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Timelike Surfaces of Constant Mean Curvature in Minkowski 3-Space

Jun-ichi INOGUCHI

Tokyo Metropolitan University
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Introduction.

Since the discovery of constant mean curvature tori in Euclidean 3-space, methods in the theory of integrable systems (or soliton theory) have been applied frequently in differential geometry. In particular, surfaces of constant mean or Gaussian curvature in Euclidean 3-space have been studied extensively [2], [3].

On the other hand, the geometry of surfaces in Minkowski 3-space has been a subject of wide interest [9], [10]. For example, a Kenmotsu-type representation formula for spacelike surfaces with prescribed mean curvature has been obtained by K. Akutagawa and S. Nishikawa, and M. A. Magid [9] has obtained such a representation formula for timelike surfaces. L. McNertney studied spacelike maximal surfaces, timelike extremal surfaces and timelike surfaces of constant positive Gaussian curvature by classical methods [10].

In our previous paper [5], we have studied spacelike surfaces with constant mean or Gaussian curvature in Minkowski 3-space via the theory of finite-type harmonic maps. To adapt the methods in the theory of integrable systems for our purposes, namely to study the geometry of surfaces, we reformulated the fundamental equations of spacelike surfaces. This allowed us to considerably simplify the computations in the study of such spacelike surfaces. The main tool of this reformulation was our use of split-quaternion numbers. In addition we obtained representation formulae for immersions in terms of loop group theory [5].

The purpose of this paper is to develop a corresponding setting for timelike surfaces of constant mean curvature in Minkowski 3-space.

In contrast to the case of spacelike surfaces or timelike extremal surfaces, no systematic theory exists for timelike constant (nonzero) mean curvature surfaces. For this reason, we shall devote Section 1 to preliminary materials. The reformulation of the fundamental equations will be carried out in Section 2. As a result we obtain a

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representation formula in this section. In the last two sections we give some applications of the representation formula. In particular we shall exhibit this surprising phenomenon that a circular cylinder and a hyperbolic cylinder can belong to a same one parameter family.

1. Fundamental equations of timelike surfaces.

We start with preliminaries on the geometry of timelike surfaces in Minkowski 3-space.

Let $\mathbf{E}_1^3$ be a *Minkowski* 3-*space* with Lorentz metric $\langle\,,\rangle$. The metric $\langle\,,\rangle$ is expressed as $\langle\,,\rangle=-d\xi_1^2+d\xi_2^2+d\xi_3^2$ in terms of natural coordinates.

Let M be a connected 2-manifold and $\varphi: M\to\mathbf{E}_1^3$ an immersion. The immersion φ is said to be *timelike* if the induced metric I of M is Lorentzian. Hereafter we may assume that M is an orientable timelike surface in $\mathbf{E}_1^3$ (immersed by φ). It is worthwhile to remark that there exists no compact timelike surface in $\mathbf{E}_1^3$. (See B. O'Neill [13 p. 125].)

On a timelike surface M, there exist local coordinates (x, y) such that

$$I=e^{\omega}(-dx^2+dy^2)\,. \tag{1.1}$$

Such local coordinates (x, y) are called *Lorentz isothermal coordinates* (See R. Kulkarni [8]). Let (u, v) be the *null coordinates derived from* (x, y). Namely (u, v) are defined by $u=x+y$, $v=-x+y$. The first fundamental form I is written by (u, v) as follows:

$$I=e^{\omega}dudv\,. \tag{1.2}$$

Partial derivatives of φ satisfy the following formulae.

$$\langle\varphi_u,\varphi_u\rangle=\langle\varphi_v,\varphi_v\rangle=0\,,\qquad \langle\varphi_u,\varphi_v\rangle=\tfrac{1}{2}e^{\omega}\,. \tag{1.3}$$

Now, let N be a local unit normal vector field to M. The vector field N is spacelike since M is timelike. The vector fields φ_u, φ_v as well as the normal N define a moving frame. We shall define a moving frame σ by $\sigma=(\varphi_u, \varphi_v, N)$. The moving frame σ satisfies the following Frenet (or Gauss-Weingarten) equations:

$$\sigma_u=\sigma\mathcal{U}\,,\qquad \sigma_v=\sigma\mathcal{V}\,, \tag{1.4}$$

$$\mathcal{U}=\begin{pmatrix}\omega_u & 0 & -H\\ 0 & 0 & -2e^{-\omega}Q\\ Q & \tfrac{1}{2}He^{\omega} & 0\end{pmatrix},\qquad \mathcal{V}=\begin{pmatrix}0 & 0 & -2e^{-\omega}R\\ 0 & \omega_v & -H\\ \tfrac{1}{2}He^{\omega} & R & 0\end{pmatrix}, \tag{1.5}$$

where $Q:=\langle\varphi_{uu}, N\rangle$, $R:=\langle\varphi_{vv}, N\rangle$, $H=2e^{-\omega}\langle\varphi_{uv}, N\rangle$. It is easy to see that $Q^{\#}:=Qdu^2$ and $R^{\#}:=Rdv^2$ are globally defined null 2-differentials on M. We shall call the pair of differentials $Q^{\#}$ and $R^{\#}$, *the Hopf pair of* M.

