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Peripheral Multiplicativity of Maps on Uniformly Closed Algebras of Continuous Functions Which Vanish at Infinity

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Abstract. We study maps between uniformly closed algebras of complex-valued continuous functions which vanish at infinity on locally compact Hausdorff spaces. Without assuming linearity nor multiplicativity on the maps we show that they are isometrical isomorphisms as Banach space operators if they satisfy that the peripheral range of the product of the images of any two elements coincides with the peripheral range of the product of those elements. Furthermore, if the underlying algebras contain approximate identities, then they are isometrically isomorphic as Banach algebras, which is a generalization of a recent result of Luttman and Tonev for the case of uniform algebras. On the other hand it is not the case without assuming the existence of approximate identities; An example is given.

1. Introduction

Molnár [8] initiated the study of multiplicatively spectrum-preserving maps on Banach algebras and proved among other theorems that a map T from a Banach algebra $C(\mathcal{X})$ of all complex-valued continuous functions on a first countable compact Hausdorff space $\mathcal{X}$ onto itself is an almost isomorphism in the sense that T is an algebra isomorphism times a weight with the values in $\{-1, 1\}$ if T is multiplicatively spectrum preserving in the sense that the spectrum of the product of any f and $g \in C(\mathcal{X})$ equals to the spectrum of the product of Tf and Tg. Rao and Roy [9] generalized the result for an arbitrary uniform algebra onto itself. Hatori, Miura and Takagi [4] studied maps from a uniform algebra A onto another one B, and show that a similar conclusion holds if the map is multiplicatively range preserving and that A is isometrically isomorphic to B as a Banach algebra. Luttman and Tonev [7] considered multiplicatively preserving property for much more smaller set; peripheral ranges. They proved the similar conclusion as the previous ones if the map between uniform algebras satisfies that the peripheral range of the product of any two functions equals to the peripheral range of the product of the images of those two functions and show that these uniform algebras are isometrically isomorphic to each other as Banach algebras. Hatori, Miura and Takagi

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[5] consider the case where underlying algebras are unital semisimple commutative Banach algebras. Rao and Roy [10] consider maps from uniformly closed algebras of continuous functions which vanish at infinity onto itself. In any case of the previous results the domain algebra and the image algebra of the given map are *algebraically* isomorphic. In this paper we show that it is not the case in general. We also show a positive result; if a map T between certain algebras of continuous functions which vanish at infinity is multiplicatively peripheral-range-preserving, then those two algebras are isometrically isomorphic as Banach *spaces*. As in the similar way as the proofs of previous results, the main object is to give a map between the Choquet boundaries, but the proof here is much involved because of lack of the unit elements in the underlying algebras, which need not be algebraically isomorphic to each other. A related result was proven by Honma [6].

2. Preliminaries

Let X be a locally compact Hausdorff space. We denote the algebra of all complex-valued continuous functions on X vanish at infinity by $C_0(X)$. A closed subalgebra A (which contains the constant functions whenever X is compact) of $C_0(X)$ is called a function algebra on X if A strongly separates the points of X in the sense that if $x, y \in X$, $x \neq y$, then there exists an $f \in A$ with $0 \neq f(x) \neq f(y)$. A function algebra is called a uniform algebra if the underlying space X is compact. (These terms are due to [11].) The maximal ideal space of a function algebra A is denoted by M_A. For $f \in A$, $\hat{f}$ is the Gelfand transform of f. Note that a function algebra A is a semi-simple commutative Banach algebra, that is, $\hat{f} = 0$ implies $f = 0$ for $f \in A$.

Let A be a function algebra on a locally compact Hausdorff space X. For a subset S of X the supremum norm on S is denoted by $\|f\|_{\infty(S)} = \sup\{|f(x)| : x \in S\}$. A peripheral range $\{z \in f(X) : |z| = \|f\|_{\infty(X)}\}$ of $f \in A$ is denoted by $\mathrm{Ran}_\pi(f)$. Note that the peripheral range of each $f \in A$ coincides with the peripheral spectrum $\{z \in \sigma(f) : |z| = \mathrm{r}(f)\}$, where $\sigma(\cdot)$ denotes the spectrum and $\mathrm{r}(\cdot)$ is the spectral radius since the Gelfand transform is an isometry for function algebras. A function $f \in A$ is said to be a peak function for A if $\mathrm{Ran}_\pi(f) = \{1\}$. For a closed subset K of X we say that K is a peak set for A if there is an f in A with $K = f^{-1}(1)$. Such a function f is called a peak function for K. If a peak set is a singleton, then the unique element of the set is called a peak point for A. A weak peak set for A is a finite or an infinite intersection of peak sets for A. If a weak peak set is a singleton, then the unique element of the set is called a weak peak point for A. The set of all weak peak points for A is denoted by $\mathrm{Ch}(A)$. Then the closure of $\mathrm{Ch}(A)$ is a Šilov boundary for A and so $\mathrm{Ch}(A)$ is a uniqueness set for A, in the sense that $f = g$ on $\mathrm{Ch}(A)$ implies $f = g$ on X for $f, g \in A$.

In a proof of the main result a version of a theorem of Bishop for function algebras on a locally compact Hausdorff space plays an important role. A theorem of Bishop for uniform algebras are well-known; See a theorem and its proof of Bishop for uniform algebra on a *compact* Hausdorff space in [1, Theorem 2.4.1]. For a convenience we show a version of the theorem and its proof due to the case of uniform algebras.

