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Morse-Smale Diffeomorphisms and the Standard Family

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Abstract. Every Morse-Smale diffeomorphism of the circle is conjugate to a diffeomorphism belonging to the set defined by

$$f_{\omega,\varepsilon,k}(x)=x+\omega+\frac{\varepsilon}{2\pi}\sin(2k\pi x) \qquad (0<\omega<1,\ 0<\varepsilon<1,\ k \text{ with } 0<\varepsilon k<1)$$

and Morse-Smale diffeomorphisms in the set is C^1 open and dense, with respect to the relative topology, in Arnol'd tongue.

Morse-Smale diffeomorphisms and the standard family are investigated. Theorem A examines that the standard family contains an analytic model for each Morse-Smale diffeomorphism, and in Theorem B the density of the set of all Morse-Smale diffeomorphisms in the Arnol'd tongue is investigated.

Let f be a diffeomorphism of the circle $S^1=\mathbf{R}/\mathbf{Z}$ and $F\colon \mathbf{R}\to\mathbf{R}$ be a lift of f such that $\mathscr{P}\circ F=f\circ\mathscr{P}$, where $\mathscr{P}$ denotes the canonical projection from $\mathbf{R}$ to S^1. A periodic point of f with the period m, x, is called a *sink* (*source*) if $0<|\frac{d}{dx}F^m(\bar{x})|<1$ ($|\frac{d}{dx}F^m(\bar{x})|>1$), where $\mathscr{P}(\bar{x})=x$. If the set of all periodic points of f, $\mathrm{Per}(f)$, is non-empty and consists of only sinks and sources, then f is called a *Morse-Smale diffeomorphism*. If diffeomorphisms $f\colon S^1\to S^1$ and $g\colon S^1\to S^1$ satisfy the relation $f\circ h=h\circ g$ for some homeomorphism $h\colon S^1\to S^1$, then we say that f is *topologically conjugate* to g and that h is a *conjugacy map* between f and g. A diffeomorphism $f\colon S^1\to S^1$ is said to be *C^1-structurally stable* if there is a C^1 neighborhood U of f such that f is topologically conjugate to every $g\in U$. We know that every Morse-Smale diffeomorphism f is C^1-structurally stable and the set of all C^1 Morse-Smale diffeomorphisms MS is open and dense in the set of all C^1 diffeomorphisms of the circle with respect to the C^1-topology (cf. [2]).

The set of circle diffeomorphisms defined by

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$$f_{\omega,\varepsilon,k}(x)=x+\omega+\frac{\varepsilon}{2\pi}\sin(2k\pi x) \pmod 1 \qquad (0<\omega<1,\ 0<\varepsilon<1,\ 0<\varepsilon k<1)$$

is called the *standard family* (cf. [2]).

THEOREM A. *Every orientation-preserving Morse-Smale diffeomorphism is topologically conjugate to a Morse-Smale diffeomorphism belonging to the standard family.*

Let $f: S^1 \to S^1$ be an orientation-preserving diffeomorphism with a lift F. Then the number $\rho_0(F)=\lim_{n\to\infty}(F^n(x)-x)/n$ is independent of x (cf. [2]). If F_1 and F_2 are two lifts of f, then $\rho_0(F_1)-\rho_0(F_2)$ is an integer, and so $\rho(f)=\rho_0(F)$ mod 1 is defined. The number $\rho(f)$ is called the *rotation number* of f. $\rho(f)$ is rational if and only if f has a periodic point (cf. [2]). Since a Morse-Smale diffeomorphism has periodic points, the rotation number of each Morse-Smale diffeomorphism is rational. On the other hand, if $f: S^1 \to S^1$ is a C^2 orientation-preserving diffeomorphism with irrational rotation number, then f is topologically conjugate to the rigid rotation (cf. [6]).

The dynamics of standard family was firstly studied by Arnol'd (cf. [1]). Herman proved in [4] that if $0<\varepsilon<1$ is fixed and $k=1$, then the set of parameter ω for which the rotation number is irrational has positive Lebesgue measure. Recently Świątek [6] proved that the analogous result is false for homeomorphisms with critical points: under rather general assumptions which admit the family $f_{\omega,\varepsilon,k}$ with $(\varepsilon, k)=(1, 1)$, he showed that the parameter ω corresponding to rational numbers constitute a set of full measure. For (ε, k) a fixed parameter Graczyk [3] proved that the set of parameter ω which corresponds to non-linearizable maps with an irrational rotation number is of Hausdorff dimension 0. If r is a rational number, then the family of C^1 diffeomorphisms of S^1 with the rotation number r, $D(r)$, is connected ($D(r)$ is called the level set). In each level set the set of Morse-Smale diffeomorphisms, $D(r)\cap MS$, is dense ([1]).

