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Some Dynamic Properties of the Modified Negative Slope Algorithm

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Abstract. S. Ferenczi and L. F. C. da Rocha introduced an algorithm which is slightly different form of the negative slope algorithm as the normalized multiplicative algorithm deduced from three interval exchange transformations. It has the form that the ceiling value is taken in the case of $x + y > 1$. We call this algorithm as the "modified negative slope algorithm". In this paper, the author shows that the modified negative slope algorithm is weak Bernoulli with respect to the absolutely continuous invariant measure and gives an algebraic characterization of periodic orbits of this algorithm using the natural extension method.

1. Introduction

The original negative slope algorithm was introduced by S. Ferenczi, C. Holton, and L. Zamboni [1] [2] as an approximation algorithm arising from three interval exchange transformations. Recently, S. Ferenczi and L. F. C. da Rocha [5] showed the existence of an absolutely continuous invariant measure which is ergodic. The author and H. Nakada [7] showed that the original negative slope algorithm is weak Bernoulli with respect to the absolutely continuous invariant measure. Then the author and S. Ito [6] characterized purely periodic points of the negative slope algorithm as pairs of quadratic numbers. S. Ferenczi and L. F. C. da Rocha [5] also introduced the modified negative slope algorithm which has a different normalizing parameter from the negative slope algorithm and showed its ergodic properties. This gave better proofs of some results on the theory of three interval exchange transformations developed in [3] [4]. In this paper, we show that the modified negative slope algorithm satisfies Yuri's conditions [10] for a map to be weak Bernoulli. This implies that the modified negative slope algorithm satisfies Rohlin's entropy formula and is weak Bernoulli with respect to the absolutely continuous invariant measure given in [5]. Then we can compute the explicit value of the entropy of the modified negative slope algorithm by Rohlin's entropy formula. We see that the invariant measure is derived from a four dimensional representation of the natural extension of the modified negative slope algorithm and obtain the exponent constant of the denominator of the n-th convergent of simultaneous approximations

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arising from the modified negative slope algorithm. Finally, we characterize purely periodic points of the modified negative slope algorithm as pairs of quadratic numbers.

In §2, we give the definition of the modified negative slope algorithm. In §3, we show some lemmas for local inverse of the modified negative slope algorithm and prove that the modified negative slope algorithm is weak Bernoulli by using Yuri's conditions [10] for a map to be weak Bernoulli. In §4, we construct four dimensional natural extension of the modified negative slope algorithm and give the absolutely continuous invariant measure of it. Then we compute the entropy of the modified negative slope algorithm by Rohlin's entropy formula. Finally, in §5, we characterize purely periodic points of the modified negative slope algorithm by using the natural extension method originally introduced by [8] for a class of continued fraction algorithms.

2. Definitions and basic notions of the modified negative slope algorithm

Let's define a map S on the unit square, which is called the modified negative slope algorithm. Let $\mathbb{X} = [0,\ 1]^2 \setminus \{(x, y) \mid x + y = 1\}$, we define

$$S(x, y) = \begin{cases} \left(\left\lceil \frac{y}{(x+y)-1} \right\rceil - \frac{y}{(x+y)-1},\ \left\lceil \frac{x}{(x+y)-1} \right\rceil - \frac{x}{(x+y)-1}\right) & \text{if } x+y>1 \\ \left(\frac{1-y}{1-(x+y)} - \left\lfloor \frac{1-y}{1-(x+y)} \right\rfloor,\ \frac{1-x}{1-(x+y)} - \left\lfloor \frac{1-x}{1-(x+y)} \right\rfloor\right) & \text{if } x+y<1\,. \end{cases}$$

We put

$$n(x, y) = \begin{cases} \left\lceil \frac{y}{(x+y)-1} \right\rceil - 1 & \text{if } x+y>1 \\ \left\lfloor \frac{1-y}{1-(x+y)} \right\rfloor & \text{if } x+y<1\,, \end{cases}$$

$$m(x, y) = \begin{cases} \left\lceil \frac{x}{(x+y)-1} \right\rceil - 1 & \text{if } x+y>1 \\ \left\lfloor \frac{1-x}{1-(x+y)} \right\rfloor & \text{if } x+y<1\,, \end{cases}$$

and

$$\varepsilon(x, y) = \begin{cases} -1 & \text{if } x+y>1 \\ +1 & \text{if } x+y<1\,. \end{cases}$$

Then we see that $n(x, y) \geq 1$ and $m(x, y) \geq 1$ for all $(x, y) \in \mathbb{X}$.