THEOREM 2.1. *Let X be a locally compact Hausdorff space and A a function algebra on X. Suppose that a closed subset K of X is a peak set for A. Then for every $f \in A$ which does not vanish on K, there exists a peak function $u \in A$ with $u^{-1}(1) = K$ such that $|fu(x)| < \|f\|_{\infty(K)}$ for every $x \in X \setminus K$.*

PROOF. A proof may be known, but we give it for a convenience. We consider the case where X is not compact. Let $X_\infty = X \cup \{\infty\}$ be the one point compactification of X and $A_\infty = A + \mathbf{C}$ the unitization of A. Then we may consider that A_∞ is a uniform algebra on X_∞. Since K is a peak set for A, K is also a peak set for A_∞. Then by a theorem of Bishop for uniform algebra, there exists a peak function $u_\infty \in A_\infty$ for K with $|fu_\infty(x)| < \|f\|_{\infty(K)}$ for every $x \in X \setminus K$. Since u_∞ is a peak function for K and $\infty \notin K$, we have $|u_\infty(\infty)| < 1$. Let w be a Möbius transform from the closed unit disk $\bar{D}$ onto itself with $w(1) = 1$ and $w(u_\infty(\infty)) = 0$. We see that $w \circ u_\infty$ is in A since w is uniformly approximated by analytic polynomials and $w(u_\infty(\infty)) = 0$, and so we see that $u_\infty \times w \circ u_\infty$ is a function in A. Put $u = u_\infty \times w \circ u_\infty$ and it is easy to see that this u is a desired function. □

3. Main Results

Let A be a function algebra on a locally compact Hausdorff space. We denote the maximal ideal space for A by M_A. The Gelfand transform of $f \in A$ is denoted by $\hat{f}$.

THEOREM 3.1. *Let A and B be function algebras on locally compact Hausdorff spaces X and Y respectively. Suppose that T is a map from A onto B such that the equality*

$$\mathrm{Ran}_\pi(TfTg) = \mathrm{Ran}_\pi(fg)$$

holds for every pair f and g in A. Then there is a continuous map N from M_B into $\{-1, 1\}$ and a homeomorphism Φ from M_B onto M_A such that

$$\widehat{Tf}(y) = N(y)\hat{f} \circ \Phi(y)\,, \quad y \in M_B$$

holds for every $f \in A$. In particular, A is isometrically isomorphic to B as Banach spaces.

Note that function algebras A and B satisfying the hypotheses in Theorem 3.1 need not be isometrically isomorphic to each other as Banach algebras. In fact, it is not the case even if the equality of peripheral range is replaced by that of spectrum.

EXAMPLE 3.2. Let $D = \{z \in \mathbf{C} : |z| < 1\}$, $D_0 = \{z \in \mathbf{C} : 0 < |z| < 1\}$, $\bar{D}_0 = \{z \in \mathbf{C} : 0 < |z| \leq 1\}$ and $L = \{z \in \mathbf{R} : 1 \leq z \leq 2\}$. Put $X_0 = \bar{D}_0 \cup L$, $\overline{X_0} = X_0 \cup \{0\}$ and $X = X_0 \times \{1, 2\}$. For $i = 1$ and 2 define maps π_i from X_0 into X such that $\pi_i(z) = (z, i)$ respectively. Then the map π_i is a homeomorphism from X_0 onto $X_0 \times \{i\}$ for $i = 1,\ 2$. Put complex-valued functions $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ on X by $f_{\mathcal{A}}(z, i) = z$ and $f_{\mathcal{B}}(z, i) = (-1)^{i+1}z$. The algebra of all complex-valued continuous functions on $\overline{X_0}$ which is analytic on D is denoted by $P(\overline{X_0})$. We denote $P(X_0)$ by the restriction of $P(\overline{X_0})$ to X_0 and $P_{00}(X_0) = \{z^2 f(z) : f \in P(X_0)\}$. The algebra of all complex-valued continuous functions which vanish at infinity is denoted by $C_0(X)$. We denote $\mathcal{A}_0 = \{f \in C_0(X) : f \circ \pi_1,\ f \circ \pi_2 \in P_{00}(X_0)\}$. Put $\mathcal{A} = \mathcal{A}_0 + \mathbf{C} f_{\mathcal{A}}$ and $\mathcal{B} = \mathcal{A}_0 + \mathbf{C} f_{\mathcal{B}}$. It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ are closed subalgebras of $C_0(X)$ which strongly separate the points of X. One can see that the maximal ideal spaces of $\mathcal{A}$ and $\mathcal{B}$ are X itself respectively. Let $N : X \to \{-1, 1\}$ be $N(x, i) = (-1)^i$. Then the map $T : \mathcal{A} \to \mathcal{B}$ defined by $T(f) = Nf$ is bijective and satisfies the equality

$$\sigma(TfTg) = \sigma(fg), \quad f, g \in \mathcal{A},$$

thus

$$\mathrm{Ran}_\pi(TfTg) = \mathrm{Ran}_\pi(fg), \quad f, g \in \mathcal{A}.$$

On the other hand $\mathcal{A}$ is not algebraically isomorphic to $\mathcal{B}$. A precise proof of the above statements is given in [3].

In the case where a function algebra A (or B) contains an approximate identity $\{e_\alpha\}$ in the sense that $\|f - fe_\alpha\|_\infty \to 0$ as a net for every $f \in A$, A is isometrically isomorphic to B as Banach algebras whenever A and B satisfy the hypotheses of Theorem 3.1.

COROLLARY 3.3. *Suppose that A and B satisfy the hypotheses in Theorem* 3.1. *If A or B contain approximate identities, then A is isometrically isomorphic to B as Banach algebras.*

For the case of uniform algebras on compact Hausdorff spaces, the corresponding result of Corollary 3.3 is proven by Luttman and Tonev [7].