It is easy to see that H is the mean curvature of M. The second fundamental form

II of M defined by

(1.6) $$II=-\langle d\varphi, dN\rangle$$

is described relative to the Lorentz isothermal coordinates (x, y) as follows:

(1.7) $$II=\begin{pmatrix} Q+R-He^{\omega} & Q-R \\ Q-R & Q+R+He^{\omega} \end{pmatrix}.$$

The Gaussian curvature K of M is given by

$$K=\det(II.I^{-1}).$$

(See [13 p. 107] or L. McNertney [10 p. 6].) The Gauss equation which describes a relation between K, H and Q takes the following form:

(1.8) $$H^2-K=4e^{-2\omega}QR.$$

It is easy to see that the common zero of Q and R coincides with the umbilic point of M. Note that the condition $QR=0$ does not imply the condition $Q=R=0$. (See T. K. Milnor [11]). The Gauss-Codazzi equation, i.e., the integrability condition of the Frenet equation,

(GC) $$\mathcal{V}_u-\mathcal{U}_v+[\mathcal{U}, \mathcal{V}]=0$$

has the following form:

(1.9) $$\omega_{uv}+\tfrac{1}{2}H^2e^{\omega}-2QRe^{-\omega}=0,$$

(1.10) $$H_u=2e^{-\omega}Q_v,$$

(1.11) $$H_v=2e^{-\omega}R_u.$$

The equations (1.10)–(1.11) show that the constancy of the mean curvature H is equivalent to the condition $Q_v=R_u=0$. As in the case of spacelike surfaces, we have the following Bonnet-type theorem.

PROPOSITION 1.1. *Every nontotally umbilic timelike constant mean curvature surface has a one parameter family of nontrivial isometric deformation preserving the mean curvature.*

PROOF. On a constant mean curvature timelike surface M, the Gauss-Codazzi equation (GC) is invariant under the deformation:

(1.12) $$Q\mapsto Q_\lambda:=\lambda Q,\quad R\mapsto R_\lambda:=\lambda^{-1}R,\qquad \lambda\in\mathbf{R}^*.$$

Integrating the deformed Gauss-Codazzi equation, one obtains a one-parameter family of timelike surfaces $\{\varphi_\lambda\}$. The deformation (1.12) does not effect the induced metric and the mean curvature. Hence all the surfaces $\{\varphi_\lambda\}$ are isometric and have the same constant mean curvature. □

REMARK. A totally umbilic timelike surface is congruent to an open portion of a pseudo-sphere $S_1^2(r)=\{\xi\in\mathbf{E}_1^3 \mid \langle\xi,\xi\rangle=r^2\}$ of radius $r>0$. (Hence if M is complete, M is congruent to $S_1^2(r)$.) Note that there is no totally umbilic timelike surface of (constant) negative curvature. (See [13 p. 116].)

Next, we shall define the Gauss map of a timelike surface. Let M be a timelike surface and N a local unit normal vector field to M. For each $p\in M$ the point $\psi(p)$ of $\mathbf{E}_1^3$ canonically corresponding to the vector N_p lies in a unit pseudo 2-sphere since N is spacelike. The resulting smooth mapping $\psi: M\to S_1^2$ is called the *Gauss map* of M.

In the geometry of spacelike surfaces, the study of constant negative curvature surfaces is the almost same as that of constant mean curvature surfaces. In contrast, to find timelike surfaces of constant positive curvature is the almost equivalent to finding timelike surfaces of constant mean curvature. We shall explain this observation by the parallel surface procedure in the final section.

2. Split-quaternion formalism.

In this section we shall reformulate the Gauss-Codazzi equations in a form familiar to the theory of integrable systems (so-called 2 by 2 matrix-formalism or AKNS-setup [1]). Our idea for this purpose is to identify the Minkowski 3-space $\mathbf{E}_1^3$ with the imaginary part $\mathrm{Im}\mathbf{H}'$ of the split-quaternion algebra $\mathbf{H}'$.

First of all, we start with some preliminaries on split-quaternion algebra. Let us denote the algebra of split-quaternions by $\mathbf{H}'$ and its natural basis by $\{\mathbf{1},\mathbf{i},\mathbf{j}',\mathbf{k}'\}$. The multiplication of $\mathbf{H}'$ is defined as follows:

$$\mathbf{i}\mathbf{j}'=-\mathbf{j}'\mathbf{i}=\mathbf{k}',\quad \mathbf{j}'\mathbf{k}'=-\mathbf{k}'\mathbf{j}'=-\mathbf{i},\quad \mathbf{k}'\mathbf{i}=-\mathbf{i}\mathbf{k}'=\mathbf{j}',$$
$$\mathbf{i}^2=-\mathbf{1},\quad \mathbf{j}'^2=\mathbf{k}'^2=\mathbf{1}. \tag{2.1}$$

An element of $\mathbf{H}'$ is called a *split-quaternion*.