We put

$$\begin{cases} n_k(x,y) = n(S^{k-1}(x,y)) \\ m_k(x,y) = m(S^{k-1}(x,y)) \\ \varepsilon_k(x,y) = \varepsilon(S^{k-1}(x,y)) \end{cases}$$

for $k \geq 1$. Then we have a sequence

$$((\varepsilon_1(x,y), n_1(x,y), m_1(x,y)), (\varepsilon_2(x,y), n_2(x,y), m_2(x,y)), \dots)$$

for each $(x, y) \in \mathbb{X}$. In §3, we see in Lemma 3.8 that if $(x, y) \neq (x', y') \in \mathbb{X}$, then there exists $k \geq 1$ such that

$$(\varepsilon_k(x,y), n_k(x,y), m_k(x,y)) \neq (\varepsilon_k(x'y'), n_k(x',y'), m_k(x',y')) . \tag{1}$$

Now we introduce the projective representation of T. We put

$$A_{(+1,n,m)} = \begin{pmatrix} n & n-1 & 1-n \\ m-1 & m & 1-m \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$A_{(-1,n,m)} = \begin{pmatrix} n+1 & n & -(n+1) \\ m & m+1 & -(m+1) \\ 1 & 1 & -1 \end{pmatrix}$$

for $m, n \geq 1$. Then we see

$$A^{-1}_{(+1,n,m)} = \begin{pmatrix} 1 & 0 & n-1 \\ 0 & 1 & m-1 \\ 1 & 1 & n+m-1 \end{pmatrix}$$

and

$$A^{-1}_{(-1,n,m)} = \begin{pmatrix} 0 & -1 & m+1 \\ -1 & 0 & n+1 \\ -1 & -1 & n+m+1 \end{pmatrix} .$$

We identify (x, y) to $\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha \end{pmatrix}$ for $\alpha \neq 0$. Then $S(x, y)$ is identified to

$$A_{(\varepsilon_1(x,y),n_1(x,y),m_1(x,y))} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

and its local inverse is given by

$$A^{-1}_{(\varepsilon_1(x,y),n_1(x,y),m_1(x,y))} .$$

In this way, we get a representation of (x, y) by

$$A^{-1}_{(\varepsilon_1,n_1,m_1)} A^{-1}_{(\varepsilon_2,n_2,m_2)} A^{-1}_{(\varepsilon_3,n_3,m_3)} \cdots .$$

For a given sequence $((\varepsilon_1, n_1, m_1),\ (\varepsilon_2, n_2, m_2), \dots, (\varepsilon_k, n_k, m_k))$, we define a cylinder set of length k by

$$\begin{aligned}&\langle(\varepsilon_1, n_1, m_1),\ (\varepsilon_2, n_2, m_2), \dots, (\varepsilon_k, n_k, m_k)\rangle\\ &\quad = \{(x, y) \mid (\varepsilon_i(x, y), n_i(x, y), m_i(x, y)) = (\varepsilon_i, n_i, m_i), 1 \le i \le k\}\,.\end{aligned}$$

In the sequel, we simply denote by Δ_k a cylinder set of length $k \ge 1$. For $(x, y) \in \Delta_k$, $S^k(x, y)$ is identified to

$$A_{(\varepsilon_k,n_k,m_k)} \cdots A_{(\varepsilon_1,n_1,m_1)} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

We denote its local inverse

$$A^{-1}_{(\varepsilon_1,n_1,m_1)} \cdots A^{-1}_{(\varepsilon_k,n_k,m_k)}$$

by Ψ_{Δ_k}.

