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Reidemeister Torsion of Seifert Fibered Spaces for $SL(2;\mathbf{C})$-Representations

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§0. Introduction.

This paper is devoted to the study of the Reidemeister torsion. It is a piecewise linear invariant for n-dimensional manifolds and originally defined by Reidemeister, Franz and de Rham. In 1985 Casson defined an interesting topological invariant of homology 3-spheres by making use of a beautiful construction on the space of $SU(2)$-representations of the fundamental group. Later Johnson developed a similar theory of Casson's one by using the Reidemeister torsion as its essential ingredient. He also derived an explicit formula for the Reidemeister torsion of Brieskorn homology 3-spheres for $SL(2;\mathbf{C})$-irreducible representations. In this paper, we call this type Reidemeister torsion the $SL(2;\mathbf{C})$-torsion following Johnson. Let M_n be a 3-manifold obtained by the $1/n$-surgery on a torus (p, q)-knot. It is a Brieskorn homology 3-sphere $\Sigma(p, q, pqn\pm 1)$. The fundamental group $\pi_1 M_n$ admits a presentation as follows;

$$\pi_1 M_n = \langle x, y \mid x^p = y^q, ml^n = 1\rangle$$

where m is a meridian of the torus knot which is a word of x and y and l is similarly a longitude. Johnson proved the following theorem.

THEOREM (Johnson). *The distinct conjugacy classes of the $SL(2;\mathbf{C})$-irreducible representations of $\pi_1 M_n$ are given by $\rho_{(a,b,k)}$ such that*

(1) $0<a<p$, $0<b<q$, $a\equiv b \bmod 2$,
(2) $0<k<N=|pqn+1|$, $k\equiv na \bmod 2$,
(3) $\operatorname{tr}\rho_{(a,b,k)}(x)=2\cos \pi a/p$,
(4) $\operatorname{tr}\rho_{(a,b,k)}(y)=2\cos \pi b/q$,
(5) $\operatorname{tr}\rho_{(a,b,k)}(m)=2\cos \pi k/N$.

In this case the $SL(2;\mathbf{C})$-torsion $\tau_{(a,b,k)}$ for $\rho_{(a,b,k)}$ is given by

$$\tau_{(a,b,k)}=\begin{cases} 2(1-\cos\pi a/p)(1-\cos\pi b/q)(1+\cos\pi kpq/N) & a\equiv b\equiv 1,\ k\equiv n \quad \bmod 2 \\ 0 & a\equiv b\equiv 0 \ \text{or}\ k\not\equiv n \quad \bmod 2\,. \end{cases}$$

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His methods can be applied to more general Seifert fibered spaces and give a way to compute the $SL(2;\boldsymbol{C})$-torsion of them.

The main result of this paper is the following theorem. Let M^3 denote the orientable Seifert fibered space given by the following Seifert index

$$\{b, (\varepsilon, g); (\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m)\}.$$

MAIN THEOREM. *Let $\rho: \pi_1 M \to SL(2;\boldsymbol{C})$ be an irreducible representation. Then the $SL(2;\boldsymbol{C})$-torsion $\tau(M; V_\rho)$ is given by*

$$\tau(M;V_\rho)=\begin{cases} 0 & \text{if } H=I \\ 2^{4-m-4g}\prod_{i=1}^{m}\left(1-(-1)^{\nu_i}\cos\dfrac{\rho_i k_i(\rho)\pi}{\alpha_i}\right) & \text{if } H\neq I,\ \varepsilon=o \\ \left(2-2\cos\dfrac{s\pi}{N+1}\right)^{4-m-2g}\prod_{i=1}^{m}\left(1-(-1)^{\nu_i}\cos\dfrac{\rho_i k_i(\rho)\pi}{\alpha_i}\right) & \text{if } H\neq I,\ \varepsilon=n \end{cases}$$

where

(1) $H=\rho(h)$,

(2) *h is a representative element of generic fiber in $\pi_1 M$,*

(3) $\rho_i, \nu_i \in \boldsymbol{Z}$ *such that* $\begin{vmatrix}\alpha_i & \rho_i \\ \beta_i & \nu_i\end{vmatrix}=-1$ *and* $0<\rho_i<\alpha_i$,

(4) $k_i(\rho)\in\boldsymbol{Z}$ *such that* $0\leq k_i\leq\alpha_i$, *and* $k_i(\rho)\equiv\beta_i \bmod 2$,

(5) $N=\beta_1/\alpha_1+\cdots+\beta_m/\alpha_m$,

(6) $s\in\boldsymbol{Z}$ *such that* $0\leq s\leq 2N+2$.

REMARK. (1) In general the dimension of the space of representations of a Seifert fibered space is not zero; in particular the distinct classes of irreducible representations are not finite. However the set of the $SL(2;\boldsymbol{C})$-torsion turns out to be a finite subset in $\boldsymbol{R}$ by this theorem; that is $SL(2;\boldsymbol{C})$-torsion is a constant function on each connected component of the space of irreducible representations.

(2) It may be a problem to determine whether there exists a 3-manifold with continuous variations of the $SL(2;\boldsymbol{C})$-torsion. In fact the answer is yes. In our paper [3], we will prove that the double of the figure-eight knot exterior in S^3 has continuous variations of the $SL(2;\boldsymbol{C})$-torsion.

Now we describe the contents of this paper. In §1 we give the necessary definitions and properties of the $SL(2;\boldsymbol{C})$-torsion following Milnor. In §2 we examine the Reidemeister torsion for the 2-dimensional torus and the solid torus. These results will be used later for the torus decomposition formula. In §3 we investigate $SL(2;\boldsymbol{C})$-irreducible representation of Seifert fibered spaces. In §4, we give a proof of Main theorem for the case of $H=-I$. In §5, we prove the non-acyclicity of the chain complex $C_*(M; V_\rho)$ in the case of $H=I$.