4. A Proof of Theorem 3.1

Let A be a function algebra on a locally compact Hausdorff space X. Let D be the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane. For an $x \in X$, put

$$P_x(A) = \{f \in A : f(X) \subset D \cup \{1\},\ f(x) = 1\},$$

$$PP_x(A) = \{f \in A : f(X) \subset D \cup \{-1, 1\},\ f(x) = 1\}.$$

Note that $P_x(A) \neq \emptyset$ and $P_x(A) \subset PP_x(A)$ for every $x \in \mathrm{Ch}(A)$. Put

$$P(A) = \bigcup_{x \in \mathrm{Ch}(A)} P_x(A)$$

and

$$PP(A) = \bigcup_{x \in \mathrm{Ch}(A)} PP_x(A).$$

Note that $P(A)$ is the set of all peak functions in A since $K \cap \mathrm{Ch}(A) \neq \emptyset$ for every peak set K for A. For an $f \in P(A)$ put

$$L_f = f^{-1}(1).$$

Then L_f is a peak set for A and every peak set for A is denoted by L_f for some $f \in P(A)$.

LEMMA 4.1. *T is an injection and so a bijection from A onto B. Thus we have that*

$$\mathrm{Ran}_\pi(T^{-1}FT^{-1}G) = \mathrm{Ran}_\pi(FG)$$

holds for every pair F and G in B.

PROOF. Let $f, g \in A$ and $Tf = Tg$. Then for every $h \in A$ we have

$$\mathrm{Ran}_\pi(fh) = \mathrm{Ran}_\pi(TfTh) = \mathrm{Ran}_\pi(TgTh) = \mathrm{Ran}_\pi(gh)\,.$$

We will show that $f(x) = g(x)$ for every $x \in \mathrm{Ch}(A)$. It will follow that $f = g$ since $\mathrm{Ch}(A)$ is a uniqueness set for A. Let $x \in \mathrm{Ch}(A)$.

We first consider the case where $f(x) \neq 0 \neq f(y)$. Since $x \in \mathrm{Ch}(A)$ and $f^{-1}(f(x))$ is a G_δ-set, there exists a peak set for A with $x \in K \subset f^{-1}(f(x))$. Then by Theorem 2.1 there exists a peak function u for K in A with

$$|fu(y)| < \|f\|_{\infty(K)} = |f(x)| \qquad y \in X \setminus K\,.$$

Thus we have that $\mathrm{Ran}_\pi(fu) = \{f(x)\}$. In a way similar we have that $\mathrm{Ran}_\pi(gv) = \{g(x)\}$ holds for a peak function v in A with $v(x) = 1$. It follows that

$$\{f(x)\} = \mathrm{Ran}_\pi(fuv) = \mathrm{Ran}_\pi(guv) = \{g(x)\}\,,$$

so we have that $f(x) = g(x)$.

Next we show that $f(x) = 0$ implies that $g(x) = 0$. Suppose that $f(x) = 0$ and $g(x) \neq 0$. We show a contradiction. As before there exists a peak function $v \in A$ with $v(x) = 1$ such that $\mathrm{Ran}_\pi(gv) = \{g(x)\}$. Then for every peak function u with $u(x) = 1$, we have that $\mathrm{Ran}_\pi(gvu) = \{g(x)\}$. On the other hand, since $f(x) = 0$ and f is continuous, we see by some calculation that there exists a peak function u_0 with $u_0(x) = 1$ such that

$$\mathrm{Ran}_\pi(fu_0) \subset \{z \in \mathbf{C} : |z| < |g(x)/2|\}\,.$$

Thus we have that

$$\begin{aligned}\{g(x)\} = \mathrm{Ran}_\pi(gvu_0) &= \mathrm{Ran}_\pi(TgT(vu_0)) \\ &= \mathrm{Ran}_\pi(TfT(vu_0)) = \mathrm{Ran}_\pi(fvu_0) \subset \{z \in \mathbf{C} : |z| < |g(x)/2|\}\end{aligned}$$

since $|v| \leq 1$ on X, which is a contradiction. In the same way we see that $g(x) = 0$ implies that $f(x) = 0$.

It follows that $f(x) = g(x)$ for every $x \in \mathrm{Ch}(A)$ and we see that T is an injection since $\mathrm{Ch}(A)$ is a uniqueness set for A. □

LEMMA 4.2. *Let $f, g \in A$ and $\alpha, \beta \in \mathbf{C} \setminus \{0\}$. Suppose that $Tf \in \alpha PP(B)$. Then we have that $(f/\alpha)^2 \in P(A)$ and $\left(\frac{Tf}{\alpha}\right)^2 \in P(B)$. We also have that $L_{\left(\frac{Tf}{\alpha}\right)^2} \subset L_{\left(\frac{Tg}{\beta}\right)^2}$ implies that $L_{\left(\frac{f}{\alpha}\right)^2} \subset L_{\left(\frac{g}{\beta}\right)^2}$ if $Tf \in \alpha PP(B)$ and $Tg \in \beta PP(B)$.*

PROOF. It is easy to see that $f \in P(A)$ if and only if $\mathrm{Ran}_\pi(f) = \{1\}$. We also see by a simple calculation that for every non-zero complex number α, $f \in \alpha PP(A)$ if and only if the inclusions $\{\alpha\} \subset \mathrm{Ran}_\pi(f) \subset \{\alpha, -\alpha\}$ hold.