For fixed $k>0$ the set

$$AT_k(r)=\{(\omega, \varepsilon): \rho(f_{\omega,\varepsilon,k})=r\}$$

is called an *Arnol'd tongue* if r is a rational number. Obviously, $AT_k(r)$ constitutes a subset of $D(r)$. Moreover define a subset $MS_k(r)=\{(\omega, \varepsilon)\in AT_k(r): f_{\omega,\varepsilon,k}\in MS\}$ of $AT_k(r)$.

THEOREM B. $MS_k(r)$ *is open and dense in an Arnol'd tongue* $AT_k(r)$ *with respect to the relative topology.*

PROOF OF THEOREM A. We begin with checking a sufficient condition under which two circle diffeomorphisms are topologically conjugate. Topologically conjugate diffeomorphisms have the same cardinality of periodic points. Thus we suppose f and g are orientation-preserving Morse-Smale diffeomorphisms such that the cardinalities of Per(f) and Per(g) are equal. Since a Morse-Smale diffeomorphism of the circle has the same numbers of sinks and sources, the cardinality of sinks of f is equal to that of g. Denote by $\mathbf{S}_f=\{s_1, s_2, \cdots, s_q\}$ and $\mathbf{S}_g=\{s'_1, s'_2, \cdots, s'_q\}$ the sets of sinks of f and

g respectively. By the definition of sinks, $f(\mathbf{S}_f)=\mathbf{S}_f$ and $g(\mathbf{S}_g)=\mathbf{S}_g$. For $x\in S^1$ we denote as $\bar{x}$ the point in $\mathbf{R}$ satisfying $0\leq\bar{x}<1$ and $\mathscr{P}(\bar{x})=x$. Without loss of generality we may assume that $\bar{s}_1<\bar{s}_2<\cdots<\bar{s}_q$ and $\bar{s}'_1<\bar{s}'_2<\cdots<\bar{s}'_q$.

CLAIM. f is topologically conjugate to g if $f(s_1)=s_{p+1}$ and $g(s'_1)=s'_{p+1}$ for a fixed $0\leq p\leq q-1$.

It is needless to say that we can find a conjugacy map between f and g by going on the similar way to prove the structural stability for Morse-Smale circle diffeomorphisms. However, to make sure we give a proof for the claim. Indeed, f is Morse-Smale and so if $\mathbf{U}_f=\{u_1, u_2, \cdots, u_q\}$ denotes the set of sources of f, then we may suppose that $\bar{s}_i<\bar{u}_i<\bar{s}_{i+1}$ for $i=1, \cdots, q$. Here $\bar{s}_{q+1}=\bar{s}_1+1$. Let $O_f(x)$ denote the orbit of $x\in S^1$ under f (i.e. $O_f(x)=\{f^j(x)\}_{j\in\mathbf{Z}}$). Since f is orientation-preserving, all periodic points have the same period, say $m>0$. Then $O_f(u_i)=\{f^j(u_i)\}_{j=0}^{m-1}$ $(1\leq i\leq q)$. The *unstable manifold* of u_i $(1\leq i\leq q)$ is defined by $W^u_f(u_i)=\{x: \lim_{n\to-\infty} f^{mn}(x)=u_i\}$ and characterized as $W^u_f(u_i)=\mathscr{P}((\bar{s}_i, \bar{s}_{i+1}))$ since f is Morse-Smale. Then $S^1=\bigcup_{i=1}^q W^u_f(u_i)\cup\mathbf{S}_f$. On the other hand, denote as $\mathbf{U}_g=\{u'_1, u'_2, \cdots, u'_q\}$ the set of sources of g, and define $O_g(x)$ $(x\in S^1)$ and $W^u_g(u'_i)$ $(i=1, \cdots, q)$ in the same way as given for f. From the assumption of the claim, s'_1 has the same period m of s_1, and so does u'_i $(i=1, \cdots, q)$. By the same argument we have $O_g(u'_i)=\{g^j(u'_i)\}_{j=0}^{m-1}$ and $W^u_g(u'_i)=\mathscr{P}((\bar{s}'_i, \bar{s}'_{i+1}))$ for $i=1, \cdots, q$.