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§1. Definition of the $SL(2;C)$-torsion.

First let us describe the definition of the $SL(2;C)$-torsion, that is, the Reidemeister torsion for $SL(2;C)$-representations. See Johnson [2] and Milnor [4], [5], [6] for details.

Let W be an n-dimensional vector space over C and let $\boldsymbol{b}=(b_1,\cdots,b_n)$ and $\boldsymbol{c}=(c_1,\cdots,c_n)$ be two bases for W. Setting $b_i=\sum_{j=1}^n p_{ij}c_j$, we obtain a nonsingular matrix $P=(p_{ij})$ with entries in C. Let $[\boldsymbol{b}/\boldsymbol{c}]$ denote the determinant of P.

Suppose

$$C_*:\ 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is an acyclic chain complex of finite dimensional vector spaces over C. We assume that a preferred basis $\boldsymbol{c}_q$ for $C_q(C_*)$ is given for each q. Choose some basis $\boldsymbol{b}_q$ for $B_q(C_*)$ and take a lift of it in $C_{q+1}(C_*)$, which we denote by $\tilde{\boldsymbol{b}}_q$.

Since $B_q(C_*)=Z_q(C_*)$, the basis $\boldsymbol{b}_q$ can serve as a basis for $Z_q(C_*)$. Furthermore the sequence

$$0\to Z_q(C_*)\to C_q(C_*)\to B_{q-1}(C_*)\to 0$$

is exact and the vectors $(\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1})$ form a basis for $C_q(C_*)$. It is easily shown that $[\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1}/\boldsymbol{c}_q]$ does not depend on the choice of the lift $\tilde{\boldsymbol{b}}_{q-1}$. Hence we simply denote it by $[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]$.

DEFINITION 1.1. The torsion of the chain complex C_* is given by the alternating product

$$\prod_{q=0}^{m}[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]^{(-1)^q}$$

and we denote it by $\tau(C_*)$.

REMARK. It is easy to see that $\tau(C_*)$ depends only on the bases $\{\boldsymbol{c}_0,\cdots,\boldsymbol{c}_m\}$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let X be a finite cell complex and $\tilde{X}$ a universal covering of X. The fundamental group π_1X acts on $\tilde{X}$ as deck transformations. Then the chain complex $C_*(\tilde{X};\boldsymbol{Z})$ has the structure of a chain complex of free $\boldsymbol{Z}[\pi_1X]$-modules. Let $\rho:\pi_1X\to SL(2;C)$ be a representation. We denote the 2-dimensional vector space C^2 by V. Using the representation ρ, V has the structure of a $\boldsymbol{Z}[\pi_1X]$-module and then we denote it by V_ρ. Define the chain complex $C_*(X;V_\rho)$ by $C_*(\tilde{X};\boldsymbol{Z})\otimes_{Z[\pi_1X]}V_\rho$ and choose a preferred basis

$$\{\sigma_1 \otimes e_1, \sigma_1 \otimes e_2, \cdots, \sigma_{k_q} \otimes e_1, \sigma_{k_q} \otimes e_2\}$$

of $C_q(X; V_\rho)$ where $\{e_1, e_2\}$ is a canonical basis of V and $\sigma_1, \cdots, \sigma_{k_q}$ are q-cells giving the preferred basis of $C_q(\tilde{X}; \boldsymbol{Z})$.

We consider the situation where $C_*(X; V_\rho)$ is acyclic. Namely all homology groups vanish; $H_*(X; V_\rho)=0$. In this case we call ρ an acyclic representation.

DEFINITION 1.2. Let $\rho: \pi_1 X \to SL(2; \boldsymbol{C})$ be an acyclic representation. Then the Reidemeister torsion of X with V_ρ-coefficients is defined to be the torsion of the chain complex $C_*(X; V_\rho)$. We denote it by $\tau(X; V_\rho)$.

REMARK. (1) We define the $SL(2; \boldsymbol{C})$-torsion $\tau(X; V_\rho)$ to be zero for a non-acyclic representation ρ.

(2) The Reidemeister torsion $\tau(X; V_\rho)$ depends on several choices. However it is well known that the Reidemeister torsion is a piecewise linear invariant. See Johnson[2], Milnor [4], [6].

The key lemma of the proof of Main theorem is the following. It gives the torus decomposition formula of the Reidemeister torsion of 3-manifolds. See Johnson [2], Milnor [6].

LEMMA 1.3. *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be an exact sequence of n-dimensional chain complexes with preferred bases $\{c'_i\}$, $\{c_i\}$ and $\{c''_i\}$ such that $[c'_i, c''_i / c_i]=1$ for $\forall i$. Suppose any two of the complexes are acyclic. Then the third one is also acyclic and the torsion of the three complexes are all well-defined. Moreover the next formula holds*:

$$\tau(C_*)=(-1)^{\sum_{i=0}^{n} \beta'_{i-1}\beta''_i}\,\tau(C'_*)\tau(C''_*)$$

where $\beta'_i=\dim \partial C'_{i+1}$ and $\beta''_i=\dim \partial C''_{i+1}$.

PROOF. It is easy to show the acyclicity of the third one from the homology long exact sequence of $0 \to C'_* \to C_* \to C''_* \to 0$.

To see the required formula, we consider the next diagram for $\forall i$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \partial C'_{i+1} & \longrightarrow & \partial C_{i+1} & \longrightarrow & \partial C''_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C'_i & \longrightarrow & C_i & \longrightarrow & C''_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \partial C'_i & \longrightarrow & \partial C_i & \longrightarrow & \partial C''_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

DIAGRAM 1.