Suppose that $Tf \in \alpha PP(B)$, $Tg \in \beta PP(B)$ and $L_{\left(\frac{Tf}{\alpha}\right)^2} \subset L_{\left(\frac{Tg}{\beta}\right)^2}$. Then we have that $\mathrm{Ran}_\pi(f^2) = \mathrm{Ran}_\pi(TfTf) = \{\alpha^2\}$, so $\mathrm{Ran}_\pi\left(\left(\frac{f}{\alpha}\right)^2\right) = \{1\}$. Thus $\left(\frac{f}{\alpha}\right)^2 \in P(A)$ and $\left(\frac{Tf}{\alpha}\right)^2 \in P(B)$.

Suppose that $L_{\left(\frac{Tf}{\alpha}\right)^2} \subset L_{\left(\frac{Tg}{\beta}\right)^2}$. Suppose that $x \in L_{\left(\frac{f}{\alpha}\right)^2} \setminus L_{\left(\frac{g}{\beta}\right)^2}$. We may assume that $x \in \mathrm{Ch}(A)$. (Let X_∞ be the one-point compactification of X and $A_\infty = A + \mathbf{C}$. Then A_∞ can be seen a uniform algebra on a compact Hausdorff space X_∞ and we see by the definition that $L_{\left(\frac{f}{\alpha}\right)^2}$ and $L_{\left(\frac{g}{\beta}\right)^2}$ are peak sets for A_∞. Then $A_\infty|L_{\left(\frac{f}{\alpha}\right)^2}$ of the restriction of A_∞ to the set $L_{\left(\frac{f}{\alpha}\right)^2}$ is a uniform algebra on $L_{\left(\frac{f}{\alpha}\right)^2}$ and $L_{\left(\frac{g}{\beta}\right)^2} \cap L_{\left(\frac{f}{\alpha}\right)^2}$ is a peak set for $A_\infty|L_{\left(\frac{f}{\alpha}\right)^2}$. Let $F \in A_\infty|L_{\left(\frac{f}{\alpha}\right)^2}$ be a peak function for the set $L_{\left(\frac{g}{\beta}\right)^2} \cap L_{\left(\frac{f}{\alpha}\right)^2}$. Then so is $\frac{F+1}{2}$ and invertible since the range of $\frac{F+1}{2}$ is in the right half-plane. Since $|\frac{F+1}{2}| < 1$ on $L_{\left(\frac{f}{\alpha}\right)^2} \setminus L_{\left(\frac{g}{\beta}\right)^2}$ and $\frac{F+1}{2} = 1$ on $L_{\left(\frac{g}{\beta}\right)^2} \cap L_{\left(\frac{f}{\alpha}\right)^2}$, we see that $\left(\frac{F+1}{2}\right)^{-1}$ takes the maximum absolute value at a point in $L_{\left(\frac{f}{\alpha}\right)^2} \setminus L_{\left(\frac{g}{\beta}\right)^2}$. Thus we see that

$$\mathrm{Ch}(A_\infty|L_{\left(\frac{f}{\alpha}\right)^2}) \cap \left(L_{\left(\frac{f}{\alpha}\right)^2} \setminus L_{\left(\frac{g}{\beta}\right)^2}\right) \neq \emptyset\,.$$

Since $\mathrm{Ch}(A_\infty|L_{\left(\frac{f}{\alpha}\right)^2}) \subset \mathrm{Ch}(A_\infty)$ and $\infty \notin L_{\left(\frac{f}{\alpha}\right)^2}$ we have that

$$\mathrm{Ch}(A) \cap \left(L_{\left(\frac{f}{\alpha}\right)^2} \setminus L_{\left(\frac{g}{\beta}\right)^2}\right) \neq \emptyset.)$$

So there exists a $u \in P_x(A)$ with $|u| < 1$ on $L_{\left(\frac{g}{\beta}\right)^2}$. Then we have

$$\emptyset \neq \mathrm{Ran}_\pi(fu) \subset \{\alpha, -\alpha\}$$

and so

$$\mathrm{Ran}_\pi((TfTu)^2) = \{\alpha^2\}$$

since $\mathrm{Ran}_\pi(fu) = \mathrm{Ran}_\pi(TfTu)$. On the other hand since $|u| < 1$ on $L_{\left(\frac{g}{\beta}\right)^2}$ we have that

$$\left|\frac{g^2}{\beta^2}u^2\right| \leq \left|\frac{g^2}{\beta^2}u\right| < 1$$

hold on X, so we see that $|gu| < |\beta|$ holds on X. It follows that

$$1 \notin \mathrm{Ran}_\pi\left(\left(\frac{Tg}{\beta}\right)^2 (Tu)^2\right)$$

since $\beta, -\beta \notin \mathrm{Ran}_\pi(gu) = \mathrm{Ran}_\pi(TgTu)$. On the other hand we have $(Tu)^2 \in P(B)$ since $\mathrm{Ran}_\pi(TuTu) = \mathrm{Ran}_\pi(u^2) = \{1\}$ hold. Then there exists a $y_0 \in Y$ such that

$$(Tu)^2(y_0) = 1 = \left(\frac{Tf}{\alpha}\right)^2 (y_0)$$

since $\mathrm{Ran}_\pi((\frac{Tf}{\alpha})^2(Tu)^2) = \{1\}$ and $(\frac{Tf}{\alpha})^2 \in P(B)$. Then $(\frac{Tg}{\beta})^2(y_0) = 1$ since $L_{(\frac{Tf}{\alpha})^2} \subset L_{(\frac{Tg}{\beta})^2}$, so we have that

$$\mathrm{Ran}_\pi\left(\left(\frac{Tg}{\beta}\right)^2 (Tu)^2\right) = \{1\},$$

which is a contradiction proving that $L_{(\frac{f}{\alpha})^2} \subset L_{(\frac{g}{\beta})^2}$. □