Take and fix an arbitrary $1\leq i\leq q$, and put $W^u_f(O_f(u_i))=\bigcup_{j=0}^{m-1} W^u_f(f^j(u_i))$ and $W^u_g(O_g(u'_i))=\bigcup_{j=0}^{m-1} W^u_g(g^j(u'_i))$. Notice that $W^u_f(O_f(u_i))$ is an f-invariant set (i.e. $f(W^u_f(O_f(u_i)))=W^u_f(O_f(u_i)))$ and that $W^u_g(O_g(u'_i))$ is g-invariant. It is checked that $f|W^u_f(O_f(u_i))$ is topologically conjugate to $g|W^u_g(O_g(u'_i))$. To show this, fix four points $a\in\mathscr{P}((\bar{s}_i, \bar{u}_i))$, $b\in\mathscr{P}((\bar{u}_i, \bar{s}_{i+1}))$, $a'\in\mathscr{P}((\bar{s}'_i, \bar{u}'_i))$ and $b'\in\mathscr{P}((\bar{u}'_i, \bar{s}'_{i+1}))$. Since u_i and u'_i are sources of f and g respectively and they have the same period $m>0$, we have

$$[\overline{f^m(a)}, \bar{a}]\cup[\bar{b}, \overline{f^m(b)}]\subset(\bar{s}_i, \bar{s}_{i+1}) , \qquad [\overline{g^m(a')}, \bar{a}']\cup[\bar{b}', \overline{g^m(b')}]\subset(\bar{s}'_i, \bar{s}'_{i+1}) .$$

Let $\phi : \mathscr{P}([\overline{f^m(a)}, \bar{a}]\cup[\bar{b}, \overline{f^m(b)}])\to\mathscr{P}([\overline{g^m(a')}, \bar{a}']\cup[\bar{b}, \overline{g^m(b')}])$ be a hemeomorphsim such that

$$\phi(a)=a' , \quad \phi(f^m(a))=g^m(a') , \quad \phi(b)=b' , \quad \phi(f^m(b))=g^m(b') ,$$

and let $D=\mathscr{P}((\overline{f^m(a)}, \bar{a}]\cup[\bar{b}, \overline{f^m(b)}))$. Then $W^u_f(u_i)\backslash\{u_i\}$ can be written as the disjoint union $W^u_f(u_i)\backslash\{u_i\}=\bigcup_{k\in\mathbf{Z}} f^{mk}(D)$ of $f^{mk}(D)$. Thus we can construct a map $\tilde{h}_i : W^u_f(u_i)\to W^u_g(u'_i)$ satisfying $\tilde{h}_i(x)=g^{mk}\circ\phi\circ f^{-mk}(x)$ for $x\in f^{mk}(D)$ $(k\in\mathbf{Z})$ and $\tilde{h}_i(u_i)=u'_i$. It is easily checked that $\tilde{h}_i$ is a homeomorphism with $\tilde{h}_i\circ f^m(x)=g^m\circ\tilde{h}_i(x)$ $(x\in W^u_f(u_i))$. Thus, if we define $h_i : W^u_f(O_f(u_i))\to W^u_g(O_g(u'_i))$ by $h_i(x)=g^j\circ\tilde{h}_i\circ f^{-j}(x)$ for $x\in f^j(W^u_f(u_i))$ $(j=0, \cdots, m-1)$, then h_i is a conjugacy map between $f|W^u_f(O_f(u_i))$ and $g|W^u_g(O_g(u'_i))$.

From the assumption of the claim it follows that $f(u_i)=u_j$ $(j=p+i \bmod q)$ and

$g(u_i')=u_j'$ ($j=p+i \bmod q$). Thus we can choose $I\subset\{1,\cdots,q\}$ such that $\mathbf{U}_f$ and $\mathbf{U}_g$ are decomposed as disjoint unions $\mathbf{U}_f=\bigcup_{i\in I}O_f(u_i)$ and $\mathbf{U}_g=\bigcup_{i\in I}O_g(u_i')$ of distinct orbits respectively. Since $S^1=\bigcup_{i\in I}W_f^u(O_f(u_i))\cup\mathbf{S}_f$ and $S^1=\bigcup_{i\in I}W_g^u(O_g(u_i'))\cup\mathbf{S}_g$ are disjoint unions, we can define a homeomorphism $h: S^1\to S^1$ by

$$h(x)=\begin{cases} h_i(x)\,, & \text{if } \ x\in W_f^u(O_f(u_i))\,,\ \ i\in I\,,\\ s_i'\,, & \text{if } \ x=s_i\,,\ \ i=1,\cdots,q\,,\end{cases}$$

where h_i ($i\in I$) is defined as above. It is easy to check that h is a conjugacy map between f and g. Therefore the above claim holds.

