TOKYO J. MATH.
VOL. 24, NO. 1, 2001

Existence and Regularity Results for Harmonic Maps with Potential

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(Communicated by Y. Maeda)

1. Introduction.

Let (M, g) and (N, h) be Riemannian m- and n- manifolds, G a smooth function on N. For a bounded domain $\Omega \subset M$ and a map $u : M \to N$ we define *the energy functional with the potential G on* Ω :

$$E_G(u; \Omega) = \int_\Omega [e(u) - G(u)]d\mu\,, \tag{1.1}$$

where $e(u)$ and $d\mu$ are the standard energy density and the volume element on M. Using local coordinate systems $(x^1, \cdots, x^m)$ and $(u^1, \cdots, u^n)$ on M and N respectively, we can write

$$E_G(u; \Omega) = \int_\Omega \left[\frac{1}{2}g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j - G(u(x))\right]\sqrt{g}dx\,, \tag{1.2}$$

where $(g^{\alpha\beta}(x)) = (g_{\alpha\beta}(x))^{-1}$, $g = \det(g_{\alpha\beta}(x))$ and $D_\alpha = \partial/\partial x^\alpha$. The Euler-Lagrange equation of E_G is given as

$$\tau(u) + \nabla G = 0\,, \tag{1.3}$$

where $\tau(u)$ denotes the tension field of u. In local

$$(\tau(u))^i = \frac{1}{\sqrt{g}}D_\alpha\{\sqrt{g}g^{\alpha\beta}D_\beta u^i\} + g^{\alpha\beta}\Gamma^i_{jk}D_\alpha u^j D_\beta u^k\,.$$

A solution $u : \Omega \to N$ of (1.3) is called to be *a harmonic map with potential G*.

The equations of type (1.3) appear also in some physical contexts. Let $\Omega \subset \mathbf{R}^m$, $N = S^2 = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = 1\}$ and $G(u) = (u, H) = u^1H^1 + \cdots + u^nH^n$ for some constant vector $H \in \mathbf{R}^3$, then the equation (1.3) becomes

$$\Delta u + u|Du|^2 - (H, u)u + H = 0 \tag{1.4}$$

Received December 1, 1999
Revised October 12, 2000

which is called as the static Landau-Lifshitz equation (see [1], [2], [3], [12] and [13]). Here and in the sequel $(\,,)$ and $|\ |$ denote the standard Euclidean inner products and norms respectively. In [2], [12] and [13], Dirichlet problems for the equation (1.4)

$$\begin{cases} \Delta u + u|Du|^2 - (H, u)u + H = 0 \\ u = f \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.5}$$

are considered.

When $\Omega = B^3$, Hong [12] showed the existence of a smooth solution to the Dirichlet problem (1.5), assuming that an extension u_0 of the boundary value f to B^3 satisfies

$$\int_{B^3} \frac{1}{2}\left(|Du_0|^2 + |H| \cdot \left|u_0 - \frac{H}{|H|}\right|^2\right) dx < \varepsilon$$

for a sufficiently small $\varepsilon > 0$.

More recently, Chen [2] showed the existence of a smooth solution to (1.5) for $H = \lambda q$ and $f \in C^\infty(\partial\Omega, S^2_q)$, where λ is a positive constant, q a point in S^2 and S^2_q the open hemisphere with the north pole q. He also showed the uniqueness of the small solutions i.e. if u_1 and u_2 are solutions of the Dirichlet problem with $u_1(\Omega), u_2(\Omega) \subset S^2_q$ then $u_1 = u_2$.

On the other hand, when $\Omega = B^2$, Hong-Lemaire [13] showed that if f is neither constant $H/|H|$ nor constant $-H/|H|$ then there are at least two different smooth solution to the Dirichlet problem (1.5). Moreover, they showed also that under a certain condition there are at least three different solutions to (1.5).

In this paper, more general cases are treated. We will prove existence of a minimizer of E_G in a suitable class of Sobolev maps with the Dirichlet boundary condition

$$u = f \quad \text{on} \quad \partial\Omega\,, \tag{1.6}$$

by the direct method of calculus of variations, assuming some conditions on G, f and N. We will prove also boundedness and regularity of the minimizer and get existence results for harmonic maps with potential.