LEMMA 4.3. *Let $y \in \mathrm{Ch}(B)$. Then we have*

$$\bigcap_{f \in T^{-1}(\beta P P_y(B)), 0 \neq \beta \in \mathbf{C}} L_{(\frac{f}{\beta})^2} \neq \emptyset.$$

PROOF. Applying the finite intersection property, we only need to show that

$$\bigcap_{k=1}^{n} L_{(\frac{f_k}{\beta_k})^2} \neq \emptyset$$

for a finite number of $f_1, \ldots, f_n \in T^{-1}(\beta_k P P_y(B))$ and non-zero complex numbers $\beta_1, \ldots \beta_n$. By the definition $T(f_k)(y) = \beta_k$ holds for each $k = 1, 2, \ldots, n$. Since T is a surjection there exists a $g \in A$ with $Tg = \prod_{k=1}^n Tf_k$, so $Tg \in \beta P P_y(B)$, where $\beta = \prod_{k=1}^n \beta_k$. Then by Lemma 4.2 we have that $(\frac{Tg}{\beta})^2, (\frac{Tf_k}{\beta_k})^2 \in P(B)$ for each $k = 1, 2, \ldots, n$. Then we see that $L_{(\frac{Tg}{\beta})^2} \subset L_{(\frac{Tf_k}{\beta_k})^2}$, so we have that $L_{(\frac{g}{\beta})^2} \subset L_{(\frac{f_k}{\beta_k})^2}$ holds for $k = 1, \ldots, n$ by Lemma 4.2. It follows that

$$\emptyset \neq L_{(\frac{g}{\beta})^2} \subset \bigcap_{k=1}^{n} L_{(\frac{f_k}{\beta_k})^2}.$$

□

LEMMA 4.4. *For every $y \in \mathrm{Ch}(B)$, there exist an $x \in \mathrm{Ch}(A)$ and an $\alpha_y \in \{-1, 1\}$ such that*

$$T^{-1}(\beta P P_y(B)) \subset \beta\alpha_y P P_x(A)$$

holds for every $\beta \in \mathbf{C}$.

PROOF. We simply write

$$\bigcap_{f \in T^{-1}(\beta P P_y(B)), 0 \neq \beta \in \mathbf{C}} L_{\left(\frac{f}{\beta}\right)^2}$$

as $\cap L_{\left(\frac{f}{\beta}\right)^2}$. By Lemma 4.3 $\cap L_{\left(\frac{f}{\beta}\right)^2}$ is not empty and so a weak peak set for A. Then there exists an $x \in \left(\cap L_{\left(\frac{f}{\beta}\right)^2}\right) \cap \mathrm{Ch}(A)$. So $f^2(x) = \beta^2$ holds for every $f \in T^{-1}(\beta P P_y(B))$ and for every non-zero complex number β. Put $\alpha_{f,y} = \frac{f(x)}{\beta}$. Then $\alpha_{f,y}$ is 1 or -1.

We will show that $\alpha_{f,y}$ does not depend on the choice of f and β indeed. Let α and β be non-zero complex numbers and $f \in T^{-1}(\alpha P P_y(B))$, $g \in T^{-1}(\beta P P_y(B))$. Then there exists a neighborhood G of y such that

$$|Tf - \alpha| < 1/2\,, \quad |Tg - \beta| < 1/2$$

hold on G since Tf and Tg is continuous. Since y is in $\mathrm{Ch}(B)$ there exists a peak function $H \in P_y(B)$ such that $y \in H^{-1}(1) \subset G$. Put $h = T^{-1}H$. Then by the above we have $(\alpha_{h,y})^2 = 1$, where $\alpha_{h,y} = h(x)$. Then by Lemmata 4.2 and 4.3 and by the definition of x we have that $\left(\frac{f}{\alpha}\right)^2$, $\left(\frac{g}{\beta}\right)^2$, $h^2 \in P_x(A)$, and so

$$\alpha\alpha_{f,y}\alpha_{h,y} \in \mathrm{Ran}_\pi(fh) = \mathrm{Ran}_\pi(TfTh) = \{\alpha\}\,,$$

$$\beta\alpha_{g,y}\alpha_{h,y} \in \mathrm{Ran}_\pi(gh) = \mathrm{Ran}_\pi(TgTh) = \{\beta\}\,.$$

Thus we have that $\alpha_{f,y} = \alpha_{g,y}$. We have shown that the value $\alpha_{f,y} = \frac{f(x)}{\beta}$ does not depend on the choice of $f \in T^{-1}(\beta P P_y(B)$, so we simply write α_y for $\alpha_{f,y}$.

Let β be a non-zero complex number and $f \in T^{-1}(\beta P P_y(B))$. Then by the above we have that $f(x) = \alpha_y \beta$ and

$$\mathrm{Ran}_\pi(f^2) = \mathrm{Ran}_\pi(TfTf) = \{\beta^2\}\,.$$

Thus $f \in \alpha_y \beta P P_x(A)$. We conclude that $T^{-1}(\beta P P_y(B)) \subset \alpha_y \beta P P_x(A)$. □

LEMMA 4.5. *For every* $y \in \mathrm{Ch}(B)$, *there exist a unique* $x \in \mathrm{Ch}(A)$ *and a unique* $\alpha_y \in \{-1, 1\}$ *such that*

$$T(\beta P P_x(A)) = \beta\alpha_y P P_y(B)$$

holds for every $\beta \in \mathbf{C}$.

