TOKYO J. MATH.
VOL. 24, NO. 2, 2001

Criterion of Proper Actions for 2-step Nilpotent Lie Groups

Salma NASRIN

University of Tokyo

(Communicated by S. Kaneyuki and R. Miyaoka)

1. Introduction.

The purpose of this note is to give a partial solution to the conjecture of Lipsman on proper actions of simply connected nilpotent Lie groups:

CONJECTURE 1.1 ([13, Conjecture 4.1(b)]). *Let G be a simply connected nilpotent Lie group, and H, L be connected subgroups of G. Then L acts properly on G/H if and only if the triplet (G, H, L) has the (CI) property.*

Here, we say the triplet (G, H, L) has the (CI) property if $L \cap gHg^{-1} = \{e\}$ for any $g \in G$ (see Definition 2.8).

If each of the triple (G, H, L) is reductive then Kobayashi [4] has proved that L acts properly on G/H if and only if the triplet (G, H, L) has the (CI) property. Conjecture 1.1 may be regarded as an analogy to nilpotent cases.

Conjecture 1.1 is known to be true in the case of lower dimensional nilpotent Lie groups ([5], [13]). There are counter examples of Conjecture 1.1 if we drop some of assumptions:

1. If H is not connected, then the implication "(CI) $\Rightarrow$ proper" may fail even though G is abelian. For example, $(G, H, L) = (\mathbf{R}^2, \mathbf{Z}^2, \mathbf{R}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix})$ is the case.

2. If G is not nilpotent then Conjecture 1.1 may fail. For instance, if $G = KAN$ is an Iwasawa decomposition of a real reductive group G and if we put $L := A$ and $H := N$, then (L, H) has the property (CI) in G, while the action of L on G/H is not proper (as we can see in [5], Example 5 when $G = SL(2, \mathbf{R})$).

The main result of this paper is briefly:

THEOREM 1.2 (see Theorem 2.11). *Conjecture 1.1 is true if G is a 2-step nilpotent Lie group.*

The point of Conjecture 1.1 is to give a criterion of proper actions by means of a simple condition called "CI property". The notion of proper actions is important in the study of discontinuous groups, motivated as follows.

Received September 8, 2000

Suppose $M = G/H$ is a homogeneous space, where G is a connected Lie group and H is a closed subgroup. According to [4] (see also [5], [8]), fundamental problems on discontinuous groups for homogeneous spaces M are:

1) to find a discrete subgroup Γ of G acting properly discontinuously on M;
2) to find a uniform latice Γ in the sense that $\Gamma\backslash M$ is a compact manifold.

Classical results concern mainly with the case where H is compact. Then, the above problems have been studied as a theory of discrete subgroups. However, if H is non-compact, discrete subgroups of G do not always act on M properly discontinuously. Kobayashi has initiated a general theory of discontinuous groups acting on G/H where G is a real reductive Lie group. We refer the readers to see the surveys [5], [8], [12] for geometric ideas of various methods in the last decade and also refer to [9], [14], [15], [19] for some recent developments.

2. Preliminary Results.

In this section we shall give a quick review on basic properties of the action of a Lie group L on G/H or, of the action of a discrete subgroup Γ on G/H. Basic references for this section are [5], [7] and [8].

In general the action of a discrete group is difficult to study. Instead, the flow of some connected Lie subgroup sometimes helps us to understand the action of a discrete group. Here is a continuous analogue of proper discontinuity:

DEFINITION 2.1 (Palais [16]). The action of a closed (connected) subgroup $L \subset G$ on M is said to be *proper* if for each compact subset $S \subset M$ the set

$$L_S = \{\gamma \in L : \gamma S \cap S \neq \emptyset\}$$

is compact.

We note that the action of L on M is *properly discontinuous* if and only if the action of L on M is proper and L is discrete, because a discrete and compact set is finite.

The following elementary observation is a bridge between the action of a discrete group and that of a connected group.

OBSERVATION 2.2 ([4, Lemma 2.3]). *Suppose a Lie group L acts on a locally compact space X. Let Γ be a cocompact discrete subgroup of L. Then*

1. *The L-action on X is proper if and only if the Γ-action is properly discontinuous.*
2. *$L\backslash X$ is compact if and only if $\Gamma\backslash X$ is compact.*

In view of Observation 2.2, Kobayashi [4] posed the following problems in a continuous setting, which is an analogy of fundamental problems on discotinuous groups.

PROBLEM 2.3. Let G be a Lie group and H and L be closed subgroups.