We are now ready to confirm the conclusion of Theorem A. Let f be a Morse-Smale diffeomorphism and let s_i ($i=1,\cdots,q$) be as above. Suppose that $f(s_1)=s_{p+1}$ for some $0\le p\le q-1$. If we define

$$F(x)=x+\frac{p}{q}+\frac{\varepsilon}{2\pi}\sin(2q\pi x)\,,$$

for ε with $0<\varepsilon q<1$, then $\mathscr{P}\circ F\circ\mathscr{P}^{-1}=f_{p/q,\varepsilon,q}$ is a circle diffeomorphism. Moreover, $f_{p/q,\varepsilon,q}$ is Morse-Smale. Indeed, write $F_0(x)=x+(\varepsilon/2\pi)\sin(2q\pi x)$. Then $\mathscr{P}\circ F_0\circ\mathscr{P}^{-1}=f_0$ is clearly a Morse-Smale diffeomorphism. Since

$$\begin{aligned} F^q(x)&=x+p+\frac{\varepsilon}{2\pi}\sum_{i=0}^{q-1}\sin[2q\pi F^i(x)]\\ &=x+p+\frac{\varepsilon}{2\pi}\sum_{i=0}^{q-1}\sin[2q\pi F_0^i(x)]=F_0^q(x)+p\,,\end{aligned}$$

we have $f^q_{p/q,\varepsilon,q}=f_0^q$. This implies that $f_{p/q,\varepsilon,q}$ is also a Morse-Smale diffeomorphism. Since $f^q_{p/q,\varepsilon,q}=f_0^q$, we have

$$\mathbf{S}_{f_{p/q,\varepsilon,q}}=\mathbf{S}_{f^q_{p/q,\varepsilon,q}}=\mathbf{S}_{f_0^q}=\mathbf{S}_{f_0}=\left\{\mathscr{P}\left(\frac{2i-1}{2q}\right): i=1,\cdots,q\right\}.$$

Write $s_i'=\mathscr{P}((2i-1)/(2q))$ ($i=1,\cdots,q$). Then

$$\begin{aligned} f_{p/q,\varepsilon,q}(s_1')&=f_{p/q,\varepsilon,q}(\mathscr{P}(1/(2q))=\mathscr{P}\circ F(1/(2q))\\ &=\mathscr{P}(1/(2q)+p/q)=\mathscr{P}((2p+1)/(2q))=s_{p+1}'\,,\end{aligned}$$

which ensures, by the above claim, that f is topologically conjugate to $f_{p/q,\varepsilon,q}$.

Proof of Theorem B. Let k be a positive integer and r be a rational number with $0\le r\le 1$. To prove the density of $MS_k(r)$ in an Arnol'd tongue, take $(\omega_0,\varepsilon)\in AT_k(r)$ and a diffeomorphism $f_{\omega_0,\varepsilon,k}$ belonging to the standard family with $\rho(f_{\omega_0,\varepsilon,k})=r$. Write $r=p/q$, where p and q are non-negative relatively prime integers. A diffeomorphism $F_{\omega_0,\varepsilon,k}$ of $\mathbf{R}$ defined by $F_{\omega_0,\varepsilon,k}(x)=x+\omega_0+(\varepsilon/2\pi)\sin(2k\pi x)$ is a lift of $f_{\omega_0,\varepsilon,k}$. For the simple notations we write f_{ω_0}, F_{ω_0} instead of $f_{\omega_0,\varepsilon,k}$, $F_{\omega_0,\varepsilon,k}$ respectively. Since $\rho(f_{\omega_0})=p/q$, each periodic point of f_{ω_0} is a fixed point of $f^q_{\omega_0}$ and so we can find $l\in\mathbf{Z}$ such that

$F^q_{\omega_0}(x)=x+l$ for $x\in\mathscr{P}^{-1}(\mathrm{Per}(f_{\omega_0}))$. Fix l and take a diffeomorphism $f_\omega=f_{\omega,\varepsilon,k}$ from the standard family. Then $F_\omega:\mathbf{R}\to\mathbf{R}$ $(\omega\in\mathbf{R})$ defined by $F_\omega(x)=x+\omega+(\varepsilon/2\pi)\sin(2k\pi x)$ is a lift of f_ω. Write

$$G(\omega, x)=F^q_\omega(x)-(x+l)$$

for $\omega, x\in\mathbf{R}$.