For a split-quaternion $\xi=\xi_0\mathbf{1}+\xi_1\mathbf{i}+\xi_2\mathbf{j}'+\xi_3\mathbf{k}'$, the *conjugate* $\bar{\xi}$ of ξ is defined by

$$\bar{\xi}=\xi_0\mathbf{1}-\xi_1\mathbf{i}-\xi_2\mathbf{j}'-\xi_3\mathbf{k}'.$$

It is easy to see that $-\xi\bar{\xi}=-\xi_0^2-\xi_1^2+\xi_2^2+\xi_3^2$. Hereafter we identify $\mathbf{H}'$ with a semi-Euclidean space $\mathbf{E}_2^4$:

$$\mathbf{E}_2^4=(\mathbf{R}^4(\xi_0,\xi_1,\xi_2,\xi_3),\ -d\xi_0^2-d\xi_1^2+d\xi_2^2+d\xi_3^2).$$

Let $G=\{\xi\in\mathbf{H}' \mid \xi\bar{\xi}=1\}$ be the multiplicative group of timelike unit split-quaternions. The Lie algebra $\mathfrak{g}$ of G is the imaginary part of $\mathbf{H}'$, that is,

$$\mathfrak{g}=\mathrm{Im}\mathbf{H}'=\{\xi_1\mathbf{i}+\xi_2\mathbf{j}'+\xi_3\mathbf{k}' \mid \xi_1,\xi_2,\xi_3\in\mathbf{R}\}.$$

The Lie bracket of $\mathfrak{g}$ is simply the commutator of split-quaternion product. Note that the commutation relation of $\mathfrak{g}$ is given by:

$$[\mathbf{i}, \mathbf{j}'] = 2\mathbf{k}', \quad [\mathbf{j}', \mathbf{k}'] = -2\mathbf{i}, \quad [\mathbf{k}', \mathbf{i}] = 2\mathbf{j}'.$$

The Lie algebra $\mathfrak{g}$ is naturally identified with a Minkowski 3-space

$$\mathbf{E}_1^3 = (\mathbf{R}^3(\xi_1, \xi_2, \xi_3), -d\xi_1^2 + d\xi_2^2 + d\xi_3^2)$$

as a metric linear space.

Next, we introduce a 2 by 2 matricial expression of $\mathbf{H}'$ as follows:

$$\xi = \xi_0\mathbf{1} + \xi_1\mathbf{i} + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' \leftrightarrow \begin{pmatrix} \xi_0 - \xi_3 & -\xi_1 + \xi_2 \\ \xi_1 + \xi_2 & \xi_0 + \xi_3 \end{pmatrix}. \tag{2.2}$$

In particular, the matricial expressions of the natural basis of $\mathbf{H}'$ are given by

$$\mathbf{1} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{j}' \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k}' \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This correspondence gives an algebra isomorphism between $\mathbf{H}'$ and the algebra $\mathbf{M}_2\mathbf{R}$ of all real matrices of degree 2. Under the identification (2.2), the group G of timelike unit split-quaternions corresponds to a special linear group:

$$\mathrm{SL}_2\mathbf{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2\mathbf{R} \;\middle|\; ad - bc = 1 \right\}.$$

The semi-Euclidean metric of $\mathbf{H}'$ corresponds to the following scalar product on $\mathbf{M}_2\mathbf{R}$.

$$\langle X, Y \rangle = \tfrac{1}{2}\{\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)\} \tag{2.3}$$

for all $X, Y \in \mathbf{M}_2\mathbf{R}$. This scalar product $\langle\,,\rangle$ is the Killing form of $\mathbf{M}_2\mathbf{R}$ up to constant multiple. The metric of G induced by (2.3) is a bi-invariant Lorentz metric of constant curvature -1. Hence the Lie group G is identified with an anti de-Sitter 3-space H_1^3 of constant curvature -1. (See [13] and K. Nomizu [12].)

As in the case of Euclidean 3-space, the *vector product operation of* $\mathbf{E}_1^3$ is defined by

$$\xi \times \eta = (\xi_3\eta_2 - \xi_2\eta_3, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_1) \tag{2.4}$$

for $\xi = (\xi_1, \xi_2, \xi_3)$, $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbf{E}_1^3$. If we regard ξ and η as elements of $\mathfrak{g}$, then the vector product of ξ and η is written in terms of the Lie bracket as follows:

$$\xi \times \eta = \tfrac{1}{2}[\xi, \eta].$$

Next, we shall define the Hopf-fibering for a pseudo-sphere S_1^2. It is easy to see that the Ad(G)-orbit of $\mathbf{k}' \in \mathfrak{g}$ is a pseudo-sphere:

$$S_1^2 = \{\xi \in \mathbf{E}_1^3 \mid \langle \xi, \xi \rangle = 1\}.$$

The Ad-action of G on S_1^2 is transitive and isometric. The isotropy subgroup of G at

$\mathbf{k}'$ is $H_0^1=\{\xi_0\mathbf{1}+\xi_3\mathbf{k}' \mid \xi_0^2-\xi_3^2=-1\}$. The group H_0^1 is a hyperbola in a Minkowski plane $\mathbf{E}_1^2(\xi_0, \xi_3)$. (This is a Lorentz analogue of $S^1\subset\mathbf{E}^2(\xi_0, \xi_1)$.) Note that the group H_0^1 is isomorphic to the multiplicative group $\mathbf{R}^*$.