Choose bases b_i' of $\partial C_{i+1}'$ and b_i'' of $\partial C_{i+1}''$ and then we get a basis of ∂C_{i+1}, $b_i=(b_i', b_i'')$. We will show that

$$\tau(C_*')\tau(C_*'')\tau(C_*)^{-1}=(-1)^{\sum_{i=0}^n \beta_{i-1}'\beta_i''} .$$

Here from the definition of the torsion,

$$\tau(C_*')\tau(C_*'')\tau(C_*)^{-1}=\prod_{i=0}^n [b_i', b_{i-1}'/c_i']^{(-1)^i}[b_i'', b_{i-1}''/c_i'']^{(-1)^i}[b_i, b_{i-1}/c_i]^{(-1)^{i+1}} .$$

Note that this value does not depend on the choice of b_i' and b_i''. Consequently we may assume that

$$[b_i', b_{i-1}'/c_i']=[b_i'', b_{i-1}''/c_i'']=1 .$$

Hence

$$\tau(C_*')\tau(C_*'')\tau(C_*)^{-1}=\prod_{i=0}^n [b_i, b_{i-1}/c_i]^{(-1)^{i+1}} .$$

Moreover, from the assumptions, we may choose identifications

$$\partial C_{i+1}\cong\partial C_i'\oplus\partial C_i'' , \quad C_i\cong C_i'\oplus C_i'' , \quad \partial C_i\cong\partial C_i'\oplus\partial C_i'' ,$$
$$C_i'\cong\partial C_{i+1}'\oplus\partial C_i' , \qquad C_i''\cong\partial C_{i+1}''\oplus\partial C_i'' .$$

Thereby we can identify C_i with $\partial C_{i+1}'\oplus\partial C_i'\oplus\partial C_{i+1}''\oplus\partial C_i''$ and get a basis for C_i

$$(b_i', b_{i-1}', b_i'', b_{i-1}'')=(c_i', c_i'')=c_i .$$

Moreover we have the following as an oriented basis,

$$\begin{aligned}(b_i', b_{i-1}', b_i'', b_{i-1}'')&=(-1)^{\beta_{i-1}'\beta_i''}(b_i', b_i'', b_{i-1}', b_{i-1}'')\\ &=(-1)^{\beta_{i-1}'\beta_i''}(b_i, b_{i-1}) .\end{aligned}$$

Hence

$$\begin{aligned}&[b_i', b_{i-1}'/c_i'][b_i'', b_{i-1}''/c_i''][b_i, b_{i-1}/c_i]^{-1}\\ &\qquad=1\cdot1\cdot(-1)^{\beta_{i-1}'\beta_i''}=(-1)^{\beta_{i-1}'\beta_i''} .\end{aligned}$$

Therefore

$$\tau(C_*')\tau(C_*'')\tau(C_*)^{-1}=(-1)^{\sum_{i=0}^n \beta_{i-1}'\beta_i''} .$$

This completes the proof of Lemma 1.3.

§2. Examples of $SL(2; C)$-torsion.

In this section, we compute the $SL(2; C)$-torsion of the torus T^2 and the solid torus S. First we consider the condition of the acyclicity of T^2. When a representation

ρ is fixed, we denote the matrix $\rho(x)$ for $\forall x$ by the corresponding capital letter X. Recall that we denote the 2-dimensional complex vector space C^2 by V and the canonical basis of V by $\{e_1, e_2\}$.

DEFINITION 2.1. A parabolic element of $SL(2; C)$ is a nontrivial element which fixes some nonzero vector in V. Equivalently an element is parabolic if it is conjugate to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $\exists t \in C - \{0\}$.

DEFINITION 2.2. Let $\rho : \pi_1 T^2 \to SL(2; C)$ be a representation. Then it is called a parabolic representation if X is either trivial or a parabolic element in $SL(2; C)$ for $\forall x \in \pi_1 T^2$.

We can easily prove the following lemma.

LEMMA 2.3. *Let $\rho : \pi_1 T^2 \to SL(2; C)$ be a representation. The following statements are equivalent*:

(1) *ρ is a parabolic representation.*

(2) $\det(X - I) = 0$ *for $\forall x \in \pi_1 T^2$ where I is the unit matrix in $SL(2; C)$.*

Now we describe the condition of acyclicity.

PROPOSITION 2.4. *Let $\rho : \pi_1 T^2 \to SL(2; C)$ be a representation. Then all homology groups vanish: $H_*(T^2, V_\rho) = 0$ if and only if ρ is a non-parabolic representation. In this case, the $SL(2; C)$-torsion is given by*

$$\tau(T^2; V_\rho) = 1 .$$

PROOF. Suppose ρ is a non-parabolic representation. We fix an orientation on T^2. By assumption, there is an element $x \in \pi_1 T^2$ such that $\det(X - I) \neq 0$. We take $y \in \pi_1 T^2$ such that the geometric intersection number $x \cdot y = 1$. We assume that a cell structure of T^2 is given by the following;

(0) one 0-cell p,

(1) two 1-cells x and y,

(2) one 2-cell w,

with the attaching map given by $\partial w = xyx^{-1}y^{-1}$. By easy computation, this chain complex is given as follows;

$$0 \longrightarrow w \otimes V \xrightarrow{\partial_2} x \otimes V \otimes y \otimes V \xrightarrow{\partial_1} p \otimes V \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} -(Y-I) \\ X-I \end{pmatrix}, \qquad \partial_1 = (X-I \quad Y-I) .$$