Now, let us prepare some notations and terminology. For a Riemannian manifold N, $\kappa_N(p;\pi)$ denotes the sectional curvature at $p \in N$ with respect to a plane section $\pi \subset T_pN$. Let $q_0 \in N$ be a fixed point and $I(q_0)$ the injectivity radius of N centered at q_0. For $p \in B_{I(q_0)}(q_0)$, let $\sigma(q_0, p)(t)$ be the geodesic curve such that $\sigma(q_0, p)(0) = q_0$ and $\sigma(q_0, p)(1) = p$. When π contains $\sigma'(q_0, p)(1)$, let us call $\kappa_N(p;\pi)$ *a radial curvature at p with respect to q_0*. We denote as $K_{\mathrm{rad}}(p; q_0)$ the maximum of radial curvatures of N at p with respect to q_0, namely

$$K_{\mathrm{rad}}(p; q_0) = \max\{\kappa_N(p;\pi) | \pi \ni \sigma'(q_0, p)(1)\}\,.$$

Throughout this paper we consider the following condition $C(q_0, R_0)$ on the radial curvatures of N.

$C(q_0, R_0)$: *Let q_0 be a fixed point of N and R_0 a positive number which is smaller than the injectivity radius $I(q_0)$ of N centered at q_0. There exists a nonnegative function*

$\rho : [0, R_0) \to [0, \infty)$ *which satisfies the following conditions:*

$$\lim_{t\to 0} \frac{\rho(t)}{t} = 1 , \tag{1.7}$$

$$\rho(t) > 0 , \quad \rho'(t) > 0 \quad \text{for all} \quad t \in (0, R_0) , \tag{1.8}$$

$$K_{\text{rad}}(p; q_0) \le -\frac{\rho''}{\rho}(\text{dist}(q_0, p)) \quad \text{for all} \quad p \in B_{R_0}(q_0) . \tag{1.9}$$

Under this condition, $\exp_{q_0}$ gives a normal coordinate system centered at q_0 on B_{R_0}. Moreover, with respect to this normal coordinate system we get an estimate on the nonlinear term of $\tau(u)$ which corresponds to the one-sided condition of Giaquinta-Giusti [6]. (See Lemma 2.1.)

EXAMPLES 1. An upper hemisphere S^n_+ satisfies $C(q_0, R_0)$ with q_0 ="the north pole", $R_0 = \pi/2$ and $\rho(t) = \sin t$.
2. A simply connected complete manifold N with nonpositive sectional curvatures satisfies $C(q_0, R_0)$ with arbitrarily fixed $q_0 \in N$, $R_0 = \infty$ and $\rho(t) = t$.

In section 2, we will show the existence of minimizers of E_G and their L^∞-bounds (Theorem 2.2). These lead us to the existence of weak solutions of (1.3).

In section 3, the regularity of the weak solutions whose existence is guaranteed in section 2 will be shown (Theorems 3.2 and 3.3). Thus we will achieve at the existence theorem of harmonic maps with potential (Theorem 3.4).

2. Existence and global boundedness of a minimizer.

First of all we prepare an auxiliary geometric lemma.

LEMMA 2.1 (Revised version of [15, Lemma 1.1]). *Let N be a Riemannian n-manifold which satisfies $C(q_0, R_0)$, $(u^1, \cdots, u^n)$ a normal coordinate system centered at q_0 and $h_{ij}(u)$ the metric tensor with respect to the normal coordinate system. Then we have the following estimates*:

$$h_{ij}(u)(X^i X^j + u^k \Gamma^i_{kl}(u) X^j X^l) \ge |\zeta|^2 + t\frac{\rho'(t)}{\rho(t)} h_{ij}(y)\xi^i \xi^j , \tag{2.1}$$

$$h_{ij}(u) X^i X^j \ge |\zeta|^2 + \frac{\rho^2(t)}{t^2} |\xi|^2 , \tag{2.2}$$

for all u, $X \in \mathbf{R}^n$, where $t = |u|$, $\zeta = t^{-2}(X, u)u$ and $\xi = X - \zeta$.