The natural projection $\pi: G\to S_1^2$, given by $\pi(g)=\mathrm{Ad}(g)\mathbf{k}'$ for all $g\in G$, defines a principal H_0^1-bundle over S_1^2. We shall call this fibering the *Hopf-fibering of* S_1^2.

For more details on split-quaternion algebra $\mathbf{H}'$ we refer to N. Hitchin [4] and I. Yokota [15].

Now, we shall rewrite the (Gauss-Codazzi) equations (1.9)–(1.11) in 2 by 2 matrix-form. Let $\varphi: M\to\mathbf{E}_1^3$ be a timelike surface as in the preceding section. The local unit normal vector field N is given by

$$N=(\varphi_x\times\varphi_y)/|\varphi_x\times\varphi_y| .$$

Define a matrix-valued function Φ by

$$\mathrm{Ad}(\Phi)(\mathbf{i}, \mathbf{j}', \mathbf{k}')=(e^{-\omega/2}\varphi_x, e^{-\omega/2}\varphi_y, N) . \tag{2.5}$$

To derive the linear differential equations corresponding to (1.4), we introduce the following $\mathbf{H}'$-valued functions U and V:

$$U=\Phi^{-1}\Phi_u , \qquad V=\Phi^{-1}\Phi_v .$$

Using the compatibility condition $\varphi_{uv}=\varphi_{vu}$ for (2.5) and the Gauss-Weingarten formulae (1.4), we get the following (Lax-) pair:

$$U=u_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-\frac{1}{4}\omega_u & -Qe^{-\omega/2}\\ \frac{H}{2}e^{\omega/2} & \frac{1}{4}\omega_u\end{pmatrix},$$
$$V=v_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}\frac{1}{4}\omega_v & -\frac{H}{2}e^{\omega/2}\\ Re^{-\omega/2} & -\frac{1}{4}\omega_v\end{pmatrix}$$

under the identification (2.2). Now only the coefficients u_0 and v_0 of $\mathbf{1}$ are still not determined. Recall that Φ was defined by (2.5) up to a multiplication by a scalar factor. Direct calculations show that the compatibility condition of the above linear differential equation is equivalent to the Gauss-Codazzi equations (1.9)–(1.11) if and only if $(\partial/\partial u)v_0=(\partial/\partial v)u_0$ holds.

PROPOSITION 2.1. *Under the identification* (2.2), *the moving frame* $f=(e^{-\omega/2}\varphi_x, e^{-\omega/2}\varphi_y, N)$ *of a timelike surface is described by the formula* (2.5). *Here* Φ *is an* $\mathbf{H}'$*-valued function which satisfies*

$$\frac{\partial}{\partial u}\Phi=\Phi\begin{pmatrix}u_0-\frac{1}{4}\omega_u & -Qe^{-\omega/2}\\ \frac{H}{2}e^{\omega/2} & u_0+\frac{1}{4}\omega_u\end{pmatrix}, \qquad \frac{\partial}{\partial v}\Phi=\Phi\begin{pmatrix}v_0+\frac{1}{4}\omega_v & -\frac{H}{2}e^{\omega/2}\\ Re^{-\omega/2} & v_0-\frac{1}{4}\omega_v\end{pmatrix}. \tag{2.6}$$

The entries u_0 *and* v_0 *are defined by* (2.5) *up to a multiplication of a scalar factor such that*

$$\frac{\partial}{\partial u} v_0 = \frac{\partial}{\partial v} u_0 .$$

Now, we shall consider a constant mean curvature surface M. As we saw in Proposition 1.1, we obtain a one-parameter deformation of M induced by a deformation of the Hopf pair $Q^{\#}$ and $R^{\#}$. Using this deformation, we get the following Bobenko-type representation formula.