Since

$$\begin{aligned}&\left\{\left(1 - \frac{y}{(x+y)-1}, 1 - \frac{x}{(x+y)-1}\right) : (x, y) \in \mathbb{X},\ x + y > 1\right\}\\ &\quad = \{(\alpha, \beta) : \alpha < 0,\ \beta < 0\}\,,\end{aligned}$$

$$\begin{aligned}&\left\{\left(\frac{1-y}{1-(x+y)}, \frac{1-x}{1-(x+y)}\right) :\ (x, y) \in \mathbb{X},\ x + y < 1\right\}\\ &\quad = \{(\alpha, \beta) : \alpha \ge 1,\ \beta \ge 1\}\,,\end{aligned}$$

we see that

$$S^l\{(x, y) \in \mathbb{X} : \varepsilon_k(x, y) = \varepsilon_k,\ n_k(x, y) = n_k, m_k(x, y) = m_k, 1 \le k \le l\} = \mathbb{X} \tag{2}$$

except for a set of Lebesgue measure 0 for any $\{(\varepsilon_k, n_k, m_k), 1 \le k \le l\}$, $\varepsilon_k = \pm 1, n_k, m_k \ge 1$.

Next we define what means that iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ stops.

DEFINITION 2.1. We define k-th iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ by $(x_k, y_k) = S^k(x, y)$. Then we say that iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ stops if there exists $k_0 \ge 0$ such that $x_{k_0} = 0$ or $y_{k_0} = 0$ or $x_{k_0} + y_{k_0} = 1$.

This implies that iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ stops if there exists $k_0 \ge 0$ s.t. $(x_{k_0}, y_{k_0}) \in \partial\mathbb{X}$. From this definition, we get the following propositions.

PROPOSITION 2.2 (Ishimura-Ito [6]). *If iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ stops, then (x, y) satisfies one of the following equations.*

$$(p+1)x + py = q$$
$$px + (p+1)y = q$$
$$px + py = q$$

for some integers $0 \le q \le 2p$.

See Proposition 2.5 of [6] for the proof. The next result gives a sufficient condition for the third equation in Proposition 2.2.

PROPOSITION 2.3 (Ishimura-Ito [6]). *If $(x, y) \in \mathbb{X}$ satisfies the following equation*

$$px + py = q$$

for any integers $0 \le q \le 2p$, then there exists $N > 0$ such that the sequence $(S^k(x, y) : k \ge 0)$ stops at $k = N$.

See Proposition 2.6 of [6] for the proof.

REMARK 2.4. From Theorem 3.3 of [5], we see that for $n_i, m_i \ge 1$, $i \ge 1$ and for any sequence $((\varepsilon_i, n_i, m_i), i \ge 1)$, there exists $(x, y) \in \mathbb{X}$ such that $(\varepsilon_i(x, y), n_i(x, y), m_i(x, y)) = (\varepsilon_i, n_i, m_i)$ unless there exists $k \ge 1$ such that either $(\varepsilon_i, m_i) = (\pm 1, 1)$ for all $i \ge k$ or $(\varepsilon_i, n_i) = (\pm 1, 1)$ for all $i \ge k$.

3. Some ergodic properties of the modified negative slope algorithm

In this section, we show that the modified negative slope algorithm is weak Bernoulli by using Yuri's conditions. See §3 of [7] for the summary of multidimensional maps for Yuri's conditions. Here we only show Yuri's conditions (C.1)–(C.9).