PROOF. Let $y \in \mathrm{Ch}(B)$. Then by Lemma 4.4 there exists $x \in \mathrm{Ch}(A)$ and α_y such that $\beta P P_y(B) \subset T(\alpha_y \beta P P_x(A))$ holds for every non-zero complex number β. Since T is a bijection from A onto B, we can apply a similar argument for T^{-1} and we see that there exists a $y' \in \mathrm{Ch}(B)$ and α_x with $(\alpha_x)^2 = 1$ such that

$$T(\beta PP_x(A)) \subset \beta\alpha_x PP_{y'}(B)$$

holds for every non-zero complex number β. Thus we see that

$$\beta PP_y(B) \subset T(\beta\alpha_y PP_x(A)) \subset (\beta\alpha_y)\alpha_x PP_{y'}(B)$$

hold and so

$$PP_y(B) \subset \alpha_x\alpha_y PP_{y'}(B)$$

holds. By a simple calculation we have that $y = y'$ and $\alpha_x\alpha_y = 1$. It follows that the equality

$$T(\beta PP_x(A)) = \beta\alpha_x PP_y(B)$$

holds for every non-zero complex number β. The equation clearly holds for $\beta = 0$. The uniqueness of x and α_y are easily derived. □

Put a function φ from $\mathrm{Ch}(B)$ into $\mathrm{Ch}(A)$ by $\varphi(y)$ which equals the corresponding x in Lemma 4.5 and put a function n from $\mathrm{Ch}(B)$ into $\{-1, 1\}$ by $n(y) = \alpha_y$ which appears in Lemma 4.5.

LEMMA 4.6. *For every $f \in A$ the equation*

$$Tf(y) = n(y) f \circ \varphi(y)$$

holds for every $y \in \mathrm{Ch}(B)$.

PROOF. Let $f \in A$ and $y \in \mathrm{Ch}(B)$. First we consider the case where $f(\varphi(y)) \neq 0$ and $Tf(y) \neq 0$. By Theorem 2.1 there exists a $u \in P_{\varphi(y)}(A)$ with $\mathrm{Ran}_\pi(fu) = \{f(\varphi(y))\}$. Since $Tu(y) = n(y)$ by Lemma 4.5 and since $\mathrm{Ran}_\pi(TfTu) = \mathrm{Ran}_\pi(fu)$, we see that $|Tf(y)| = |Tf(y)Tu(y)|$ so $|Tf(y)| \leq |f(\varphi(y))|$ holds. On the other hand there exists a $U \in P_y(B)$ with $\mathrm{Ran}_\pi(fT^{-1}U) = \mathrm{Ran}_\pi(TfU) = \{Tf(y)\}$ by Theorem 2.1. Putting $\beta = \alpha_y$ in Lemma 4.5 we see that $T^{-1}(U) \in \alpha_y PP_{\varphi(y)}(A)$ so that $T^{-1}U(\varphi(y)) = \alpha_y$. Thus we have that

$$|f(\varphi(y))| = |f(\varphi(y))T^{-1}U(\varphi(y))| \leq |Tf(y)|\,.$$

It follows that $|Tf(y)Tu(y)| = |Tf(y)| = |f(\varphi(y))|$, so we have that $Tf(y)Tu(y) = f(\varphi(y))$ since $\mathrm{Ran}_\pi(TfTu) = \{f(\varphi(y))\}$. Thus $Tf(y) = n(y)f(\varphi(y))$ holds since $Tu(y) = n(y)$ and $n(y)^2 = 1$.

Suppose that $Tf(y) = 0$ and $f(\varphi(y)) \neq 0$. Then there exists a sequence $\{U_n\}$ in $P_y(B)$ with $\|TfU_n\|_{\infty(Y)} \to 0$ as $n \to \infty$. On the other hand we have $|u_n(\varphi(y))| = |U_n(y)|$ by Lemma 4.5, where $u_n = T^{-1}U_n$. So we see that $|fu_n(\varphi(y))| = |f(\varphi(y))| \neq 0$. Thus if $z \in \mathrm{Ran}_\pi(fu_n)$, then $|z| \geq |f(\varphi(y))|$. On the other hand $\mathrm{Ran}_\pi(fu_n) = \mathrm{Ran}_\pi(TfU_n) \subset \{z \in \mathbf{C} : |z| \leq \|TfU_n\|_{\infty(Y)}\}$, which is a contradiction.

In a way similar to the above we see that $Tf(y) \neq 0$ and $f(\varphi(y)) = 0$ are incompatible. □

Applying a similar argument to T^{-1} we have that there exist a map φ' from $\mathrm{Ch}(A)$ into $\mathrm{Ch}(B)$ and a map n' from $\mathrm{Ch}(A)$ into $\{-1, 1\}$ such that

$$T^{-1}F(x) = n'(x)F \circ \varphi'(x)$$

holds for every $F \in B$ and $x \in \mathrm{Ch}(A)$. It follows that $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are identity maps, so we see that φ is a bijection.

LEMMA 4.7. *The function n is extended to a function N from M_B into* $\{-1, 1\}$ *and* φ *is extended to a homeomorphism Φ from M_B onto M_A such that the equation*

$$\widehat{Tf}(y) = N(y)\hat{f} \circ \Phi(y)\,, \qquad y \in M_B$$

holds for every $f \in A$.