It is checked that $\frac{\partial}{\partial\omega}G(\omega, x)>0$. Indeed, if we write $F(\omega, x)=F_\omega(x)$ and $F^j_\omega(x)=F(\omega, F^{j-1}_\omega(x))$ for $j\geq 1$, then

$$\begin{aligned}\frac{\partial}{\partial\omega}F^j_\omega(x)&=\frac{\partial}{\partial\omega}(F(\omega, F^{j-1}_\omega(x)))\\&=\left(\frac{\partial}{\partial\omega}F\right)(\omega, F^{j-1}_\omega(x))+\left(\frac{\partial}{\partial x}F\right)(\omega, F^{j-1}_\omega(x))\frac{\partial}{\partial\omega}F^{j-1}_\omega(x)\\&=1+(1+k\varepsilon\cos(2k\pi F^{j-1}_\omega(x)))\frac{\partial}{\partial\omega}F^{j-1}_\omega(x)\,.\end{aligned}$$

Thus if $\frac{\partial}{\partial\omega}F^{j-1}_\omega(x)>0$ for $j\geq 2$, then we have $\frac{\partial}{\partial\omega}F^j_\omega(x)>0$ by $0<\varepsilon k<1$. $\frac{\partial}{\partial\omega}F_\omega(x)=\frac{\partial}{\partial\omega}F(\omega, x)=1$ and so $\frac{\partial}{\partial\omega}G(\omega, x)>0$.

Take and fix $x_0\in\mathscr{P}^{-1}(\mathrm{Per}(f_{\omega_0}))$. Then, $G(\omega_0, x_0)=0$ and $\frac{\partial}{\partial\omega}G(\omega_0, x_0)>0$. By the implicit function theorem there is an open interval J_{x_0} containing x_0 and an analytic function $\omega: J_{x_0}\to\mathbf{R}$ such that $\omega(x_0)=\omega_0$ and $G(\omega(x), x)=0$ for $x\in J_{x_0}$. Thus,

$$\left(\frac{\partial}{\partial\omega}G\right)(\omega(x), x)\frac{d}{dx}\omega(x)+\left(\frac{\partial}{\partial x}G\right)(\omega(x), x)=0$$

for $x\in J_{x_0}$, which implies that $\frac{d}{dx}\omega(x)=0$ if and only if $(\frac{\partial}{\partial x}G)(\omega(x), x)=0$.

There is an open subinterval $\hat{J}_{x_0}\subset J_{x_0}$ containing x_0 such that $G(\omega(x), x)=0$ and $\frac{d}{dx}\omega(x)\neq 0$ for $x\in\hat{J}_{x_0}\setminus\{x_0\}$. Indeed, a complex function $\tilde{F}_{\omega_0}(z)=z+\omega_0+\frac{\varepsilon}{2\pi}\sin(2k\pi z)$ $(z\in\mathbf{C})$ is a transcendental entire function and so $\tilde{F}_{\omega_0}$ is not univalent. If there is an interval $J\subset\mathbf{R}$ and $F^q_{\omega_0}(x)=x+l$ for $x\in J$, then it holds that $\tilde{F}^q_{\omega_0}(z)=z+l$ $(z\in\mathbf{C})$ and so $\tilde{F}^q_{\omega_0}$ is univalent, which is a contradiction. Thus $F^q_{\omega_0}(x)\neq x+l$ for some $x\in J$. Since $G(\omega, x)$ and $\frac{d}{dx}\omega$ is analytic on J_{x_0}, there is an open subinterval $\hat{J}_{x_0}\subset J_{x_0}$ containing x_0 such that $G(\omega(x), x)=0$ and $\frac{d}{dx}\omega(x)\neq 0$ for $x\in\hat{J}_{x_0}\setminus\{x_0\}$.