3.1. Yuri's conditions and Rényi cylinders. First we fix a constant $L \ge 1$ and define the set of "Rényi cylinders" for a map S of a bounded domain X of $\mathbb{R}^d$ onto itself with its countable partition $\{X_a : a \in I\}$ by

$$R(S) = \left\{ \langle a_1, \dots, a_k \rangle : \sup_{x \in S^k \langle a_1, \dots, a_k \rangle} |\det D\Psi_{\langle a_1, \dots, a_k \rangle}(x)| \right.$$
$$\left. \le L \cdot \inf_{x \in S^k \langle a_1, \dots, a_k \rangle} |\det D\Psi_{\langle a_1, \dots, a_k \rangle}(x)|, k \ge 1 \right\}$$

where $\langle a_1, \dots, a_k \rangle$ indicates a cylinder set of length k for $a_1, \dots, a_k \in I$. Moreover we put

$$\mathcal{D}_k = \{\langle a_1, \dots, a_k \rangle : \langle a_1, \dots, a_i \rangle \notin R(S) \text{ for } 1 \le i \le k\},$$

$$\mathbf{D}_k = \bigcup_{\langle a_1, \dots, a_k \rangle \in \mathcal{D}_k} \langle a_1, \dots, a_k \rangle,$$

$$\mathcal{B}_k = \{\langle a_1, \dots, a_k \rangle \in R(S) : \langle a_1, \dots, a_{k-1} \rangle \in \mathcal{D}_{k-1}\},$$

and

$$\mathbf{B}_k = \bigcup_{\langle a_1, \dots, a_k \rangle \in \mathcal{B}_k} \langle a_1, \dots, a_k \rangle .$$

Yuri's conditions

(C.1) (S, Q) separates points, that is, for any $x \neq x' \in X$ there exists $n \geq 0$ such that $S^n(x)$ and $S^n(x')$ are not the same elements in Q where Q denotes the countable partitions on X arising from S.

(C.2) For each j, $0 \leq j \leq N$, there exists $\langle a_1, \dots, a_{s_j} \rangle \subset U_j$ such that $\langle a_1, \dots, a_{s_j} \rangle \in R(S)$ and $S^{s_j} \langle a_1, \dots, a_{s_j} \rangle = X$.

(C.3) If $\langle a_1, \dots, a_k \rangle \in R(S)$, then $\langle b_1, \dots, b_l, a_1, \dots, a_k \rangle \in R(S)$ unless $\langle b_1, \dots, b_l, a_1, \dots, a_k \rangle$ is a set of Lebesgue measure 0.

(C.4)
$$\sum_{k=1}^{\infty} \lambda(\mathbf{D}_k) < \infty$$

where $\lambda(A)$ denotes the d-dimensional Lebesgue measure of a Borel set A of $\mathbb{R}^d$.

(C.4)* $\displaystyle\sum_{k=1}^{\infty} \lambda(\mathbf{D}_k) \cdot \log k < \infty.$

(C.5) For any $l \geq 1$,

$$\sum_{k=0}^{\infty} \left(\sum_{\langle a_1, \dots, a_k \rangle \in \mathcal{D}_k} \left(\sup_{x \in S^k \langle a_1, \dots, a_k \rangle \cap \left(\bigcup_{j=1}^{l} \mathbf{B}_j \right)} |\det D\Psi_{\langle a_1, \dots, a_k \rangle}(x)| \right) \right) < +\infty .$$

(C.6) $\sharp \mathcal{D}_1 < \infty$.

(C.7) There exists a positive integer l such that for all $k > 0$ and all $\langle a_1, \dots, a_k \rangle \in \mathcal{D}_k$,

$$\frac{\sup_{x \in S^k \langle a_1, \dots, a_k \rangle} |\det D\Psi_{\langle a_1, \dots, a_k \rangle}(x)|}{\inf_{x \in S^k \langle a_1, \dots, a_k \rangle} |\det D\Psi_{\langle a_1, \dots, a_k \rangle}(x)|} = O(k^l) .$$

(C.8) $\log |\det DS(\cdot)|$ is Lebesgue integrable.