1. Find a criterion on the triplet (L, G, H) such that the action of L on G/H is proper.
2. Find a criterion on the triplet (L, G, H) such that the double coset $L\backslash G/H$ is compact in the quotient topology.

There is a complete answer to Problem 2.3 in terms of Lie algebras in [8] in the following cases:

(i) Problem 2.3(1) when G is reductive (see [7], [8, Section 2])

(ii) Problem 2.3(2) when the groups G, H, L are real reductive (see [4], [8, Section 5])

Inspired by [5] Lipsman has discussed Problem 2.3 in the case where G is a simply connected nilpotent Lie group. This case is considered as an opposite extremal to the semisimple case. We will give an answer to Problem 2.3(1) in the case where G is a simply connected 2-step nilpotent Lie group. Now we introduce some notations that are useful for a further study of Problem 2.3(1).

Relations $\sim$ and $\pitchfork$. Suppose that H and L are subsets of a locally compact topological group G.

DEFINITION 2.4 ([7, Definition 2.1.1]). We denote by $H \sim L$ in G if there exists a compact set S of G such that $L \subset SHS^{-1}$ and $H \subset SLS^{-1}$. Here, $SHS^{-1} := \{ahb^{-1} \in G : a, b \in S, h \in H\}$. Then the relation $H \sim L$ in G defines an equivalence relation.

We say the pair (H, L) is *proper* in G, denoted by $H \pitchfork L$ in G if the set $SHS^{-1} \cap L$ is relatively compact for any compact set S in G.

Definition 2.4 is motivated by the following:

OBSERVATION 2.5 ([7, Observation 2.1.3]). *Let H and L be closed subgroups of G, and Γ a discrete subgroup of G.*

1. *The action of L is proper on the homogeneous manifold G/H if and only if $H \pitchfork L$ in G.*

2. *The action of Γ on the homogeneous manifold G/H is properly discontinuous if and only if $H \pitchfork \Gamma$ in G.*

Here are some elementary properties of the relations $\sim$, $\pitchfork$:

LEMMA 2.6 ([7, Lemma 2.2]). *Suppose G is a locally compact topological group and that H, H' and L are subsets of G.*

(i) *$H \pitchfork L$ if and only if $L \pitchfork H$.*

(ii) *If $H \sim H'$ and if $H \pitchfork L$ in G, then $H' \pitchfork L$ in G.*

The following is a reformulation of Problem 2.3(1) as:

PROBLEM 2.7 (a reformulation of Problem 2.3(1)). Let G be a Lie group, and H and L subsets of G. Find the criterion on the pair (L, H) (or on the pair of their equivalence classes with respect to $\sim$) such that $L \pitchfork H$ in G.

Property (CI). If a discrete group Γ acts on X properly discontinuously, then every isotropy subgroup is finite and every Γ orbit is closed ([8, Lemma 2.3]). The latter condition corresponds to the fact that each point is closed in the quotient topology of $\Gamma\backslash X$. In general, the converse implication does not hold. Kobayashi has singled out an intermediate property in a continuous setting. In fact, let H, L be closed subgroups of a locally compact topological group G. If L acts properly on G/H, then any L-orbit $LgH \simeq L/L \cap gHg^{-1} \subset G/H$ is a closed submanifold, and each isotropy subgroup $L \cap gHg^{-1}$ is compact. In general, this condition is not sufficient for the properness of the L-action (see [8, Example 2.9] for a counter example). However, we pick up the second condition because of its simplicity:

DEFINITION 2.8 ([5], [8], [13]). Suppose that H and L are subsets of a locally compact topological group G. We say that the pair (L, H) has the *property (CI)* in G if $L \cap gHg^{-1}$ is compact for any $g \in G$.

Here (CI) stands for that action of L has a compact isotropy subgroup $L \cap gHg^{-1}$ at each point $gH \in G/H$, or stands for that L and gHg^{-1} has a compact intersection ($g \in G$).

If $H \pitchfork L$ in G, then the pair (L, H) has the property (CI) in G. For actual calculation, it is much easier to check the property (CI) than properness. So, we are interested in how and to which extent the property (CI) implies the proper action.

PROBLEM 2.9 ([5, Problem 2], see [8, Open problem 6(ii)]). For which Lie groups, does the following equivalence hold ?

$$H \pitchfork L \text{ in } G \Longleftrightarrow \text{the pair } (L, H) \text{ has the property (CI) in } G. \tag{2.1}$$

The equivalence (2.1) holds if G, H, L are real reductive algebraic groups (see [8, Theorem 3.18]). Lipsman's conjecture (Conjecture 1.1) can be restated as:

CONJECTURE 2.10 (Lipsman). *The equivalence* (2.1) *holds for a simply connected nilpotent Lie group.*

Theorem 2.11 is a partial solution to this conjecture in the case where G is a 2-step simply connected nilpotent Lie group.