PROOF. We can proceed as in the proof of [15, Lemma 1.1] or [16, Lemma 2.1], noticing that the assumptions only on the radial curvatures like $C(q_0, R_0)$ are sufficient to apply Rauch's comparison theorem. □

REMARK. The estimate (2.1) corresponds to the one-sided condition of Giaquinta-Giusti [6] (see the proof of Theorem 3.3).

In the following we assume that the target manifold N satisfies the condition $C(q_0, R_0)$ and always use a normal coordinate $(u^1, \cdots, u^n)$ centered at q_0 on $B_{R_0}(q_0)$. For some positive constant $R < R_0$ and a boundary data $f \in H^{1,2} \cap L^\infty(\Omega, \mathbf{R}^n)$ with $\|f\|_{L^\infty(\Omega)} < R$ we seek a minimizer of E_G in the class

$$X_{f,R} := \{v \in H^{1,2}(\Omega, \mathbf{R}^n);\ v - f \in H_0^{1,2}(\Omega, \mathbf{R}^n),\ \|v\|_{L^\infty(\Omega)} \le R\}. \tag{2.3}$$

and show that the minimizer u satisfies the equation (1.3) weakly. (In order to see that minimizer u in the class $X_{f,R}$, we must show the strict inequality $\|u\|_{L^\infty} < R$.)

To find a minimizer, we consider the following condition on G:

$$|G(u)| \le b_0 + b_1|u|^\gamma \quad \text{for some} \quad \gamma \in [0, 2^*). \tag{2.4}$$

Moreover, in order to show the boundedness of $|u|$ we put the following conditions on $\partial G/\partial s (s = |u|)$:

$$\left|\frac{\partial G}{\partial s}(u)\right| \le b_2 + b_3|u|^{\gamma-1} \quad \text{for some} \quad \gamma \in \left[0, \frac{4}{m-2}\right). \tag{2.5}$$

Here, b_0, b_1, b_2 and b_3 are positive constants.

Now, we can state our results on the existence of minimizers of E_G in the class $X_{f,R}$ (Theorem 2.2) and L^∞-estimate of them (Theorem 2.3).

THEOREM 2.2. *Let (M, g) be a smooth Riemannian m-manifold, and Ω a bounded domain of M with the smooth boundary $\partial\Omega$. Let (N, h) be a smooth Riemannian n-manifold which satisfies the condition $C(q_0, R_0)$ for some $q_0 \in N$ and $R_0 \in (0, +\infty]$. Assume that G satisfies* (2.4). *Then for any $R < R_0$ and $f \in H^{1,2} \cap L^\infty(\Omega, \mathbf{R}^n)$ with $\|f\|_{L^\infty(\Omega)} < R$ there exists a minimizer of E_G in the class $X_{f,R}$.*

PROOF. Let $\{v_k\}$ be a minimizing sequence of E_G in the class $X_{f,R}$. Since the condition (2.4) implies that

$$E_G(v; \Omega) \ge c_0(g, h) \int_\Omega |Dv|^2 dx - c_1(b_0, b_1, \gamma, R, \Omega) \tag{2.6}$$

for every $v \in X_{f,R}$, we see that the sequence $\{v_k\}$ is equibounded in $H^{1,2}$. Therefore, taking subsequence if necessary, we see that

$$v_k \rightharpoonup u \quad \text{weakly in} \quad H^{1,2}, \tag{2.7}$$

$$v_k \to u \quad \text{strongly in} \quad L^\gamma \quad \text{for} \quad \gamma \in [1, 2^*) \tag{2.8}$$

for some $u \in \{v \in H^{1,2} \cap L^\gamma(\Omega, \mathbf{R}^n);\ v - f \in H^{1,2}\}$. Here, we used the Kondrachov compactness theorem also. From (2.7) and (2.8), we see easily that

$$\liminf_{k\to\infty} E_G(v_k; \Omega) \ge E_G(u; \Omega).$$

On the other hand, L^γ-strong convergence implies the almost everywhere convergence, therefore the limit map u belongs to the class $X_{f,R}$. Thus, u minimizes E_G in the class $X_{f,R}$. □

THEOREM 2.3 *Let* (M, g), (N, h), Ω *and* f *be as in Theorem 2.2. Suppose that* $2 \le m \le 4$ *and that* G *satisfies* (2.4) *and* (2.5). *If we have*

$$b_0, b_1, b_2, b_3, E_G(f), \|f\|_{L^\infty(\Omega)} < r_0 \tag{2.9}$$

for a sufficiently small constant $r_0 > 0$, *then a minimizer* u *in the class* $X_{f,R}$ *satisfies* $\|u\|_{L^\infty(\Omega)} < R$ *for some* $R < R_0$ *and solves* (1.3) *weakly. Here,* r_0 *depends only on* g, h, m, Ω *and* R_0.