THEOREM 2.2 (Bobenko-type formula). *Let $\Phi_\lambda(u, v)$ be a solution of the following linear differential equations:*

$$\frac{\partial}{\partial u} \Phi_\lambda = \Phi_\lambda U(\lambda) , \qquad \frac{\partial}{\partial v} \Phi_\lambda = \Phi_\lambda V(\lambda) , \tag{2.7}$$

$$U(\lambda) = \begin{pmatrix} u_0 - \frac{1}{4}\omega_u & -Q_\lambda e^{-\omega/2} \\ \frac{H}{2} e^{\omega/2} & u_0 + \frac{1}{4}\omega_u \end{pmatrix}, \qquad V(\lambda) = \begin{pmatrix} v_0 + \frac{1}{4}\omega_v & -\frac{H}{2} e^{\omega/2} \\ R_\lambda e^{-\omega/2} & v_0 - \frac{1}{4}\omega_v \end{pmatrix}, \tag{2.8}$$

$$\frac{\partial}{\partial u} v_0 = \frac{\partial}{\partial v} u_0 ,$$

where H is a real constant, $Q_\lambda = \lambda Q$ and $R_\lambda = \lambda^{-1} R$ for $\lambda = \pm e^t$. Then

$$\varphi_\lambda = \frac{2}{H} \left\{ \frac{\partial}{\partial t} \Phi_\lambda . \Phi_\lambda^{-1} - \frac{1}{2} N_\lambda \right\}, \qquad N_\lambda = \mathrm{Ad}(\Phi_\lambda)\mathbf{k}' , \tag{2.9}$$

describes a loop of timelike constant mean curvature H surfaces with first fundamental form $I = e^{\omega} dudv$ and Hopf pair $Q_\lambda^{\#}$ and $R_\lambda^{\#}$. The Gauss mapping of each φ_λ is N_λ.

Conversely, let φ_λ be a loop of timelike constant mean curvature H surfaces with first fundamental form $I = e^{\omega} dudv$ and Hopf pair $Q_\lambda^{\#}$ and $R_\lambda^{\#}$. Then φ_λ is given by the formula (2.9) *via a solution Φ_λ of* (2.7)–(2.8).

PROOF. We use the identification (2.2). Differentiating (2.9), we have

$$\frac{\partial}{\partial u} \varphi_\lambda = \frac{2}{H} \mathrm{Ad}(\Phi_\lambda) \left\{ \frac{1}{2} (\mathbf{i} + \mathbf{j}') \right\}, \qquad \frac{\partial}{\partial v} \varphi_\lambda = \frac{2}{H} \mathrm{Ad}(\Phi_\lambda) \left\{ \frac{1}{2} (-\mathbf{i} + \mathbf{j}') \right\}. \tag{2.10}$$

These formulae coincide with (2.5). □

REMARK. If we use a substitution $\lambda = \pm e^{2t}$, then the formula (2.9) become:

$$\varphi_\lambda = \frac{1}{H} \left\{ \frac{\partial}{\partial t} \Phi_\lambda . \Phi_\lambda^{-1} - N_\lambda \right\}.$$

REMARK. A Bobenko-type formula for spacelike surfaces of constant mean curvature was obtained by T. Taniguchi [14]. However, his formula is slightly different from our (2.9) in [5]. Our formula for spacelike or timelike surfaces describes a loop of such surfaces.

Let us denote the moving frame $(e^{-\omega/2}\varphi_x, e^{-\omega/2}\varphi_y, N)$ by f as before and let $p: \mathrm{SL}_2\mathbf{R}\to \mathrm{O}_1(3)^\circ$ be a double covering projection. Here $\mathrm{O}_1(3)^\circ$ denotes the identity component of the Lorentz group $\mathrm{O}_1(3)$. Then we can normalise the coefficients u_0 and v_0 such that Φ takes value in G. In fact, there exists a smooth map $\Phi: \mathfrak{D}\subset M\to G$ defined on a simply-connected region $\mathfrak{D}$ of M such that $p\circ\Phi=f$ (cf. Appendix in J. Dorfmeister and G. Haak [4]).

The mapping Φ is defined uniquely up to a sign. Under this normalisation, the linear differential equations (2.6) then become

$$(2.6')\qquad \frac{\partial}{\partial u}\Phi=\Phi\begin{pmatrix} -\frac{1}{4}\omega_u & -Qe^{-\omega/2} \\ \frac{H}{2}e^{\omega/2} & \frac{1}{4}\omega_u , \end{pmatrix},\qquad \frac{\partial}{\partial v}\Phi=\Phi\begin{pmatrix} \frac{1}{4}\omega_v & -\frac{H}{2}e^{\omega/2} \\ Re^{-\omega/2} & -\frac{1}{4}\omega_v \end{pmatrix}.$$

3. The vacuum solution.

In this section we give an example of loops of timelike constant mean curvature surfaces.

Choose $H=1/2$ and $Q=R=-1/4$. Then $\omega\equiv 0$ is a global solution of the Gauss-Codazzi equations (1.7)–(1.8) defined on a whole plane $\mathbf{R}^2(x, y)$. The solution $\omega\equiv 0$ is called the *vacuum solution* of the Gauss-Codazzi equation (GC). The deformed equations of (2.6′) corresponding to the vacuum solution are

$$\frac{\partial}{\partial u}\Phi_\lambda=\Phi_\lambda\begin{pmatrix} 0 & \lambda/4 \\ 1/4 & 0 \end{pmatrix},\quad \frac{\partial}{\partial v}\Phi_\lambda=\Phi_\lambda\begin{pmatrix} 0 & -1/4 \\ -\lambda^{-1}/4 & 0 \end{pmatrix},\qquad \lambda\in\mathbf{R}^* .$$

We can solve the above equations explicitly under the initial condition $\Phi_\lambda(0, 0)\equiv\mathbf{1}$.