There are some further cases, specially in the context of a continuous analog of the Auslander conjecture (see [8], the remark following Example 1.9), where the equivalence 2.1 is known to hold. See Example 5 and Proposition A.2.1 in [5]; Theorem 3.1 and Theorem 5.4 in [13].

Now, we are ready to state a more precise verson of Theorem 1.2.

THEOREM 2.11. *Let G be a simply connected 2-step nilpotent Lie group, H and L be connected subgroups of G. Then the following six conditions are equivalent*:

1) *L acts on G/H properly.*

1)$'$ *H acts on G/L properly.*

1)$''$ *$H \pitchfork L$ in G.*

2) *The triplet (L, G, H) has the property (CI). That is, $L \cap gHg^{-1}$ is compact for any $g \in G$.*

2)$'$ *$L \cap gHg^{-1} = \{e\}$ for any $g \in G$.*

2)$''$ *$\mathfrak{l} \cap \mathrm{Ad}(g)\mathfrak{h} = \{0\}$ for any $g \in G$. Here, $\mathfrak{l}$ and $\mathfrak{h}$ are Lie algebras of L and H, respectively.*

In the above theorem, we refer to [8], Theorem 3.18 for the equivalence (1) $\Leftrightarrow$ (1)$'$ $\Leftrightarrow$ (1)$''$ and to [10], Lemma 3.2 for (2) $\Leftrightarrow$ (2)$'$ $\Leftrightarrow$ (2)$''$. The implication (1) $\Rightarrow$ (2) is trivial. The rest of this paper is devoted to the proof of the implication $(2)'' \Rightarrow (1)$ in the case where G is a simply connected 2-step nilpotent Lie group.

3. Proof of the main result Theorem 1.2.

First of all we recall the following definition:

DEFINITION 3.1. A Lie algebra $\mathfrak{g}$ is said to be a 2-step nilpotent Lie algebra if $[X, [Y, Z]] = 0$ for all $X, Y, Z \in \mathfrak{g}$. We say that a connected Lie group G is 2-step nilpotent Lie group if its Lie algebra is a 2-step nilpotent Lie algebra.

EXAMPLE 3.2. Any abelian Lie algebra is a 2-step nilpotent Lie algebra.

EXAMPLE 3.3. Consider the simplest non-abelian three dimensional Heisenberg Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbf{R} \right\}.$$

Then $\mathfrak{g}$ is a 2-step nilpotent Lie algebra.

EXAMPLE 3.4 (Lie algebra $\mathbf{R} \ltimes \mathbf{R}^n$). We fix a linear map

$$\psi : \mathbf{R} \longrightarrow \mathrm{End}(\mathbf{R}^n) = M_n(\mathbf{R})$$

with $\psi(1)^2 = 0$. Let $\mathfrak{g} = \mathbf{R} \oplus \mathbf{R}^n$. We define a Lie bracket on $\mathfrak{g}$ by

$$[(a, \vec{x}), (b, \vec{y})] = (0, \psi(a)\vec{y} - \psi(b)\vec{x}) \tag{3.1}$$

for $(a, \vec{x}), (b, \vec{y}) \in \mathfrak{g}$. Then $\mathfrak{g}$ is a 2-step nilpotent Lie algebra.

PROOF. To prove that $\mathfrak{g}$ is a Lie algebra with the bracket operation (3.1) it is sufficient to check the Jacobi identity.

For any $(a, \vec{x}), (b, \vec{y}), (c, \vec{z}) \in \mathfrak{g}$, we have

$$\begin{aligned} [[(a, \vec{x}), (b, \vec{y})], (c, \vec{z})] &= [(0, \psi(a)\vec{y} - \psi(b)\vec{x}), (c, \vec{z})] \\ &= (0, -\psi(c)(\psi(a)\vec{y} - \psi(b)\vec{x})) \end{aligned} \tag{3.2}$$

$$\begin{aligned} [[(b, \vec{y}), (c, \vec{z})], (a, \vec{x})] &= [(0, \psi(b)\vec{z} - \psi(c)\vec{y}), (a, \vec{x})] \\ &= (0, -\psi(a)(\psi(b)\vec{z} - \psi(c)\vec{y})) \end{aligned} \tag{3.3}$$

$$\begin{aligned} [[(c, \vec{z}), (a, \vec{x})], (b, \vec{y})] &= [(0, \psi(c)\vec{x} - \psi(a)\vec{z}), (b, \vec{y})] \\ &= (0, -\psi(b)(\psi(c)\vec{x} - \psi(a)\vec{z})). \end{aligned} \tag{3.4}$$

Now if we add (3.2), (3.3) and (3.4) together then it equals to zero. Hence the bracket relation (3.1) in $\mathfrak{g}$ satisfies the Jacobi identity. Therefore $\mathfrak{g}$ is a Lie algebra.