Since $G(\omega, x)$ is analytic with respect to x, the set $R_\omega=\{x\in[0, 1): G(\omega, x)=0\}$ is a finite set for each fixed ω, and so write $R_{\omega_0}=\{x_0, x_1, \cdots, x_m\}$. Then we can find an open interval $\hat{J}_{x_i}$ and an analytic function ω_i defined on $\hat{J}_{x_i}$ such that $G(\omega_i(x), x)=0$ and $\frac{d}{dx}\omega_i(x)\neq 0$ for $x\in\hat{J}_{x_i}\setminus\{x_i\}$ $(i=0, 1, \cdots, m)$. Taking each $\hat{J}_{x_i}$ small enough we may suppose $\hat{J}_{x_i}\cap\hat{J}_{x_j}=\varnothing$ if $i\neq j$. Denote as $\Omega_{x_i}=\{\omega_i(x): x\in\hat{J}_{x_i}\}$ an interval containing ω_0 for $i=0, \cdots, m$. Notice that Ω_{x_i} is not always an open interval. Since R_{ω_0} is a finite set, there is a maximal subset R of R_{ω_0} such that $\bigcap_{x_i\in R}\Omega_{x_i}\supsetneqq\{\omega_0\}$. Thus $I_{\omega_0}=\bigcap_{x_i\in R}\Omega_{x_i}$ is an interval containing ω_0 and satisfies that if $\omega\in I_{\omega_0}$ and $x_i\in R$, then $G(\omega, x)=0$ for some $x\in\hat{J}_{x_i}$. Thus f_ω has a periodic point of the period q if $\omega\in I_{\omega_0}$, from which $\rho(f_\omega)$

is a rational number. Since $\rho(f_\omega)$ is continuous with respect to ω (under the distance between $\rho(f_\omega)$ and $\rho(f_{\omega'})$ defined by $|\rho(f_\omega)-\rho(f_{\omega'})|$ (mod 1)) and $\rho(f_{\omega_0})=r$, we have $\rho(f_\omega)=r$ for every $\omega\in I_{\omega_0}$. Thus f_ω $(\omega\in I_{\omega_0})$ is a member of $AT_k(r)$.

It remains to check that f_ω $(\omega\in I_{\omega_0})$ is Morse-Smale. Since every periodic points of f_ω has the same period q, it suffices to show that $\frac{d}{dx}F_\omega^q(x)\neq 1$ for each $x\in\mathscr{P}^{-1}(\mathrm{Per}(f_\omega))$. For a fixed $\omega\in I_{\omega_0}$ we have that $x\in\mathscr{P}^{-1}(\mathrm{Per}(f_\omega))$ if and only if $G(\omega, x)=0$. Thus f_ω is Morse-Smale if $\frac{\partial}{\partial x}G(\omega, x)\neq 0$ for each $x\in\{z: G(\omega, z)=0\}$. Suppose that $\frac{\partial}{\partial x}G(\omega, x)=0$ for some $x\in\{z: G(\omega, z)=0\}$ and $\omega\in I_{\omega_0}$. If ω is sufficiently close to ω_0, then $G(\omega, x)=0$ implies that $x\in\hat{J}_{x_i}$ for some $0\leq i\leq m$. Since $\frac{\partial}{\partial\omega}G(\omega, x)>0$, we have that $\omega=\omega_i(x)$ and that $x\in\hat{J}_{x_i}\setminus\{x_i\}$ when $\omega\neq\omega_0$. By the assumption $\frac{\partial}{\partial x}G(\omega_i(x), x)=0$ and so $\frac{d}{dx}\omega_i(x)=0$, which contradicts the fact that $G(\omega_i(x), x)=0$ and $\frac{d}{dx}\omega_i(x)\neq 0$ for $x\in\hat{J}_{x_i}\setminus\{x_i\}$. If ω is sufficiently close to ω_0 and $G(\omega, x)=0$, then $\frac{\partial}{\partial x}G(\omega, x)\neq 0$. Thus f_ω is Morse-Smale.

Take $\omega_n\in I_{\omega_0}\setminus\{\omega_0\}$ and suppose that $\omega_n\to\omega_0(n\to\infty)$. Then $f_{\omega_n,\varepsilon,k}\in AT_k(r)\cap MS$ and $f_{\omega_n,\varepsilon,k}\to f_{\omega_0,\varepsilon,k}$ $(n\to\infty)$ with respect to the C^1-topology. Thus $MS_k(r)$ is C^1-dense in $AT_k(r)$. The C^1-openness of $MS_k(r)$ in $AT_k(r)$ (with respect to the relative topology) is easily checked from the fact that MS is open in the set of C^1 diffeomorphisms of the circle.

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