Since $\det(X - I) \neq 0$, ∂_1 is surjective and then $\dim(\operatorname{Ker} \partial_1) = 2$. Similarly ∂_2 is injective and $\dim(\operatorname{Im}\partial_2)=2$. On the other hand, we have

$$\operatorname{Im}\partial_2 \subset \operatorname{Ker}\partial_1$$

by the definition of the boundary operators. Hence

$$\operatorname{Im}\partial_2 = \operatorname{Ker}\partial_1 .$$

Therefore this chain complex $C_*(T^2; V_\rho)$ is acyclic. Then $\tau(T^2; V_\rho)$ is given by the following. Since a canonical basis of $V \oplus V$ is given by $\{(\boldsymbol{e}_1, \boldsymbol{0}), (\boldsymbol{e}_2, \boldsymbol{0}), (\boldsymbol{0}, \boldsymbol{e}_1)(\boldsymbol{0}, \boldsymbol{e}_2)\}$, we may identify the bases

$$\begin{aligned} \boldsymbol{c}_2 &= \{\boldsymbol{e}_1, \boldsymbol{e}_2\} , \\ \boldsymbol{c}_1 &= \{(\boldsymbol{e}_1, \boldsymbol{0}), (\boldsymbol{e}_2, \boldsymbol{0}), (\boldsymbol{0}, \boldsymbol{e}_1), (\boldsymbol{0}, \boldsymbol{e}_2)\} , \\ \boldsymbol{c}_0 &= \{\boldsymbol{e}_1, \boldsymbol{e}_2\} . \end{aligned}$$

We take a basis $\boldsymbol{b}_i$ of B_i for ${}^\forall i \in \{0, 1\}$ which satisfies $\boldsymbol{b}_1 = \partial\boldsymbol{c}_2$, $\boldsymbol{b}_0 = \partial\boldsymbol{c}_1$. Then by the definition of the $SL(2; \boldsymbol{C})$-torsion,

$$\tau(T^2; V_\rho) = [\boldsymbol{b}_1/\boldsymbol{c}_2][\boldsymbol{b}_1, \boldsymbol{b}_0/\boldsymbol{c}_1]^{-1}[\boldsymbol{b}_0/\boldsymbol{c}_0] .$$

By straightforward computation,

$$\begin{aligned} &[\boldsymbol{b}_1/\boldsymbol{c}_2] = 1 , \\ &[\boldsymbol{b}_1, \boldsymbol{b}_0/\boldsymbol{c}_0] = \det\begin{pmatrix} -(Y-I) & 0 \\ X-I & I \end{pmatrix} = \det(Y-I) , \\ &[\boldsymbol{b}_0/\boldsymbol{c}_0] = \det(Y-I) . \end{aligned}$$

Therefore the $SL(2; \boldsymbol{C})$-torsion is given by

$$\tau(T^2; V_\rho) = 1 .$$

Conversely we assume that ρ is a parabolic representation. If ρ is a trivial representation, it is clear that $C_*(T^2; V_\rho)$ is a usual V-coefficient chain complex and not acyclic. Hence we may assume ρ is nontrivial. Then there is an element $x \in \pi_1 T^2$ such that $X = \rho(x) \neq I$. Let $v \in V$ denote the fixed vector of X and L the complex line spanned by v. Let $y \in \pi_1 T^2$ be any other element such that $Y = \rho(y) \neq I$. Since Y commutes with X, they have a common eigenvector which must be v or its multiple. Since Y is a parabolic element of $SL(2; \boldsymbol{C})$, Y also fixes the vector v. Then we have

$$\operatorname{Im}\partial_1 \subset L$$

and then ∂_1 is not surjective. Hence $H_0(T^2; V_\rho) \neq 0$. This completes the proof.

REMARK. If $\tau(M; V_\rho)$ is well-defined for an even dimensional closed orientable manifold M, then the absolute value of the Reidemeister torsion

$$|\tau(M;V_\rho)|=1\,.$$

See Ray-Singer [8] for details.

Next we consider the solid torus $S=S^1\times D^2$ with $\pi_1 S\cong \boldsymbol{Z}$ generated by x.

PROPOSITION 2.5. *Let $\rho:\pi_1 S\to SL(2;\boldsymbol{C})$ be a representation. The representation ρ is non-parabolic if and only if the chain complex $C_*(S;V_\rho)$ is acyclic. In this case the $SL(2;\boldsymbol{C})$-torsion of S is given by*

$$\tau(S;V_\rho)=\det(X-I)\,.$$

PROOF. It is easy to see that S has the same simple homotopy type as S^1. We may assume that a cell structure of S^1 is given by one 0-cell p and one 1-cell x. Then the corresponding chain complex is given by

$$0\longrightarrow x\otimes V\xrightarrow{\partial=X-I}p\otimes V\longrightarrow 0\,.$$

Hence $C_*(S;V_\rho)$ is acyclic if and only if $\det(X-I)\neq 0$. Therefore ρ is a non-parabolic representation. If we take a basis $\boldsymbol{b}_0=\{\partial\boldsymbol{e}_1,\partial\boldsymbol{e}_2\}$ for $B_0(C_*)$, then the $SL(2;\boldsymbol{C})$-torsion is given by

$$\tau(S;V_\rho)=[\boldsymbol{b}_0/\boldsymbol{c}_1]^{-1}[\boldsymbol{b}_0/\boldsymbol{c}_0]=1\cdot\det(X-I)=\det(X-I)\,.$$

This completes the proof of Proposition 2.5.

§3. Irreducible representations of Seifert fibered spaces.

In this section, we investigate the $SL(2;\boldsymbol{C})$-irreducible representation of the Seifert fibered space M given by the Seifert index $\{b,(\varepsilon,g),(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m)\}$. It is well known that the fundamental group of M has a presentation as follows. If $\varepsilon=o$, that is, if the orbit surface is orientable, then

$$\pi_1 M=\langle a_1,b_1,\cdots,a_g,b_g,q_1,\cdots,q_m,h\mid[a_i,h]=[b_i,h]=[q_i,h]=1\,,$$
$$q_i^{\alpha_i}h^{\beta_i}=1,\ q_1\cdots q_m[a_1,b_1]\cdots[a_g,b_g]=h^b\rangle\,.$$

If $\varepsilon=n$, that is, if the orbit surface is nonorientable, then

$$\pi_1 M=\langle v_1,\cdots,v_g,q_1,\cdots,q_m,h\mid v_ihv_i^{-1}=h^{-1},\ q_ihq_i^{-1}=h\,,$$
$$q_i^{\alpha_i}h^{\beta_i}=1,\ q_1\cdots q_mv_1^2\cdots v_g^2=h\rangle\,.$$

REMARK. In the case of $\varepsilon=o$ generators a_i, b_i and q_i come from the fundamental group of the orbit surface. Then we can choose the representative closed curves on the orbit surface $q_1,\cdots,q_m$ such that $q_1\cdots q_m[a_1,b_1]\cdots[a_g,b_g]=1$. Similarly we choose the curves in the case of $\varepsilon=n$.