(1) $\lambda=\nu^2=e^{2t}>0$:

The solution Φ_λ is

$$\Phi_\lambda(u, v)=\begin{pmatrix} \cosh\{\frac{1}{4}(\nu u-\nu^{-1}v)\} & \nu\sinh\{\frac{1}{4}(\nu u-\nu^{-1}v)\} \\ \nu^{-1}\sinh\{\frac{1}{4}(\nu u-\nu^{-1}v)\} & \cosh\{\frac{1}{4}(\nu u-\nu^{-1}v)\} \end{pmatrix}.$$

The following formulae are easily obtained from Theorem 2.2 and its remark:

$$\varphi_\lambda(u, v)=\begin{pmatrix} \frac{\nu^{-1}-\nu}{4}(\nu u+\nu^{-1}v)+\frac{\nu+\nu^{-1}}{2}\sinh\{\frac{1}{2}(\nu u-\nu^{-1}v)\} \\ \frac{\nu^{-1}+\nu}{4}(\nu u+\nu^{-1}v)+\frac{\nu-\nu^{-1}}{2}\sinh\{\frac{1}{2}(\nu u-\nu^{-1}v)\} \\ -\cosh\{\frac{1}{2}(\nu u-\nu^{-1}v)\}-1 \end{pmatrix};$$

$$N_\lambda(u, v)=\begin{pmatrix} -\frac{\nu+\nu^{-1}}{2}\sinh\{\frac{1}{2}(\nu u-\nu^{-1}v)\} \\ \frac{\nu-\nu^{-1}}{2}\sinh\{\frac{1}{2}(\nu u-\nu^{-1}v)\} \\ \cosh\{\frac{1}{2}(\nu u-\nu^{-1}v)\} \end{pmatrix};$$

$$I_\lambda\equiv I=-dx^2+dy^2 ,\quad H_\lambda\equiv 1/2 ,\quad Q_\lambda=-\lambda/4 ,\quad R_\lambda=-\lambda^{-1}/4 ;$$

$$II_\lambda=-\cosh^2 t\,dx^2-2\sinh t\cosh t\,dxdy-\sinh^2 t\,dy^2 .$$

The one-parameter family $\{\varphi_\lambda\}_{\lambda>0}$ consists of isometric (i.e. flat) timelike immersions from $\mathbf{E}^2_1(x, y)$ into $\mathbf{E}^3_1$. In particular,

$$\varphi_1(x, y)=(\sinh x, y, -\cosh x)+(0, 0, -1).$$

The timelike surface $\varphi_1: -\xi_1^2+(\xi_3+1)^2=1$, $\xi_3<0$ is called a *timelike hyperbolic cylinder.*

(2) $\lambda=-v^2=-e^{2t}<0$:

In this case, the solution Φ_λ is

$$\Phi_\lambda(u, v)=\begin{pmatrix} \cos\{\frac{1}{4}(vu+v^{-1}v)\} & -v\sin\{\frac{1}{4}(vu+v^{-1}v)\} \\ v^{-1}\sin\{\frac{1}{4}(vu+v^{-1}v)\} & \cos\{\frac{1}{4}(vu+v^{-1}v)\} \end{pmatrix}.$$

Hence we get the following formulae:

$$\varphi_\lambda(u, v)=\begin{pmatrix} \frac{v+v^{-1}}{4}(vu-v^{-1}v)-\frac{v-v^{-1}}{2}\sin\{\frac{1}{2}(vu+v^{-1}v)\} \\ \frac{v^{-1}-v}{4}(vu-v^{-1}v)+\frac{v+v^{-1}}{2}\sin\{\frac{1}{2}(vu+v^{-1}v)\} \\ -\cos\{\frac{1}{2}(vu-v^{-1}v)\}-1 \end{pmatrix};$$

$$N_\lambda(u, v)=\begin{pmatrix} \frac{v-v^{-1}}{2}\sin\{\frac{1}{2}(vu+v^{-1}v)\} \\ -\frac{v+v^{-1}}{2}\sin\{\frac{1}{2}(vu+v^{-1}v)\} \\ \cos\{\frac{1}{2}(vu+v^{-1}v)\} \end{pmatrix};$$

$$I_\lambda\equiv I=-dx^2+dy^2, \quad H_\lambda\equiv 1/2, \quad Q_\lambda=-\lambda/4, \quad R_\lambda=-\lambda^{-1}/4;$$