PROOF. Suppose that M is a maximal regular ideal of B. We show that $T^{-1}M$ is a maximal regular ideal of A. Since T is linear and isometric we see by Lemma 4.6 that $T^{-1}M$ is a closed subspace of A since $\mathrm{Ch}(A)$ is a uniqueness set for A in the sense that $f, g \in A$ with $f = g$ on $\mathrm{Ch}(A)$ implies that $f = g$. We see that the codimension of $T^{-1}M$ in A is 1 since that of M in B is 1 and since T is a linear bijection. Let $f \in A$ and $g \in T^{-1}M$. Then we have that $TfTg \in M$ since M is an ideal of B. On the other hand by Lemma 4.6 we have the equations

$$\begin{aligned}(T(fg))^2 &= (n(fg) \circ \varphi)^2 = ((fg) \circ \varphi)^2 \\ &= (nf \circ \varphi)^2(ng \circ \varphi)^2 = (TfTg)^2\end{aligned}$$

hold. Since $\mathrm{Ch}(B)$ is a uniqueness set for B we see that

$$(T(fg))^2 = (TfTg)^2 \in M\,.$$

There exists a unique multiplicative linear functional ϕ on B such that $\phi^{-1}(0) = M$. So $0 = \phi((T(fg))^2) = (\phi(T(fg)))^2$ and thus we have that $T(fg) \in M$ so that $fg \in T^{-1}M$. It follows that $T^{-1}M$ is an ideal of A, so $T^{-1}M$ is a maximal ideal since the codimension of $T^{-1}M$ in A is 1.

Next we show that $T^{-1}M$ is regular ideal in the sense that $A/T^{-1}M$ has the unit element. Let e be an element in B such that $e + M$ is the unit in B/M. We show that

$$(T^{-1}(e))^2T^{-1}(f) - T^{-1}(f) \in T^{-1}M$$

holds for every $f \in B$. It will follow that $(T^{-1}(e))^2 + T^{-1}M$ is the unit in $A/T^{-1}M$. Let $f \in B$. Since $ef - f \in M$, we have that $e^2f - f = e(ef - f) + ef - f \in M$. It follows that

$$(n \circ \varphi^{-1}e \circ \varphi^{-1})^2 n \circ \varphi^{-1}f \circ \varphi^{-1} - n \circ \varphi^{-1}f \circ \varphi^{-1} \in n \circ \varphi^{-1}M \circ \varphi^{-1}$$

on $\mathrm{Ch}(A)$ since $n^2 = 1$. By Lemma 4.6 $T^{-1}g = n \circ \varphi^{-1} g \circ \varphi^{-1}$ holds on $\mathrm{Ch}(A)$ since $n^2 = 1$ on $\mathrm{Ch}(B)$. Thus we see that the equation

$$(T^{-1}e)^2 T^{-1} f - T^{-1} f \in T^{-1} M$$

holds for every $f \in B$, so $(T^{-1}e)^2 + T^{-1}M$ is the unit in $A/T^{-1}M$. We conclude that $T^{-1}M$ is a regular ideal of A.

We define the map Φ from M_B into M_A by the above correspondence and show that it is an extension of φ. Let $y \in M_B$ (resp. $x \in M_A$). The point y (resp. x) is pompously denoted by the maximal regular ideal $M_B(y)$ (resp. $M_A(x)$) which corresponds to y (resp. x). We define the map Φ from M_B into M_A; for $y \in M_B$, $\Phi(y) \in M_A$ corresponds to the maximal regular ideal $T^{-1}M_B(y)$, which is well-defined by the above. Thus $T^{-1}(M_B(y)) = M_A(\Phi(y))$ for every $y \in M_B$, so $M_B(y) = TM_A(\Phi(y))$ holds for every $y \in M_B$. We also have by Lemma 4.6 that $TM_A(\varphi(y)) = nM_A(\varphi(y)) \circ \varphi$ holds on $\mathrm{Ch}(B)$. Thus we have that $TM_A(\varphi(y)) \subset M_B(y)$ for every $y \in \mathrm{Ch}(B)$. It follows that $M_A(\varphi(y)) \subset M_A(\Phi(y))$ holds for every $y \in \mathrm{Ch}(B)$. Since $M_A(\varphi(y))$ is a maximal regular ideal, we see that $M_A(\varphi(y)) = M_A(\Phi(y))$ and thus we have that $\varphi(y) = \Phi(y)$ holds for every $y \in \mathrm{Ch}(B)$.

Next we extend n to the function N on M_B. For $x \in M_A$ (resp. $y \in M_B$), put the evaluational functional $\phi_x : A \to \mathbf{C}$ by $\phi_x(f) = \hat{f}(x)$ (resp. $\phi_y : B \to \mathbf{C}$ by $\phi_y(f) = \hat{f}(y)$). We see that $\ker\phi_y \circ T = \ker\phi_{\Phi(y)}$, where $\ker\cdot$ denotes the kernel of the functional. (Let $f \in \ker\phi_{\Phi(y)}$. Then $Tf \in M_B(y)$ since $\ker\phi_{\Phi(y)} = M_A(\Phi(y))$, so $\phi_y(Tf) = 0$ and we have that $f \in \ker\phi_y \circ T$. On the other hand, let $f \in \ker\phi_y \circ T$. Then we have $Tf \in M_B(y)$ and so $f \in T^{-1}M_B(y) = M_A(\Phi(y)) = \ker\phi_{\Phi(y)}$.) It follows that there exists a non-zero complex number $N(y)$ such that the equality $\phi_y \circ T = N(y)\phi_{\Phi(y)}$ holds and so

$$\widehat{Tf}(y) = (\phi_y \circ T)(f) = N(y)\phi_{\Phi(y)}(f) = N(y)\hat{f} \circ \Phi(y)$$

hold for every $f \in A$ and $y \in M_B$. On the other hand we have

$$(Tf^2)^2 = (nf^2 \circ \varphi)^2 = f^4 \circ \varphi = (nf \circ \varphi)^4 = (Tf)^4$$

hold on $\mathrm{Ch}(B)$ by Lemma 4.6, so we have that $(Tf^2)^2 = (Tf)^4$ since $\mathrm{Ch}(B)$ is a uniqueness set for B. On the other hand we see that