We fix this presentation for $\pi_1 M$ and consider only $SL(2; \boldsymbol{C})$-irreducible representations. The next lemma gives us a clue to compute the $SL(2; \boldsymbol{C})$-torsion.

LEMMA 3.1. *Let $\rho: \pi_1 M \to SL(2; \boldsymbol{C})$ be an irreducible representation. Then the image of the generic fiber h is given by*

$$H = \rho(h) = \begin{cases} \pm I & (\varepsilon = o) \\ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} & (\varepsilon = n) \end{cases}$$

where I is the unit matrix in $SL(2; \boldsymbol{C})$, $\lambda \in \boldsymbol{C}$ such that $\lambda^{2N+2} = 1$, $N = \beta_1/\alpha_1 + \cdots + \beta_m/\alpha_m$.

PROOF. By the irreducibility of ρ, it is easy to see that H is a non-parabolic element.

Case 1: $\varepsilon = o$. Suppose $H \neq \pm I$. Let u be an eigenvector for an eigenvalue λ of H. Since H commutes with $A_i = \rho(a_i)$, $B_i = \rho(b_i)$ and $Q_j = \rho(q_j)$, all vectors $A_i u$, $B_i u$ and $Q_j u$ is contained in the vector space spanned by u. It contradicts the irreducibility of ρ. Thus $H = \pm I$.

Case 2: $\varepsilon = n$. Since we consider the conjugacy classes of representations, we may suppose H is the diagonal matrix $H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Subcase 1: $m = 0$. In this case M has no exceptional fibers; it is an S^1-bundle over a non-orientable surface of genus g. By the relation $V_i H = H^{-1} V_i$,

$$V_i H \boldsymbol{e}_1 = \lambda V_i \boldsymbol{e}_1 = H^{-1} V_i \boldsymbol{e}_1 .$$

Accordingly we get

$$H V_i \boldsymbol{e}_1 = \lambda^{-1} V_i \boldsymbol{e}_1$$

and $V_i \boldsymbol{e}_1$ is contained in the eigenspace for an eigenvalue λ^{-1} as in Case 1. Similarly $V_i \boldsymbol{e}_2$ is contained in the eigenspace for λ. Thus we may set for each i

$$V_i = \begin{pmatrix} 0 & a_i \\ b_i & 0 \end{pmatrix} \qquad \text{such that } a_i b_i = -1 .$$

By simple computation, we have

$$V_i^2 = -I .$$

The relation of $\pi_1 M$ implies

$$H = V_1^2 V_2^2 \cdots V_g^2 = (-I)^g .$$

Hence

$$H = \pm I .$$

Subcase 2: $m \geq 1$. Then M has the exceptional fibers $q_1, \cdots, q_m$. For $\forall q_j$, we set the corresponding matrix

$$Q_j=\begin{pmatrix} s_j & t_j \\ u_j & v_j \end{pmatrix}.$$

The condition $HQ_j=Q_jH$ implies

$$\begin{pmatrix} \lambda s_j & \lambda t_j \\ \lambda^{-1}u_j & \lambda^{-1}v_j \end{pmatrix}=\begin{pmatrix} \lambda s_j & \lambda^{-1}t_j \\ \lambda u_j & \lambda^{-1}v_j \end{pmatrix}.$$

If we compare each entry of the left-side with the one of the right-side,

$$\lambda=\lambda^{-1} \qquad \text{or} \qquad t_j=u_j=0\,.$$

If $\lambda=\lambda^{-1}$, then we get $\lambda=\pm 1$ and consequently $H=\pm I$. If $\lambda\neq\lambda^{-1}$, then every Q_j is a diagonal matrix. In this case, the relation $q_j^{\alpha_j}h^{\beta_j}=1$ implies

$$\begin{pmatrix} s_j^{\alpha_j} & 0 \\ 0 & v_j^{\alpha_j} \end{pmatrix}=\begin{pmatrix} \lambda^{-\beta_j} & 0 \\ 0 & \lambda^{\beta_j} \end{pmatrix}.$$

Hence we get

$$s_j=\lambda^{-\beta_j/\alpha_j} \quad \text{and} \quad v_j=\lambda^{\beta_j/\alpha_j}\,.$$

On the other hand, we get

$$V_i=\begin{pmatrix} 0 & a_i \\ b_i & 0 \end{pmatrix} \qquad \text{such that } V_i^2=-I$$

as in the subcase 1. The relation $h=q_1\cdots q_m v_1^2\cdots v_g^2$ implies

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}=(-I)^g\begin{pmatrix} s_1\cdots s_m & 0 \\ 0 & v_1\cdots v_m \end{pmatrix}$$

$$=(-1)^g\begin{pmatrix} \lambda^{-(\beta_1/\alpha_1+\cdots+\beta_m/\alpha_m)} & 0 \\ 0 & \lambda^{\beta_1/\alpha_1+\cdots+\beta_m/\alpha_m} \end{pmatrix}.$$

Hence the following holds:

$$\lambda^{-(\beta_1/\alpha_1+\cdots+\beta_m/\alpha_m)}=(-1)^g\lambda\,.$$

Therefore setting $N=\beta_1/\alpha_1+\cdots+\beta_m/\alpha_m$, we get

$$\lambda^{2N+2}=1\,.$$

This completes the proof.