When we can take $R_0 = +\infty$, *the smallness condition* (2.9) *is not necessary.*

PROOF. Let u be a minimizer of E_G in the class $X_{f,R}$, then u satisfies

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} E_G(u + t\varphi; \Omega) \\ &= \int_\Omega \left[g^{\alpha\beta} h_{ij} D_\alpha u^i \{D_\beta \varphi^j + \Gamma^j_{kl} D_\beta u^l \varphi^k\} - \varphi^i \frac{\partial G}{\partial u^i}\right] \sqrt{g}dx \end{aligned} \tag{2.10}$$

for all $\varphi \in H_0^{1,2}(\Omega, \mathbf{R}^n)$ with

$$u + t\varphi \in X_{f,R} \quad \text{for all } t \quad \text{with } |t| < \varepsilon \quad \text{for some } \varepsilon > 0\,. \tag{2.11}$$

Because of the restriction (2.11), we can not say that (2.10) holds for every test function $\varphi \in C_0^\infty(\Omega, \mathbf{R}^n)$. Therefore, we can not call u a weak solution yet.

Now, let us show the estimate

$$\|u\|_{L^\infty(\Omega)} < R\,, \tag{2.12}$$

which enables us to call u as a weak solution.

For any nonnegative function $\eta \in C_0^1(\Omega)$, if we take $\varepsilon > 0$ sufficiently small, $u+(-\varepsilon)\eta u$ belongs to the class $X_{f,R}$. Therefore we can take $\varphi = -\varepsilon\eta u$ in (2.10) and get

$$\begin{aligned} 0 = \int_\Omega \Big[& g^{\alpha\beta}(x) h_{ij}(u) \{D_\alpha u^i D_\beta u^j + u^k \Gamma^j_{lk} D_\alpha u^i D_\beta u^l\}\eta \\ & + g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i u^j D_\beta \eta - u^i \frac{\partial G}{\partial u^i}\eta \Big] d\mu\,. \end{aligned} \tag{2.13}$$

Since we are using a normal coordinate on N, (2.13) implies that

$$\begin{aligned} & \int_\Omega g^{\alpha\beta}(x) h_{ij}(u) \{D_\alpha u^i D_\beta u^j + u^k \Gamma^j_{lk} D_\alpha u^i D_\beta u^l\}\eta d\mu \\ & = - \int_\Omega \left\{ \frac{1}{2} g^{\alpha\beta}(x) D_\alpha |u|^2 D_\beta \eta - u^i \frac{\partial G}{\partial u^i}\eta \right\} d\mu\,. \end{aligned} \tag{2.14}$$

On the other hand, from (2.1) of Lemma 2.1, we have

$$g^{\alpha\beta}(x)h_{ij}(u)\{D_\alpha u^i D_\beta u^j + u^k \Gamma^j_{lk} D_\alpha u^i D_\beta u^l\} \geq \delta |Du|^2 \tag{2.15}$$

for

$$\delta = \min\{1, \inf_{0<t<R} t\rho'(t)/\rho(t)\} > 0\,.$$

Thus, under the condition $C(q_0, R_0)$, we get for all $\eta \in C_0^1(\Omega)$ with $\eta \geq 0$ that

$$0 \geq \int_\Omega \left\{ \frac{1}{2} g^{\alpha\beta}(x) D_\alpha |u|^2 D_\beta \eta - |u| \frac{\partial G}{\partial s} \eta \right\} d\mu\,, \tag{2.16}$$

where $s = |u| = \mathrm{dist}(q_0, u(x))$. Since u belongs to the class $X_{f,R}$, u is essentially bounded and therefore $D|u|^2$ is in the class L^2, namely $|u|^2 \in H^{1,2}$.