(C.9) there exists a positive integer k_0 such that if $\langle a_1, \dots, a_k \rangle \in \mathcal{D}_k^c$ and $\langle a_2, \dots, a_k \rangle \in \mathcal{D}_{k-1}$, then

$$\langle a_1, \dots, a_k \rangle \subset \bigcup_{j=1}^{k_0} \mathbf{B}_j .$$

For the modified negative slope algorithm S, we define the set $R(S)$ by

$$\begin{aligned} R(S) = \{ & \langle (\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \dots, (\varepsilon_k, n_k, m_k) \rangle \mid \\ & (\varepsilon_k, n_k, m_k) \neq (\pm 1, 1, 1) \\ & \text{or for } k \geq 2 \ (\varepsilon_k, n_k, m_k) = (+1, 1, 1) \ \text{ and } \ (\varepsilon_{k-1}, n_{k-1}, m_{k-1}) \neq (+1, 1, 1) , \end{aligned}$$

$$(\varepsilon_k, n_k, m_k) = (-1, 1, 1) \quad \text{and} \quad (\varepsilon_{k-1}, n_{k-1}, m_{k-1}) \neq (-1, 1, 1)\} .$$

Then we see that $R(S)$ satisfies the definition of Rényi cylinders in Lemma 3.7 and the modified negative slope algorithm S satisfies Yuri's conditions (C.1)–(C.9) with (C.4)* in §3.2. To show these facts, we prepare some lemmas in the following.

3.2. Some properties for Ψ_{Δ_k}. We put

$$\Psi_{\Delta_k} = \begin{pmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{pmatrix}$$

for any cylinder Δ_k, $k \geq 1$. Then we have some lemmas for $p_i^{(k)}$, $r_i^{(k)}$, $q_i^{(k)}$, $i = 1, 2, 3$, $k \geq 1$ in the following.

LEMMA 3.1. *For entries of Ψ_{Δ_k}, we have*

$$\begin{cases} p_1^{(k)} = p_2^{(k)} + 1 \\ r_1^{(k)} = r_2^{(k)} - 1 \\ q_1^{(k)} = q_2^{(k)} . \end{cases}$$

PROOF. By simple calculation, we see that

$$A^{-1}_{(\pm 1, n, m)} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} .$$

Then we see that

$$A^{-1}_{(\varepsilon_1, n_1, m_1)} \cdots A^{-1}_{(\varepsilon_k, n_k, m_k)} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

for $k \geq 1$. Therefore, we obtain

$$\begin{pmatrix} p_1^{(k)} - p_2^{(k)} \\ r_1^{(k)} - r_2^{(k)} \\ q_1^{(k)} - q_2^{(k)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} .$$

□

LEMMA 3.2. *For all $k \geq 1$, we have $q_3^{(k)} > 0$ and $2q_1^{(k)} + q_3^{(k)} > 0$.*

PROOF. From the previous lemma, we see that

$$
(q_1^{(k)},\ q_2^{(k)},\ q_3^{(k)})
= \begin{cases}
\big(q_1^{(k-1)}+q_3^{(k-1)},\ q_2^{(k-1)}+q_3^{(k-1)},\ (n_k+m_k-2)q_1^{(k-1)}+(n_k+m_k-1)q_3^{(k-1)}\big) \\
\hfill \text{if } \ \varepsilon_k=+1 \\
\big(-q_2^{(k-1)}-q_3^{(k-1)},\ -q_1^{(k-1)}-q_3^{(k-1)},\ (n_k+m_k+2)q_1^{(k-1)}+(n_k+m_k+1)q_3^{(k-1)}\big) \\
\hfill \text{if } \ \varepsilon_k=-1\,.
\end{cases}
$$

We see that $q_3^{(1)} > 0$ and $2q_1^{(1)} + q_3^{(1)} > 0$ from §1. Assume that $q_3^{(k-1)} > 0$ and $2q_1^{(k-1)} + q_3^{(k-1)} > 0$ for $k \geq 2$. Then, from the above relations and Lemma 3.1, we have $q_3^{(k)} > 0$ and

$$
2q_1^{(k)}+q_3^{(k)} = \begin{cases}
(n_k+m_k)(q_1^{(k-1)}+q_3^{(k-1)})+q_3^{(k-1)} & \text{if } \ \varepsilon_k=+1 \\
(n_k+m_k-2)(q_1^{(k-1)}+q_3^{(k-1)})+(2q_1^{(k-1)}+q_3^{(k-1)}) & \text{if } \ \varepsilon_k=-1
\end{cases}
$$
$$
> 0\,.
$$

This is the assertion of this lemma. □

Then we have similar results for $p_i^{(k)}$ and $r_i^{(k)}$, $i = 1, 2, 3$.