$$II_\lambda=\sinh^2 t dx^2+2\sinh t\cosh t dxdy+\cosh^2 t dy^2.$$

The one-parameter family $\{\varphi_\lambda\}_{\lambda<0}$ consists of isometric (i.e. flat) timelike immersions of $\mathbf{E}^2_1(x, y)$ into $\mathbf{E}^3_1$. In particular,

$$\varphi_{-1}(x, y)=(x, \sin y, -\cos y)+(0, 0, -1).$$

The immersion φ_{-1} induces an imbedding of a warped product manifold $\mathbf{E}^1(x)_{-1}\times S^1=\mathbf{E}^1(x)_{-1}\times(\mathbf{E}^1(y)/2\pi\mathbf{Z})$ into $\mathbf{E}^3_1$. The resulting timelike surface is a circular cylinder $\xi_2^2+(\xi_3+1)^2=1$ with the warped product (Lorentz) metric $-dx^2+dy^2$. This surface φ_{-1} is called a *timelike circular cylinder.*

In Minkowski 3-space, a (timelike) hyperbolic cylinder and a (timelike) circular cylinder belong to same one-parameter family. This surprising phenomenon tells us that the geometry of nonzero constant mean curvature surfaces is of some interest.

REMARK. Timelike hyperbolic cylinders and timelike circular cylinders are found in [9]. However M. Magid did not mention the above phenomenon.

4. Parallel surface procedure for timelike surfaces.

The notion of parallel surface can be dealt with in much the same way as that of a surface in Euclidean 3-space or a spacelike surface in Minkowski 3-space.

A theorem of O. Bonnet says surfaces of parallel distances $\pm 1/(2H)$ of a constant mean curvature H surface have constant positive curvature. We shall prove a Bonnet-type theorem for parallel surfaces of a timelike surface in $\mathbf{E}^3_1$ by using the Bobenko-type formula (2.9). The immersion is a sum of two vector-valued functions—the logarithmic derivative of the frame and the unit normal vector field. The parallel surface procedure provides a differential geometric meaning of the form of (2.9).

First we consider the logarithmic derivative part of (2.9).

PROPOSITION 4.1. *Let φ_λ be a loop of timelike constant mean curvature surfaces as in Theorem 2.2. Then*

$$\varphi^{[1]}_\lambda = \frac{2}{H}\frac{\partial}{\partial t}\Phi_\lambda . \Phi_\lambda^{-1} = \varphi_\lambda + \frac{1}{H} N_\lambda \tag{4.1}$$

is a loop of branched timelike surfaces with Gauss map $N^{[1]}_\lambda = -N_\lambda$, first fundamental form $\tilde{I}_\lambda = e^{\tilde{\omega}} du dv$, $e^{\tilde{\omega}} = (4/H^2)QRe^{-\omega}$, mean curvature H and Hopf pair $Q^\#_\lambda$, $R^\#_\lambda$. The branch point of $\varphi^{[1]}_\lambda$ corresponds to the point φ_λ such that $QR=0$ holds.

PROOF. Differentiating (4.1), we have

$$\frac{\partial}{\partial u}\varphi^{[1]}_\lambda = \frac{2}{H} Q_\lambda e^{-\omega}\frac{\partial}{\partial v}\varphi_\lambda , \qquad \frac{\partial}{\partial v}\varphi^{[1]}_\lambda = \frac{2}{H} R_\lambda e^{-\omega}\frac{\partial}{\partial u}\varphi_\lambda .$$

Direct calculations using these formulae give the required result. □

The original timelike surfaces of constant mean curvature are rewritten in a logarithmic derivative form as follows.

PROPOSITION 4.2. *Let φ_λ be a loop of spacelike constant mean curvature surfaces as in the preceding proposition* 4.1. *Then φ_λ is written in a logarithmic derivative form as follows*:

$$\varphi_\lambda = \frac{2}{H}\frac{\partial}{\partial t}\Phi^{[2]}_\lambda . (\Phi^{[2]}_\lambda)^{-1} , \qquad \Phi^{[2]}_\lambda = \Phi_\lambda \begin{pmatrix} \sqrt{|\lambda|} & 0 \\ 0 & 1/\sqrt{|\lambda|} \end{pmatrix} . \tag{4.2}$$

PROOF. By the definition of $\Phi^{[2]}_\lambda$, we get

$$U^{[2]}(\lambda) := (\Phi^{[2]}_\lambda)^{-1}\frac{\partial}{\partial z}\Phi^{[2]}_\lambda = \begin{pmatrix} u_0 - \frac{1}{4}\omega_u & -\mathrm{sgn}(\lambda) Q e^{-\omega/2} \\ |\lambda| \frac{H}{2} e^{\omega/2} & u_0 + \frac{1}{4}\omega_u \end{pmatrix},$$

$$V^{[2]}(\lambda) := (\Phi^{[2]}_\lambda)^{-1}\frac{\partial}{\partial \bar{z}}\Phi^{[2]}_\lambda = \begin{pmatrix} v_0 + \frac{1}{4}\omega_v & -\frac{H}{2}|\lambda|^{-1} e^{\omega/2} \\ \mathrm{sgn}(\lambda) R e^{-\omega/2} & v_0 - \frac{1}{4}\omega_v \end{pmatrix}.$$

