P), we use the same vector U as in (7) where z_1 is replaced by z_2:

$$z_2(t,x')=\sqrt{\frac{1-c(t,x')}{1+c(t,x')}} \qquad (\sqrt{1}=1). \tag{12}$$

Then, we have

THEOREM B. *The problem* (P) *under the conditions* (C.1) *and* (C.2) *is transformed into the system*:

$$\begin{cases} \tilde{\tilde{M}}_2U_t=\tilde{\tilde{A}}_{21}U_{x_1}+\sum\limits_{j=2}^n\tilde{\tilde{A}}_{2j}U_{x_j}+\tilde{\tilde{E}}_2U+\tilde{\tilde{F}}_2 \\ U(0,x)=U_0(x) \\ \tilde{\tilde{P}}_2U|_{x_1=0}=\tilde{\tilde{G}}_2(t,x') \\ (t,x)=(t,x_1,x')\in\boldsymbol{R}^1_+\times\boldsymbol{R}^1_+\times\boldsymbol{R}^{n-1} \end{cases} \tag{13}$$

where

$$\tilde{\tilde{M}}_2=\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{1/2}\operatorname{diag}(1,1,1+|z_2|^2,\cdots,1+|z_2|^2,1)$$
$$-\frac{h_1(t,x)}{\sqrt{a_{11}(t,x)}}\operatorname{diag}(-1,1,1-|z_2|^2,\cdots,1-|z_2|^2,1),$$

$$\tilde{\tilde{A}}_{21}=\sqrt{a_{11}(t,x)}\operatorname{diag}(-1,1,1-|z_2|^2,\cdots,1-|z_2|^2,1),$$

$$\tilde{\tilde{\tilde{A}}}_{2j}=\frac{a_{1j}(t,x)}{a_{11}(t,x)}\tilde{\tilde{\tilde{A}}}_{21}+\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{-1/2}\left(h_j(t,x)-\frac{h_1(t,x)}{a_{11}(t,x)}a_{1j}(t,x)\right)$$

$$\cdot\operatorname{diag}(1,1,1+|z_2|^2,\cdots,1+|z_2|^2,1)$$

$$+\sum_{l=2}^{n} p_{lj}\ \begin{array}{r c} & \scriptstyle (l+1)\ \text{column} \\ \scriptstyle (l+1)\ \text{row} & \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & 1 & & & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & z_2 & & & \cdot \\ \cdot & & & & \cdot & 0 & & & \cdot \\ \cdot & & & & \cdot & \cdot & & & \cdot \\ \cdot & & & & \cdot & \cdot & \mathbf{0} & & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 0 & & & \cdot \\ 1 & \bar{z}_2 & 0 & \cdot & 0 & 0 & & & \cdot \\ & & & & & & & & \cdot \\ & & & \mathbf{0} & & & & & \cdot \\ & & & & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix} \end{array}$$

$$(j=2,\cdots,n),$$

$\tilde{\tilde{\tilde{E}}}_2\cdots$ *an* $(n+2)\times(n+2)$ *matrix which the same property as* $\tilde{\tilde{\tilde{E}}}_1$,

$\tilde{\tilde{\tilde{F}}}_2={}^t(f, z_2 f, 0,\cdots,0)$,

$\tilde{\tilde{\tilde{P}}}_2=\left(1,-z_2,-\frac{2}{1+c}\tilde{b}_2,\cdots,-\frac{2}{1+c}\tilde{b}_n,0\right)$,

$\tilde{\tilde{\tilde{G}}}_2=-\tilde{\tilde{\tilde{G}}}_1$,

and there exists a positive constant C such that

$$[\tilde{\tilde{\tilde{A}}}_{21}U,U]\geq C[U,U]\qquad \textit{for all}\quad U\in\operatorname{Ker}\tilde{\tilde{\tilde{P}}}_2 . \tag{14}$$

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Present Address:
MASARU TANIGUCHI
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY
OKUBO, SHINJUKU-KU, TOKYO 169–50, JAPAN

FUKUZO SUZUKI
GUNMA COLLEGE OF TECHNOLOGY
580 TORIBA-MACHI, MAEBASHI, GUNMA 371, JAPAN