LEMMA 3.3. *For $k \geq 1$, we have $p_3^{(k)} \geq 0$ and $2p_i^{(k)} + p_3^{(k)} > 0$, $i = 1, 2$.*

PROOF. We see that $p_3^{(1)} \geq 0$ and $2p_1^{(1)} + p_3^{(1)} > 0$ from §1. Assume that $p_3^{(k-1)} \geq 0$ and $2p_1^{(k-1)} + p_3^{(k-1)} > 0$ for $k \geq 2$. Then, from Lemma 3.1, we have

$$
p_3^{(k)} = \begin{cases}
(n_k+m_k-2)(p_1^{(k-1)}+p_3^{(k-1)})+p_3^{(k-1)}+n-1 & \text{if } \ \varepsilon_k=+1 \\
(n_k+m_k)(p_1^{(k-1)}+p_3^{(k-1)})+(2p_1^{(k-1)}+p_3^{(k-1)})+m+1 & \text{if } \ \varepsilon_k=-1
\end{cases}
$$
$$
\geq 0
$$

and

$$
2p_1^{(k)}+p_3^{(k)} = \begin{cases}
(n_k+m_k)(p_1^{(k-1)}+p_3^{(k-1)})+p_3^{(k-1)}+n & \text{if } \ \varepsilon_k=+1 \\
(n_k+m_k-2)(p_1^{(k-1)}+p_3^{(k-1)})+(2p_1^{(k-1)}+p_3^{(k-1)})+m & \text{if } \ \varepsilon_k=-1
\end{cases}
$$
$$
> 0\,.
$$

This is the assertion of this lemma. □

By the same way, we have $r_3^{(k)} \geq 0$ and $2r_i^{(k)} + r_3^{(k)} > 0$ for $k \geq 1$, $i = 1, 2$. Moreover, we see that the signs of $p_i^{(k)}, r_i^{(k)}, q_i^{(k)}$, $i = 1, 2$ are same for all $k \geq 1$.

LEMMA 3.4. *For* $(x, y) \in \mathbb{X}$ *and* $k \geq 1$, *we have*

$$x + y = \frac{(p_2^{(k)} + r_2^{(k)})(x_k + y_k) + (p_3^{(k)} + r_3^{(k)})}{q_2^{(k)}(x_k + y_k) + q_3^{(k)}}$$

$$\text{and} \qquad (p_2^{(k)} + r_2^{(k)})q_3^{(k)} - (p_3^{(k)} + r_3^{(k)})q_2^{(k)} = -1\,.$$

PROOF. By taking the determinant of Ψ_{Δ_k}, we have

$$\begin{vmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{vmatrix} = p_1^{(k)} \begin{vmatrix} r_2^{(k)} & r_3^{(k)} \\ q_2^{(k)} & q_3^{(k)} \end{vmatrix} - p_2^{(k)} \begin{vmatrix} r_1^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_3^{(k)} \end{vmatrix} + p_3^{(k)} \begin{vmatrix} r_1^{(k)} & r_2^{(k)} \\ q_1^{(k)} & q_2^{(k)} \end{vmatrix}.$$

From Lemma 3.1, the right hand side is equal to

$$(p_2^{(k)} + 1) \begin{vmatrix} r_2^{(k)} & r_3^{(k)} \\ q_2^{(k)} & q_3^{(k)} \end{vmatrix} - p_2^{(k)} \begin{vmatrix} r_2^{(k)} - 1 & r_3^{(k)} \\ q_1^{(k)} & q_3^{(k)} \end{vmatrix} + p_3^{(k)} \begin{vmatrix} r_2^{(k)} - 1 & r_2^{(k)} \\ q_1^{(k)} & q_2^{(k)} \end{vmatrix}.$$