From the above lemma, we get easily the following corollary.

COROLLARY 3.2. *$Q_i=\rho(q_i)$ has only eigenvalues which are roots of unity.*

§4. Proof of Main theorem (1).

In this section, we give a proof of Main theorem. Here we decompose M into tubular neighborhoods of exceptional fibers and their complement. Then we compute the $SL(2; C)$-torsion for each part and apply Lemma 1.3 to our situations. Since we can compute the $SL(2; C)$-torsion for $\varepsilon = n$ as in the case of $\varepsilon = o$, we will prove only the case of $\varepsilon = o$.

We put

$$\Sigma^* = \Sigma - (D_0^2 \cup \cdots \cup D_m^2)$$

where Σ is an orientable closed surface of genus g and $D_0^2, \cdots, D_m^2$ are disjoint embedded open 2-disks. Also let M_m denote the trivial S^1-bundle $\Sigma^* \times S^1$. We give a canonical torus decomposition of Seifert fibered space M as follows:

$$M \cong M_m \cup S_0 \cup S_1 \cdots \cup S_m$$

where any S_i is the solid torus. The solid torus S_0 is the one corresponding to the triviality obstruction b and S_i for $\forall i \in \{1, \cdots, m\}$ is the one corresponding to the exceptional fiber.

Lemma 4.1. *Let $\rho : \pi_1(M) \to SL(2; C)$ be an irreducible representation. Suppose all homology groups of the boundary vanish: $H_*(\partial M_m; V_\rho) = 0$. Then $H_*(M; V_\rho) = 0$ if and only if $H_*(M_m; V_\rho) = H_*(S_0; V_\rho) = \cdots = H_*(S_m; V_\rho) = 0$. In this case, we have*

$$\tau(M; V_\rho) = \tau(M_m; V_\rho)\tau(S_0; V_\rho) \cdots \tau(S_m; V_\rho).$$

Proof. Apply Lemma 1.3 to the short exact sequence of the chain complex given by the torus decomposition of M;

$$0 \to \bigoplus_{i=0}^{m} C_*(\partial S_i; V_\rho) \to C_*(M_m; V_\rho) \oplus \bigoplus_{i=0}^{m} C_*(S_i; V_\rho) \to C_*(M; V_\rho) \to 0.$$

By the proof of Proposition 2.4, $\dim \partial C_*(\partial S_i; V_\rho)$ is even. Therefore we have Lemma 4.1.

Proposition 4.2. *Let $\rho : \pi_1(M) \to SL(2; C)$ be an irreducible representation. We denote the restriction of ρ to $\pi_1(M_m)$ by the same symbol ρ. Then all homology groups vanish: $H_*(M_m; V_\rho) = 0$ if and only if $H = \rho(h) = -I$. In this case the $SL(2; C)$-torsion is given by*

$$\tau(M_m; V_\rho) = 2^{2-2m-4g}.$$

Proof. It is easy to see that M_m has the same simple homotopy type as the direct product of the one point union of $2g+m$ circles $S^1 \vee \cdots \vee S^1$ and S^1. We denote this space by $(\bigvee_i S_i) \times S^1$. Then $\bigvee_i S_i$ has a natural cell decomposition given by one 0-cell u and $2g+m$ 1-cells a_i, b_i, q_j. It gives a cell decomposition of $(\bigvee_i S_i) \times S^1$ by

(1) 0-cell u,

(2) 1-cells $a_1, \cdots, a_g, b_1, \cdots, b_g, q_1, \cdots, q_m, h$ corresponding to the generators of $\pi_1 M$.

(3) 2-cells $v_{a_1}, v_{a_2}, \cdots, v_{a_g}, v_{b_1}, \cdots, v_{b_g}, v_{q_1}, \cdots, v_{q_m}$ respectively with boundary a_i, b_i and q_i.

By using this cell structure, we can determine the structure of $C_*(M_m; V_\rho)$. Recall that $\{e_1, e_2\}$ is a canonical basis of V. The 2-chain module $C_2(M_m; V_\rho)$ is a free $\boldsymbol{Z}[\pi_1 M_m]$-module on $\{v_{a_j} \otimes e_i, v_{b_j} \otimes e_i, v_{q_j} \otimes e_i\}$ for $\forall i \in \{1, 2\}$ and $\forall\, j \in \{1, \cdots, g\}$. Similarly $C_1(M_m; V_\rho)$ is a free $\boldsymbol{Z}[\pi_1 M_m]$-module on $\{a_j \otimes e_i, b_j \otimes e_i, q_j \otimes e_i, h \otimes e_i\}$ and $C_0(M_m)$ is a free $\boldsymbol{Z}[\pi_1 M_m]$-module on $\{u \otimes e_i\}$. Then the boundary operators are given by

$$\partial_2 = \begin{pmatrix} I-H & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & I-H & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & I-H \\ A_1-I & A_2-I & \cdots & B_1-I & \cdots & Q_1-I & \cdots & Q_m-I \end{pmatrix},$$

$$\partial_1 = (A_1-I \ \cdots \ B_1-I \ \cdots \ Q_1-I \ \cdots \ Q_m-I \quad H-I)\,.$$

It is easy to see that $C_*(M_m; V_\rho)$ is acyclic if and only if $H=-I$. Let $\boldsymbol{b}_i$ be a basis of the boundary $B_i(M_m; V_\rho)$ for $i=0, 1$. Then the $SL(2; \boldsymbol{C})$-torsion is given by

$$\tau(M_m; V_\rho) = [\boldsymbol{b}_1/\boldsymbol{c}_2][\boldsymbol{b}_1, \boldsymbol{b}_0/\boldsymbol{c}_1]^{-1}[\boldsymbol{b}_0/\boldsymbol{c}_0]\,.$$

We may choose a lift of $\boldsymbol{b}_1$ which coincides with $\boldsymbol{c}_2$ and the one of $\boldsymbol{b}_0$ which coincides with $\{h \otimes e_1, h \otimes e_2\}$. By simple computation,

$$\tau(M_m; V_\rho) = 1 \cdot (\det(I-H))^{-(2g+m)} \cdot \det(H-I) = (\det(I-H))^{-(2g+m+1)}\,.$$

Then substituting $-I$ for H, we have

$$\begin{aligned} \tau(M_m; V_\rho) &= \left(\det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)^{-(2g+m)+1} \\ &= 2^{-2(2g+m)+2}\,. \end{aligned}$$

This completes the proof of Proposition 4.2.