Let $w = |u|^2 - |f|^2$, then from (2.16), w satisfies

$$\int_\Omega \left\{ g^{\alpha\beta}(x) D_\alpha w D_\beta \eta - g^{\alpha\beta}(x) D_\alpha |f|^2 D_\beta \eta + |u| \frac{\partial G}{\partial s} \eta \right\} d\mu \leq 0 \tag{2.17}$$

for any $\eta \in C_0^1(\Omega)$ with $\eta \geq 0$. Assume that $|u|\partial G/\partial s \in L^q(\Omega)$ for some $q > m/2$, then using [9, Theorem 8.15] we get

$$\sup_\Omega |w| \leq c_2(m, g, \Omega) \left(\|u\|_{L^4} + \|f\|_{L^{2q}} + \left\| |u| \frac{\partial G}{\partial s} \right\|_{L^q} \right)\,. \tag{2.18}$$

Now, let us estimate the right hand side of (2.18). Since we are assuming (2.4), the minimality of u implies that

$$\begin{aligned}
\int_\Omega e(u) d\mu &\leq \int_\Omega G(u) d\mu + E_G(f) \\
&\leq E_G(f) + b_0 \mathrm{vol.}(\Omega) + b_1 \int_\Omega |u|^\gamma d\mu \\
&\leq E_G(f) + b_0 \mathrm{vol.}(\Omega) + b_1 \int_\Omega \{\varepsilon |u|^{2^*} + \varepsilon^{-\frac{\gamma}{2^*-\gamma}}\} d\mu \\
&\leq c_3(E_G(f), \Omega, g, \varepsilon, \gamma, b_0, b_1) + \varepsilon c_4(\Omega, g, h, b_1) \int_\Omega e(u) d\mu\,.
\end{aligned}$$

Here, we used Young's inequality and the Sobolev inequality. By choosing $\varepsilon > 0$ sufficiently small, we get the following a-priori estimate:

$$\int_\Omega |Du|^2 dx \leq c_5(g, h, \gamma, b_0, b_1, \Omega, E_G(f))\,. \tag{2.19}$$

Using the Sobolev inequality and the assumption that $2 \leq m \leq 4$, from (2.19) we get

$$\|u\|_{L^4} \leq c_6(\Omega, m) \|u\|_{L^{2^*}} \leq c_6 K_0(g, h, \gamma, b_0, b_1, \Omega, E_G(f))\,. \tag{2.20}$$

for some positive constants c_6 and K_0. Here, it is nothing to see that K_0 satisfies

$$\lim_{b_0, b_1, E_G(f) \to 0} K_0(g, h, \gamma, b_0, b_1, \Omega, E_G(f)) = 0\,. \tag{2.21}$$

On the other hand, using the condition (2.5), we see that

$$\left\| |u| \frac{\partial G}{\partial s} \right\|_{L^q} \leq c_7(b_2, b_3, q, \Omega)\|u\|_{L^{2^*}} \quad \text{for} \quad q = \min\{2^*, 2^*/\gamma\} > m/2 .$$

Thus, if $2 \leq m \leq 4$ and (2.5) holds, we obtain from (2.18)

$$\sup_{\Omega} |u|^2 \leq c_4\{c_6(1 + c_7)K_0 + \|f\|_{L^{2q}}\} + \sup_{\Omega} |f|^2 . \tag{2.22}$$

Now, from (2.21) and (2.22), we can see that if b_0, b_1, b_2, b_3, $E_G(f)$ and $\|f\|_{L^\infty(\Omega)}$ are sufficiently small we have (2.12).

When we can take $R_0 = +\infty$, for any given b_0, b_1, b_2, b_3 and f we can choose R sufficiently large so that R is greater than the right hand side of (2.22). □

3. Regularity of minimizers.

In this section we show the $C^{0,\alpha}$-regularity of a minimizer u under the condition (2.5).