Also, $\mathfrak{g}$ is 2-step nilpotent because for any $(a, \vec{x})$, $(b, \vec{y})$, $(c, \vec{z})$ in $\mathfrak{g}$, we see that

$$\begin{aligned}[[(a, \vec{x}), (b, \vec{y})], (c, \vec{z})] &= [(0, \psi(a)\vec{y} - \psi(b)\vec{x}), (c, \vec{z})] \\ &= (0, -\psi(c)(\psi(a)\vec{y} - \psi(b)\vec{x})) \\ &= (0, 0).\end{aligned}$$

Here, the last equality follows from $\psi(c)\psi(a) = ca\psi(1)^2 = 0$ because $\psi(1)^2 = 0$. □

Next, we collect some elementary lemmas those are needed later.

LEMMA 3.5. *If G is a 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ then*
1. *The map* $\exp : \mathfrak{g} \to G$ *is diffeomorphic.*
2. $\exp A \exp X = \exp(A + X + \frac{1}{2}[A, X])$ *for any* $A, X \in \mathfrak{g}$.
3. $\exp A \exp X \exp B = \exp(X + \frac{1}{2}[A - B, X] + A + B + \frac{1}{2}[A, B])$ *for* $A, X, B \in \mathfrak{g}$.
4. *If we put $B = -A$ then we have*

$$\exp A \exp X \exp(-A) = \exp(X + [A, X]) = \exp((\mathrm{Id} + \mathrm{ad}\, A)X).$$

In particular, we have

$$(\mathrm{Id} + \mathrm{ad}(A))X = \mathrm{Ad}(\exp A)X.$$

PROOF. The lemma is a direct consequence of the Campbell-Hausdorff formula and the fact that $\mathfrak{g}$ is a 2-step nilpotent Lie algebra. □

Here is a key formulation on the part of general topology.

LEMMA 3.6. *Suppose $\mathbf{R}^n$ is an n-dimensional vector space and V, W are its subspaces. Let $K \subset \mathrm{GL}(n, \mathbf{R})$ be a compact subset and $T \subset \mathbf{R}^n$ be a compact subset. If the set*

$$V \cap (K \cdot W + T) \tag{3.5}$$

is not compact, then there exists $k \in K$ such that

$$V \cap k \cdot W \neq \{0\}.$$

PROOF. We note that $V \cap (K \cdot W + T)$ is a closed subset of $\mathbf{R}^n$ because K is compact. Now if the set (3.5) is not compact, then there are sequences $v_j \in V$, $w_j \in W$, $s_j \in T$, $k_j \in K$ $(\subset \mathrm{GL}(n, \mathbf{R}))$ $(j \in \mathbf{N})$, such that

$$v_j = k_j w_j + s_j \tag{3.6}$$

where v_j and w_j are unbounded. Then we can find its subsequences (with the same notation) such that

$$\lim_{j\to\infty} |v_j| = \lim_{j\to\infty} |w_j| = \infty, \quad \lim_{j\to\infty} \frac{v_j}{|v_j|} = v, \quad \lim_{j\to\infty} \frac{w_j}{|w_j|} = w$$
$$\lim_{j\to\infty} k_j = k, \quad \lim_{j\to\infty} s_j = s,$$

for some $v, w \in S^{n-1}$, and $k \in K$, $s \in T$.

Then from equation (3.6), we have

$$\frac{v_j}{|v_j|} = \frac{|w_j|}{|v_j|} \cdot \frac{k_j w_j}{|w_j|} + \frac{s_j}{|v_j|} \,. \tag{3.7}$$

Taking the limit as j tends to ∞ in both sides of (3.7), we obtain,

$$v = \Big(\lim_{j \to \infty} \frac{|w_j|}{|v_j|} \Big) k \cdot w + 0 \,. \tag{3.8}$$

Now, since $k \in \mathrm{GL}(n, \mathbf{R})$ and $w \neq 0$, we have $|k \cdot w| \neq 0$. Then (3.8) implies that the limit $\lim_{j \to \infty} \frac{|w_j|}{|v_j|}$ exists, for which we write $\lambda \in \mathbf{R}_+^\times$.