Because ∂M_m is the disjoint union of tori, the fundamental group $\pi_1 M$ is generated by h and $\{q_1, \cdots, q_m\}$. Then $C_*(\partial M_m; V_\rho)$ is acyclic if and only if $H=-I$ by Proposition 2.4.

PROPOSITION 4.3. *If $H=-I$, then the $SL(2; \boldsymbol{C})$-torsion of S_0 is given by*

$$\tau(S_0; V_\rho) = 2^2\,.$$

PROOF. Let ρ_0 and ν_0 be integers such that $\begin{vmatrix} 1 & \rho_0 \\ b & \nu_0 \end{vmatrix} = -1$. We define an element $l_0 \in \pi_1 M_m$ by $q_0^{\rho_0} h^{\nu_0}$. The sewing of the solid torus S_0 makes the curve $m_0 = q_0 h^b$ on the component of ∂M_m null-homotopic in S_0. On the other hand the closed curve l_0 is the generator in $\pi_1 S_0 \cong \boldsymbol{Z}$. Then the relation implies

$$L_0 = \rho(l_0) = Q_0^{\rho_0} H^{\nu_0} .$$

Since $q_0 = (h^b)^{-1} = (q_1 \cdots q_m [a_1, b_1] \cdots [a_g, b_g])^{-1}$ and $\nu_0 - b\rho_0 = -1$,

$$\begin{aligned} L_0 &= (Q_1 \cdots Q_m [A_1, B_1] \cdots [A_g, B_g])^{-\rho_0} H^{\nu_0} \\ &= H^{-b\rho_0 + \nu_0} = H^{-1} = -I . \end{aligned}$$

Therefore the $SL(2; \boldsymbol{C})$-torsion of S_0 is given as follows;

$$\begin{aligned} \tau(S_0 ; V_\rho) &= \det(L_0 - I) \\ &= \det\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \\ &= 2^2 . \end{aligned}$$

This completes the proof.

PROPOSITION 4.4. *If $H = -I$, then the $SL(2; \boldsymbol{C})$-torsion of S_i is given by*

$$\tau(S_i ; V_\rho) = 2\left(1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}\right).$$

PROOF. Let ρ_i and ν_i be integers such that $\begin{vmatrix} \alpha_i & \rho_i \\ \beta_i & \nu_i \end{vmatrix} = -1$ and $0 < \rho_i < \alpha_i$. We define the generator $l_i \in \pi_1 S_i$ by $q_i^{\rho_i} h^{\nu_i}$. Here the image of l_i is given by

$$L_i = \rho(l_i) = Q_i^{\rho_i} H^{\nu_i} = (-1)^{\nu_i} Q_i^{\rho_i} .$$

By Proposition 2.5, we have

$$\tau(S_i ; V_\rho) = \det(L_i - I) = \det((-1)^{\nu_i} Q_i^{\rho_i} - I) = 2 - (-1)^{\nu_i} \operatorname{tr} Q_i^{\rho_i} .$$

In view of relations $q_i^{\alpha_i} h^{\beta_i} = 1$ and $H = -I$, the identity $Q_i^{\alpha_i} = (-I)^{\beta_i}$ holds. Then we may denote the eigenvalues of Q_i by $\exp(\sqrt{-1} k_i(\rho)\pi/\alpha_i)$ and $\exp(-\sqrt{-1} k_i(\rho)\pi/\alpha_i)$ where $0 \leq k_i(\rho) \leq \alpha_i$ and $k_i(\rho) \equiv \beta_i \bmod 2$. Hence we get

$$\tau(S_i ; V_\rho) = 2\left(1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}\right).$$

This completes the proof of Proposition 4.4.

By using Lemma 4.1, the $SL(2; \boldsymbol{C})$-torsion $\tau(M ; V_\rho)$ of the Seifert fibered space is given by

$$\begin{aligned}\tau(M; V_\rho) &= \tau(M_m; V_\rho)\tau(S_0; V_\rho)\cdots\tau(S_m; V_\rho)\\ &= 2^{2-2m-4g}\cdot 2^2\cdot 2^m\cdot\prod_{i=1}^{m}\left(1-(-1)^{\nu_i}\cos\frac{\rho_i k_i(\rho)\pi}{\alpha_i}\right)\\ &= 2^{4-m-4g}\prod_{i=1}^{m}\left(1-(-1)^{\nu_i}\cos\frac{\rho_i k_i(\rho)\pi}{\alpha_i}\right).\end{aligned}$$

We have a proof of Main theorem for the case of $H=-I$.

§5. Proof of Main theorem (2).

If $H=I$, we cannot apply Lemma 4.1 to our situations because a given representation is not acyclic when we restrict it to the complement of exceptional fibers. However then the representation ρ is not acyclic. Now we prove the following proposition.