When the boundedness of a minimizer u of E_G is given, we can easily see that the results of [6] and [14] are valid for our case. More precisely we have the following theorems:

THEOREM 3.1. *Let M, N, Ω and f be as in Theorem 2.2 and G a smooth function defined on N. Assume that u minimize E_G in the class $X_{f,R}$ and that $\|u\|_{L^\infty(\Omega)} < R$. Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0, \mathbf{R}^n)$ for every $\alpha \in (0, 1)$. Moreover,*

$$\Omega \backslash \Omega_0 = \left\{ x_0 \in \Omega ; \liminf_{r \to 0} r^{2-n} \int_{B_r(x_0)} |Du|^2 dx > \varepsilon_0 \right\} \tag{3.1}$$

where ε_0 is a positive constant independent of u. Finally

$$\mathcal{H}^{n-q}(\Omega \backslash \Omega_0) = 0$$

for some $q > 2$, $\mathcal{H}^{n-q}$ denoting $(n-q)$-dimensional Hausdorff measure.

PROOF. It is enough to proceed as the proof of [6, Theorem 5.1], adding $\int G(u)dx$ to their quadratic functional. We will get

$$\int_{B\rho} (1 + |Du|^2) dx$$
$$\leq c_8 \left[\left(\frac{\rho}{r}\right)^m + \omega \left(r^2 + c_9 r^{2-m} \int_{B_r} |Du|^2 dx \right)^{1-2/q} \right] \int_{B_{2r}} |Du|^2 dx + c_{10} r^m ,$$

instead of [6, (5.11)]. Now, the assertion follows from the above estimate using "a useful lemma" on [5, p. 44]. □

THEOREM 3.2. *Let M, N, Ω, G, f and u be as in Theorem 3.1. Assume that the boundary data f is in the class $H^{1,s}(\Omega, N)$ for some $s > m$. Then u is Hölder continuous in a neighborhood of $\partial\Omega$.*

PROOF. Let x_0 be an arbitrary point on $\partial\Omega$ and choose a local coordinate system so that $x_0 = 0$. As in [14], let us consider the blown-up functions

$$u_{(\nu)}(x) = u(x/\nu)\,, \quad g^{\alpha\beta}_{(\nu)}(x) = g^{\alpha\beta}(x/\nu)\,, \quad (\nu = 1, 2, 3, \dots)\,.$$

Then $u_{(\nu)}$ minimizes the functional

$$\int \left[\frac{1}{2} g^{\alpha\beta}_{(\nu)}(x) h_{ij}(v) D_\alpha v^i D_\beta v^j - \frac{1}{\nu^2} G(v(x))\right] \sqrt{g_{(\nu)}}dx\,,$$

and converges to a minimizer v of the functional

$$\int \left[\frac{1}{2} g^{\alpha\beta}(0) h_{ij}(v) D_\alpha v^i D_\beta v^j\right] \sqrt{g}dx\,.$$

Namely, the potential term disappears in the blowing-up process. Thus, we can proceed as in [14] and get the assertion. □

Now, we prove the following regularity theorem for minimizers of E_G.

THEOREM 3.3. *Suppose that all assumptions in Theorem* 3.2 *are satisfied. Let u be a minimizer of E_G in the class $X_{f,R}$ which satisfies $\|u\|_{L^\infty(\Omega)} < R$. Then u is Hölder continuous on $\bar{\Omega}$.*

PROOF. By virtue of Theorems 3.1 and 3.2, it is enough to show that for every $\varepsilon_0 > 0$ and $x \in \Omega$ there exists a positive constant $\rho > 0$ such that

$$\rho^{2-n} \int_{B_\rho(x)} |Du|^2 dx \le \varepsilon_0\,. \tag{3.2}$$

To show (3.2) we can proceed similarly as in [6] by remarking that the estimate (2.15) plays the role of the one-sided condition of [6].

Let $x \in \Omega$ be an arbitrarily fixed point and r a positive constant such that $B_{2r}(x) \subset\subset \Omega$. Choosing $\eta \ge 0$ in (2.14) so that spt $\eta \subset B_{2r}(x)$ and using (2.15) we get

$$\delta \int_{B_{2r}(x)} |Du|^2 \eta d\mu \le -\int_{B_{2r}(x)} \left\{\frac{1}{2} g^{\alpha\beta}(x) D_\alpha |u|^2 D_\beta \eta - |u|\frac{\partial G}{\partial s}\right\} d\mu\,. \tag{3.3}$$