Hence, we have proved that there exists $k \in K$ such that

$$v = k(\lambda w) \,,$$

that is ,

$$V \cap k \cdot W \neq \{0\} \,.$$

$\square$

REMARK 3.7. The point of the assumption of Lemma 3.6 is that not only $K \subset M(n, \mathbf{R})$ but also $K \subset \mathrm{GL}(n, \mathbf{R})$. Here is an illustrative example for the reason why we need $K \subset \mathrm{GL}(n, \mathbf{R})$. Suppose $K := \{(x^2, x) : 0 \leq x \leq 1\}$. We note that K is compact. Then

$$\overline{\mathbf{R}K} \supsetneqq \mathbf{R}K$$

because

$$\begin{aligned} \mathbf{R}K &= \bigcup_{k \in K} \{rk : r \in \mathbf{R}\} \\ &= \{(x, y) \in \mathbf{R}^2 : y \geq x \geq 0 \text{ or } y \leq x \leq 0\} \setminus \{(0, y) : y \neq 0\} \,. \end{aligned}$$

This is because $K \subset \mathbf{R}^2$ but $K \not\subset \mathbf{R}^2 \setminus \{(0, 0)\}$.

Now we are ready to prove the implication $(2)'' \Rightarrow (1)$ in Theorem 2.11.

PROOF OF $(2)'' \Rightarrow (1)$. Suppose we are in the setting of Theorem 2.11. We shall prove that if (1) fails then so does $(2)''$. Assume that there is a compact subset S of G such that $L \cap SHS^{-1}$ is not relatively compact. Then we can find unbounded sequences

$$Y_j \in \mathfrak{l}, \quad X_j \in \mathfrak{h} \quad (j = 1, 2, \cdots)$$

and bounded sequences $A_j, B_j \in \mathfrak{g}$ such that

$$\exp Y_j = \exp A_j \exp X_j \exp B_j \,.$$

Using Lemma 3.5 and the injectivity of the exponential map, we have

$$
\begin{aligned}
Y_j &= X_j + \left[\frac{1}{2}(A_j - B_j), X_j\right] + D_j \\
&= \left(\mathrm{Id} + \mathrm{ad}\left(\frac{A_j - B_j}{2}\right)\right) \cdot X_j + D_j
\end{aligned}
\tag{3.9}
$$

where $D_j = A_j + B_j + (1/2)[A_j, B_j]$.

Now, we define the following subset of $\mathrm{End}_{\mathbf{R}}(\mathfrak{g})$

$$
K := \left\{\mathrm{Id} + \mathrm{ad}\left(\frac{A - B}{2}\right) : A, B \in \log S\right\}.
$$

Note that each element of K is invertible, because $\mathrm{ad}((A - B)/2)$ is a nilpotent linear transformation. Hence $K \subset GL(\mathfrak{g})$. Since S is compact K is a compact subset of $GL(\mathfrak{g})$.

Since A_j, B_j are bounded sequences, we can take a compact subset T of $\mathfrak{g}$ such that

$$
\{D_j : j = 1, 2, \cdots\} \subset T.
$$

Then the following subset

$$
\mathfrak{l} \cap (K \cdot \mathfrak{h} + T)
$$

is non-compact by (3.9).

Applying Lemma 3.6, we find $k \in K$ such that

$$
\mathfrak{l} \cap k \cdot \mathfrak{h} \neq \{0\}.
$$

If we write $k = \mathrm{Id} + \mathrm{ad}(C)$ for $C \in \mathfrak{g}$, then $k = \mathrm{Ad}(\exp C)$ by Lemma 3.5(4). Therefore, we have proved $\mathfrak{l} \cap \mathrm{Ad}(\exp C)\mathfrak{h} \neq \{0\}$.

Thus, if the set $L \cap SHS^{-1}$ is not compact then we have $\mathfrak{l} \cap \mathrm{Ad}(g)\mathfrak{h} \neq \{0\}$, with $g = \exp C$. Thus, we have proved $(2)'' \Rightarrow (1)$. This completes the proof of Theorem 1.2.

ACKNOWLEDGEMENT. I am grateful to my supervisor Professor Toshiyuki Kobayashi for much valuable advice and guidance in the preparation of this manuscript.

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Present Address:
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO,
KOMABA, MEGURO, TOKYO, 153–8914 JAPAN.
e-mail: nasrin@ms.u-tokyo.ac.jp