Calculations similar to the proof of (2.9) show the result. □

Next we consider the following transform of the frame:

$$\Phi_\lambda^{[3]} := \Phi_\lambda \begin{pmatrix} \sqrt[4]{|\lambda|} & 0 \\ 0 & 1/\sqrt[4]{|\lambda|} \end{pmatrix}. \tag{4.3}$$

The following formulae can be easily verified:

$$U^{[3]}(\lambda) := (\Phi_\lambda^{[3]})^{-1} \frac{\partial}{\partial z} \Phi_\lambda^{[3]} = \begin{pmatrix} u_0 - \frac{1}{4}\omega_u & -\operatorname{sgn}(\lambda)\sqrt{|\lambda|}Qe^{-\omega/2} \\ \sqrt{|\lambda|}\frac{H}{2}e^{\omega/2} & u_0 + \frac{1}{4}\omega_u \end{pmatrix}, \tag{4.4}$$

$$V^{[3]}(\lambda) := (\Phi_\lambda^{[3]})^{-1} \frac{\partial}{\partial \bar{z}} \Phi_\lambda^{[3]} = \begin{pmatrix} v_0 + \frac{1}{4}\omega_v & -\frac{1}{\sqrt{|\lambda|}}\frac{H}{2}e^{\omega/2} \\ \frac{\operatorname{sgn}(\lambda)}{\sqrt{|\lambda|}}Re^{-\omega/2} & v_0 - \frac{1}{4}\omega_v \end{pmatrix}.$$

A somewhat tedious and lengthy but elementary computation using (4.4) yields the following result.

PROPOSITION 4.3. *The mapping*

$$\varphi_\lambda^{[3]} := \frac{2}{H}\frac{\partial}{\partial t}\Phi_\lambda^{[3]}.(\Phi_\lambda^{[3]})^{-1} \tag{4.5}$$

defines a loop of branched timelike immersions. The fundamental associated quantities of $\varphi_\lambda^{[3]}$ *are given as follows*:

$$I_\lambda^{[3]} = \begin{pmatrix} \frac{e^\omega}{4} + \frac{QR}{H^2}e^{-\omega} - \frac{Q_\lambda + R_\lambda}{2H} & \frac{R_\lambda - Q_\lambda}{2H} \\ \frac{R_\lambda - Q_\lambda}{2H} & -\frac{e^\omega}{4} - \frac{QR}{H^2}e^{-\omega} - \frac{Q_\lambda + R_\lambda}{2H} \end{pmatrix},$$

$$II_\lambda^{[3]} = 2H\left(\frac{e^\omega}{4} - \frac{QR}{H^2}e^{-\omega}\right)du\,dv,$$

$$H_\lambda^{[3]} = -2H\frac{\frac{H^2}{4}e^\omega + QRe^{-\omega}}{\frac{H^2}{4}e^\omega - QRe^{-\omega}}, \quad K_\lambda^{[3]} = 4H^2, \quad N_\lambda^{[3]} = N_\lambda.$$

In particular, a branch point of $\varphi_\lambda^{[3]}$ *corresponds to a flat point of* φ_λ. *Each branched surface* $\varphi_\lambda^{[3]}$ *is a parallel surface of* φ_λ:

$$\varphi_\lambda^{[3]} = \varphi_\lambda + \frac{1}{2H}N_\lambda. \tag{4.6}$$

Since each mean curvature $H_\lambda^{[3]}$ of $\varphi_\lambda^{[3]}$ is independent of the deformation parameter (spectral parameter) λ and the Gaussian curvature $K_\lambda^{[3]}$ is positive constant $4H^2$, both principal curvatures of $\varphi_\lambda^{[3]}$ are independent of λ. The second fundamental form $II_\lambda^{[3]}$ is also independent of λ. The loops $\{\varphi_\lambda^{[1]}\}$ and $\{\varphi_\lambda^{[2]}\} = \{\varphi_\lambda\}$ are isometric-mean curvature-preserving families, but $\{\varphi_\lambda^{[3]}\}$ is a II-isometric and K-preserving family.

Propositions 4.1–4.3 give a Bonnet-type theorem on parallel surface procedure for timelike surfaces in $\mathbf{E}_1^3$.

These propositions provides the equivalence between the study of constant mean curvature timelike surfaces and that of constant positive curvature timelike surfaces.

The theory of finite-type harmonic maps can be applied to the geometry of constant mean curvature timelike surfaces. In addition so-called Bäcklund-Darboux transformations are applicable to such surfaces. These topics will be developed in the forthcoming papers [6], [7].

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Present Address:
Department of Applied Mathematics, Fukuoka University,
Nanakuma, Jonan-ku, Fukuoka, 814–0180 Japan.
e-mail: inoguchi@bach.sm.fukuoka-u.ac.jp