Since we are assuming that $\|u\|_{L^\infty} < R$, we have

$$\left| |u|\frac{\partial G}{\partial s} \right| \le K \tag{3.4}$$

for some positive constant K which depends only g, h, G and f. Let $M(r) = \sup_{B_r(x)} |u|$ and $z = M^2(2r) - |u|^2$. Then from (3.3) and (3.4) we get

$$0 \le \int_{B_r(x)} (g^{\alpha\beta}(x) D_\alpha z D_\beta \eta + K) d\mu\,. \tag{3.5}$$

Thus, z is a nonnegative supersolution of a uniformly elliptic equation and therefore, using the weak Harnack inequality (see [9, Theorem 8.18]), we obtain

$$r^{-m} \int_{B_{2r}(x)} |z| dx \le c_{11}(g, m, K) \left(\inf_{B_r(x)} z + r^2\right)\,. \tag{3.6}$$

Let $w \in C^2(B_{2r}(x)) \cup C(B_{2\bar{r}}(x))$ be a solution of the Dirichlet problem

$$\begin{cases} D_\beta\{\sqrt{g}g^{\alpha\beta}(x)D_\alpha w\} = -\frac{1}{r^2} \text{ in } B_{2r}(x)\,, \\ \qquad\qquad w = 0 \text{ on } \partial B_{2r}(x)\,. \end{cases}$$

Then, w is bounded from above by a positive constant α_1 in $B_{2r}(x)$ (see for example [9, Theorem 3.7]). On the other hand, since the right hand side of the above equation is negative, w is a positive supersolution of the equation $D_\beta\{\sqrt{g}g^{\alpha\beta}(x)D_\alpha w\} = 0$, and therefore by the weak Harnack inequality we have that $w \geq \alpha_2$ in $B_r(x)$ for some positive constant α_2. Here, α_1 and α_2 does not depend on r. Indeed, if w_1 is a solution of the above Dirichlet problem for $r = 1$, then $w(x) = w_1(tx)$ solves the Dirichlet problem for $r = t$.

Now, w is in the class $H_0^{1,2}(B_{2r}(x))$ clearly and satisfies the following weak from of the equation

$$\int_{B_{2r}(x)} g^{\alpha\beta}(x)D_\beta w D_\alpha\varphi\sqrt{g}dx = r^{-2}\int_{B_{2r}(x)}\varphi dx \quad \text{for all} \quad \varphi \in H_0^{1,2}(B_{2r}(x))\,. \tag{3.7}$$

Let $\varphi = wz$ in (3.7), then we have

$$\frac{1}{2}\int_{B_{2r}(x)} g^{\alpha\beta}D_\beta w^2 D_\alpha z d\mu + \int_{B_{2r}(x)} g^{\alpha\beta}D_\beta w D_\alpha w z d\mu = r^{-2}\int_{B_{2r}(x)} wz dx\,,$$

and therefore

$$\frac{1}{2}\int_{B_{2r}(x)} g^{\alpha\beta}D_\beta w^2 D_\alpha z dx \leq \alpha_1 r^{-2}\int_{B_{2r}(x)} z dx\,. \tag{3.8}$$

Since w^2 is in the class $H_0^{1,2}$ also, we can take $\eta = w^2$ in (3.3). Taking $\eta = w^2$ in (3.3) and using (3.6) and (3.8), we get

$$\begin{aligned} \delta\alpha_2^2\int_{B_r(x)}|Du|^2 d\mu &\leq \frac{1}{2}\int_{B_{2r}(x)} g^{\alpha\beta}D_\alpha z D_\beta\eta d\mu + \int_{B_{2r}(x)} Kw^2 d\mu \\ &\leq \alpha_1 r^{-2}\int_{B_{2r}(x)} z dx + K\alpha_1^2(2r)^m \\ &\leq \alpha_1 c_{12} r^{m-2}\inf_{B_r(x)} z + c_{13}r^m\,. \end{aligned} \tag{3.9}$$

Thus we obtain

$$r^{2-m}\int_{B_r(x)}|Du|^2 dx \leq c_{14}\left\{\inf_{B_r(x)} z + r^2\right\} \leq c_{14}\{M^2(2r) - M^2(r) + r^2\}\,. \tag{3.10}$$

On the other hand u is bounded and therefore

$$\sum_{k=0}^{+\infty}[M^2(2^{1-k}r) - M^2(2^k r)] \leq M^2(2r) \leq \sup_\Omega |u|^2\,.$$

Thus (3.10) implies (3.2) with $\rho = 2^{-k}r$ for some k. □

Now, combining Theorems 2.2, 3.3 and the standard Schauder estimates, we get the following existence theorem.