PROPOSITION 5.1. *Let $\rho: \pi_1(M)\to SL(2; C)$ be an irreducible representation such that $H=\rho(h)=I$. Then ρ is not acyclic; that is, $H_*(M; V_\rho)\neq 0$.*

PROOF. The proof is by contradiction. We assume all homology groups of M vanish: $H_*(M; V_\rho)=0$. Then the following sequences given by the Mayer-Vietoris sequence are exact.

$$0\to H_2(\partial M_m; V_\rho)\to H_2(M_m; V_\rho)\to 0\,,$$

$$0\to H_1(\partial M_m; V_\rho)\to H_1(M_m; V_\rho)\oplus\bigoplus_{i=0}^{m} H_1(S_i; V_\rho)\to 0\,,$$

$$0\to H_0(\partial M_m; V_\rho)\to H_0(M_m; V_\rho)\oplus\bigoplus_{i=0}^{m} H_0(S_i; V_\rho)\to 0\,.$$

Case 1: There exists a non-parabolic element in $\{A_i, B_i, Q_j\}$. From the proof of Proposition 4.2, in the chain complex $C_*(M_m; V_\rho)$,

$$\operatorname{rank}(\partial_2)=\operatorname{rank}(\partial_1)=2\,.$$

In this case, by easy computation, the homology groups of M_m are given as follows;

$$\begin{aligned}&H_2(M_m; V_\rho)\cong V^{2g+m-1}\,,\\ &H_1(M_m; V_\rho)\cong V^{2g+m-1}\,,\\ &H_0(M_m; V_\rho)=0\,.\end{aligned}$$

By the above exact sequences and the Poincaré duality, we have the following identifications;

$$H_0(\partial M_m;\, V_\rho)\cong H_2(\partial M_m;\, V_\rho)\cong H_2(M_m;\, V_\rho)\cong V^{2g+m-1}\,.$$

On the other hand, we have

$$\begin{aligned} H_0(\partial M_m;\, V_\rho)&\cong H_0(M_m;\, V_\rho)\oplus \bigoplus_{i=0}^{m} H_0(S_i;\, V_\rho)\\ &\cong \{0\}\oplus V^{m+1-k}\\ &\cong V^{m+1-k} \end{aligned}$$

where k is the number of the solid tori with non-trivial 0-dimensional homology group. Hence we have

$$k=2-2g\,.$$

Because k is a non-negative integer, the genus $g=0$ or 1.

First we assume $g=0$; that is, $k=2$. In this case,

$$\pi_1 M=\langle q_1, \cdots, q_m, h \,|\, [q_i, h]=1,\, q_i^{\alpha_i}h^{\beta_i}=1,\, q_1\cdots q_m=h^b\rangle\,.$$

Then we have

$$\bigoplus_{i=0}^{m} H_0(S_i;\, V_\rho)\cong V^{m-1}$$

by Propositions 4.3 and 4.4. For simplicity, we may assume

$$\operatorname{rank}(L_i-I)=0 \qquad \text{for} \quad \forall i\in\{0, \cdots, m-2\}$$

and

$$\operatorname{rank}(L_i-I)=2 \qquad \text{for} \quad \forall i\in\{m-1, m\}\,.$$

For $\forall i\in\{0, \cdots, m-2\}$, that is $L_i\in SL(2;\, C)$ is a parabolic element. On the other hand, from the relations of $\pi_1 M$, we have

$$L_i=Q_i^{\rho_i}H^{\nu_i}=Q_i^{\rho_i}=I\,.$$

Hence

$$Q_i=I \qquad \text{for} \quad \forall i\in\{0, \cdots, m-2\}$$

and

$$Q_{m-1}Q_m=I\,.$$

Hence the representation ρ is reducible because Q_{m-1} and Q_m have a common eigenvector.

Next we assume $g=1$; that is, $k=0$. In this case, we have

$$\pi_1 M = \langle a_1, b_1, q_1, \cdots, q_m, h \mid [a_1, h] = [b_1, h] = [q_i, h] = 1,\\ q_i^{\alpha_i} h^{\beta_i} = 1, [a_1, b_1] q_1 \cdots q_m = h^b \rangle .$$

Then we have

$$\bigoplus_{i=0}^{m} H_0(S_i; V_\rho) \cong V^{m+1} .$$

Then for $\forall i \in \{0, \cdots, m\}$

$$\mathrm{rank}(L_i - I) = 0$$

and $L_i \in SL(2; C)$ is parabolic or trivial. On the other hand, we have

$$L_i = Q_i^{\rho_i} H^{\nu_i} = Q_i^{\rho_i} = I .$$

Hence we have

$$Q_i = I \qquad \text{for} \quad \forall i \in \{0, \cdots, m\} .$$

Then ρ factors through a representation of the group $\langle a_1, b_1 \mid [a_1, b_1] = 1 \rangle$. Since this group is abelian, this representation is reducible. This is a contradiction.

Case 2: All A_i, B_i, Q_i are parabolic elements. In this case, we have

$$Q_i = I \qquad \text{for } \forall i \in \{0, \cdots, m\} .$$

Then we have

$$\mathrm{rank}(\partial_2) = 2 \quad \text{or} \quad 0$$

for $C_*(M_m; V_\rho)$. Hence

$$H_2(M_m; V_\rho) \cong \begin{cases} V^{2g+m-1} & \text{if} \quad \mathrm{rank}(\partial_2) = 2 \\ V^{2g+m} & \text{if} \quad \mathrm{rank}(\partial_2) = 0 . \end{cases}$$

By Poincaré duality and the exact sequence, we obtain

$$H_2(M_m; V_\rho) \cong H_2(\partial M_m; V_\rho) \cong H_0(M_m; V_\rho) \oplus V^{m+1} .$$

Then we get the genus $g = 1$. Hence this representation ρ is reducible since ρ factors through the representation of the group $\langle a_1, b_1 \mid [a_1, b_1] = 1 \rangle$ as in Case 1. This completes the proof of Proposition 5.1.

By the lemmas and the propositions, we get a proof of Main theorem.

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