$$(\widehat{Tf^2})^2(y) = (N(y)\widehat{f^2} \circ \Phi(y))^2 = (N(y))^2 \hat{f}^4 \circ \Phi(y)$$

and

$$(\widehat{Tf})^4(y) = (N(y))^4 \hat{f}^4 \circ \Phi(y)$$

hold for every $f \in A$ and $y \in M_B$. Since $N(y)$ is non-zero, we have that $(N(y))^2 = 1$ for every $y \in M_B$. In particular the equation

$$n(y)f \circ \varphi(y) = Tf(y) = N(y)\hat{f} \circ \Phi(y) = N(y)f \circ \varphi(y)$$

hold for every $f \in A$ and $y \in \mathrm{Ch}(B)$, it follows that $n(y) = N(y)$ holds for every $y \in \mathrm{Ch}(B)$.

We will show that Φ is a homeomorphism from M_B onto M_A. By Lemma 4.1 T^{-1} is well-defined from B onto A and satisfies the condition

$$\mathrm{Ran}_\pi(T^{-1}FT^{-1}G) = \mathrm{Ran}_\pi(FG) \qquad F, G \in B\,.$$

As in the same way as the case of T we see that there exist maps Φ' from M_A into M_B and N' from M_A with the values in $\{-1, 1\}$ such that

$$\widehat{T^{-1}F} = N'\hat{F} \circ \Phi'$$

holds for every $F \in B$. It follows that

$$\hat{f} = \widehat{T^{-1}Tf} = N'(N \circ \Phi')\hat{f} \circ (\Phi \circ \Phi')$$

holds for every $f \in A$. Since $|N| = 1$ and $|N'| = 1$ on M_B and M_A respectively, we have that $\Phi \circ \Phi'(x) = x$ holds for every $x \in M_A$. Applying the similar argument to $T \circ T^{-1}$ we have that $\Phi' \circ \Phi(y) = y$ holds for every $y \in M_B$. It follows that Φ and Φ' are bijections and $\Phi' = \Phi^{-1}$. Let $y_\alpha \to y$ be a converging net in M_B. Then

$$|\hat{f}(\Phi(y_\alpha))| = |\widehat{Tf}(y_\alpha)| \to |\widehat{Tf}(y)| = |\hat{f}(\Phi(y))|$$

holds for every $f \in A$ since $\widehat{Tf}$ is continuous and $|N| = 1$. Since the weak topology on M_A induced by the set of functions $\{|\hat{f}| : f \in A\}$ coincides with the original topology, we see that $\Phi(y_\alpha) \to \Phi(y)$ holds. Thus Φ is a continuous map. In the same way we see that $\Phi' = \Phi^{-1}$ is continuous, and so we see that Φ is a homeomorphism from M_B onto M_A.

Finally we show that N is continuous. Let $y_\alpha \to y$ be a converging net in M_B. Let $f \in A$ be such that $\hat{f} \circ \Phi(y) \neq 0$. Such an f exists since T is a surjection and $|N| = 1$ on M_B. We may assume that $\hat{f} \circ \Phi(y_\alpha) \neq 0$ since $\hat{f} \circ \Phi$ is continuous. Then we have that

$$N(y_\alpha) = \frac{\widehat{Tf}(y_\alpha)}{\hat{f} \circ \Phi(y_\alpha)} \to \frac{\widehat{Tf}(y)}{\hat{f} \circ \Phi(y)} = N(y)\,.$$

Thus we see that N is continuous on M_B. □

We show a proof of Corollary here.

A PROOF OF COROLLARY. We consider the case where A contains an approximate identity. (In a way similar we can prove the conclusion for the case where B contains an approximate identity by considering T^{-1} instead of T.) Let $\{e_\alpha\}$ be an approximate identity, that is $\{e_\alpha\}$ is a net in A and $\|e_\alpha f - f\|_{\infty(X)} \to 0$. We show that $\hat{A} \circ \Phi = \hat{B}$. It will follow that $S : \hat{A} \to \hat{B}$ defined by $S(\hat{f}) = \hat{f} \circ \Phi$ for $f \in A$ gives an isometrical and algebraical isomorphism from $\hat{A}$ onto $\hat{B}$. Since the Gelfand transforms are isometries for uniformly closed algebras, we see that A is isometrically isomorphic to B as Banach algebras. Let $f \in A$. Then we have

$$\|\widehat{Te_\alpha}\widehat{Tf} - \hat{f}\circ\Phi\|_{\infty(M_B)} = \|N\widehat{e_\alpha}\circ\Phi N\hat{f}\circ\Phi - \hat{f}\circ\Phi\|_{\infty(M_B)}$$
$$= \|\widehat{e_\alpha}\circ\Phi\hat{f}\circ\Phi - \hat{f}\circ\Phi\|_{\infty(M_B)} = \|\widehat{e_\alpha}\hat{f} - \hat{f}\|_{\infty(M_A)} = \|e_\alpha f - f\|_{\infty(X)} \to 0$$

since the Gelfand transform is an isometry here. Thus we have that $\hat{f}\circ\Phi \in \hat{B}$ since $Te_\alpha, Tf \in B$. Thus by Lemma 4.7 we have that

$$\hat{B} = N\hat{A}\circ\Phi \subset N\hat{B}\,,$$

so $\hat{B} \subset N\hat{B}$. Since $N^2 = 1$ we also have that $N\hat{B} \subset N^2\hat{B} = \hat{B}$, so $\hat{B} = N\hat{B}$. It follows that $\hat{A}\circ\Phi = N^2\hat{A}\circ\Phi = N\hat{B} = \hat{B}$. □

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