THEOREM 3.4. *Let M, N and Ω be as in Theorem* 2.2 *and* r_0 *as in Theorem* 2.3. *Suppose that a smooth function* $G(u)$ *satisfies* (2.4) *and* (2.5) *with* $b_i < r_0$ $(i = 0, 1, 2, 3)$ *and that the boundary data* f *is in the class* $H^{1,s} \cap L^\infty(\Omega, \mathbf{R}^n)$ *for some* $s > m$ *and satisfies* $E_G(f)$, $\|f\|_{L^\infty(\Omega)} < r_0$. *Then there exists a minimizer* u *of* E_G *in the class* $X_{f,R}$ *for some* $R < R_0$. *Moreover, the minimizer* u *is in the class* $C^{2,\alpha}(\Omega, B_R) \cap C^{0,\alpha}(\bar{\Omega}, B_R)$ *and a harmonic map with potential* G.

If we can take $R_0 = \infty$, *the smallness conditions on* b_i $(i = 0, 1, 2, 3)$, $E_G(f)$ *and on* $\|f\|_{L^\infty(\Omega)}$ *are not necessary.*

References

[1] Q. CHEN, Liouville theorem for harmonic maps with potential, Manuscripta Math. **95** (1998), 507–517.

[2] Q. CHEN, Maximum principle, uniqueness and existence for harmonic maps with potential and Landau-Lifshitz equations, Calc. Var. **8** (1999), 91–107.

[3] A. FARDOUN and A. RATTO, Harmonic maps with potential, Calc. Var. **5** (1997), 183–197.

[4] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Non-linear Elliptic Systems*, Ann. of Math. Stud. **105** (1983), Princeton Univ. Press.

[5] M. GIAQUINTA, *Introduction to regularity theory for nonlinear elliptic equations*, Lectures in Mathematics ETH Zürich, Birkhäuser (1993).

[6] M. GIAQUINTA and E. GIUSTI, On the regularity of the minima of variational integrals, Acta Math. **148** (1982), 31–46.

[7] M. GIQUINTA and E. GIUSTI, Differentiability of minima of non-differentiable functionals, Invent. Math. **72** (1983), 285–298.

[8] D. GROMOLL, W. KLINGENBERG and W. MEYER, *Rimannsche Geometrie im Großen*, Lecture Notes in Math. **55** (1968), Springer.

[9] D. GILBERG and N. S. TRUDINGER, *Elliptic partial differential equations of second order* (2nd ed. revised 3rd printing), Springer (1998).

[10] S. HILDEBRANDT, J. JOST and K.-O. WIDMAN, Harmonic mappings and minimal submanifolds, Invent. Math. **62** (1980), 269–298.

[11] S. HILDEBRANDT and H. KAUL, Two-Dimensional variational problems with obstructions and Plateau's problem for H-surfaces in a Riemannian manifold, Comm. Pure Appl. Math. **25** (1972), 187–223.

[12] M.-C. HONG, The Landau-Lifshitz equation with the external field—a new extension for harmonic maps with values in S^2, Math. Z. **220** (1995), 171–188.

[13] M.-C. HONG and L. LEMAIRE, Multiple solutions of the static Landau-Lifshitz equation from B^2 into S^2, Math. Z. **220** (1995), 295–306.

[14] J. JOST and M. MEIER, Boundary reguralty for minima of certain quadratic functionals, Math. Ann. **262** (1983), 549–561.

[15] A. TACHIKAWA, On interior regularity and Liouville's theorem for harmonic mappings, Manuscripta Math. **42** (1983), 11–40.

[16] A. TACHIKAWA, Harmonic mappings from $\mathbf{R}^m$ into an Hadamard manifold, J. Math. Soc. Japan **42** (1990), 147–153.

[17] A. TACHIKAWA, Harmonic maps from a Riemannian manifold with a pole into an Hadamard manifold with negative sectional curvatures, Manuscripta Math. **74** (1992), 69–81.

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