Since $\det \Psi_{\Delta_k} = 1$, we have

$$(r_2^{(k)} q_3^{(k)} - r_3^{(k)} q_2^{(k)}) + (p_2^{(k)} q_3^{(k)} - p_3^{(k)} q_2^{(k)}) = 1\,. \tag{3}$$

Substituting $p_1^{(k)} = p_2^{(k)} + 1$, $r_1^{(k)} = r_2^{(k)} - 1$ and $q_1^{(k)} = q_2^{(k)}$ for (3), we see that

$$(r_1^{(k)} q_3^{(k)} - r_3^{(k)} q_1^{(k)}) + (p_1^{(k)} q_3^{(k)} - p_3^{(k)} q_1^{(k)}) = 1\,. \tag{4}$$

From (3) and (4), we have

$$\frac{p_1^{(k)} + r_1^{(k)}}{q_1^{(k)}} = \frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} = \frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} + \frac{1}{q_2^{(k)} q_3^{(k)}}\,. \tag{5}$$

Then we see that

$$\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha \end{pmatrix} = \begin{pmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \\ 1 \end{pmatrix}$$

for $\alpha \neq 0$. Therefore, we obtain

$$x = \frac{p_1^{(k)} x_k + p_2^{(k)} y_k + p_3^{(k)}}{q_1^{(k)} x_k + q_2^{(k)} y_k + q_3^{(k)}}\,, \tag{6}$$

$$y = \frac{r_1^{(k)} x_k + r_2^{(k)} y_k + r_3^{(k)}}{q_1^{(k)} x_k + q_2^{(k)} y_k + q_3^{(k)}}\,. \tag{7}$$

Since $q_3^{(k)} > 0$ for $k \geq 1$, the denominators of the above two equations are not equal to 0. From $p_1^{(k)} = p_2^{(k)} + 1$, $r_1^{(k)} = r_2^{(k)} - 1$ and $q_1^{(k)} = q_2^{(k)}$, we have

$$x + y = \frac{(p_2^{(k)} + r_2^{(k)})(x_k + y_k) + (p_3^{(k)} + r_3^{(k)})}{q_2^{(k)}(x_k + y_k) + q_3^{(k)}} . \tag{8}$$

This is the assertion of this lemma. □

LEMMA 3.5. *For $k \geq 1$, we have*

$$\max\left\{ \frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}}, \frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} \right\} < 2 .$$

PROOF. From (5) of Lemma 3.4, we see that

$$\begin{cases} \dfrac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} < \dfrac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} & \text{if } q_2^{(k)} > 0 \\ \dfrac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} < \dfrac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} & \text{if } q_2^{(k)} < 0 . \end{cases}$$

(I) Suppose that $q_2^{(k-1)} > 0$ and $\frac{p_2^{(k-1)} + r_2^{(k-1)}}{q_2^{(k-1)}} < 2$ for $k \geq 2$. Then we have the following.

(i) If $\varepsilon_k = +1$, which means $q_2^{(k)} > 0$, then we have

$$\frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} = \frac{p_2^{(k-1)} + r_2^{(k-1)} + p_3^{(k-1)} + r_3^{(k-1)}}{q_2^{(k-1)} + q_3^{(k-1)}} .$$

Since

$$\Psi_{\Delta_{k-1}}(0, 1) = \left(\frac{p_2^{(k-1)} + p_3^{(k-1)}}{q_2^{(k-1)} + q_3^{(k-1)}}, \frac{r_2^{(k-1)} + r_3^{(k-1)}}{q_2^{(k-1)} + q_3^{(k-1)}} \right) \neq (1, 1) ,$$

then we obtain

$$\frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} < 2 .$$

(ii) If $\varepsilon_k = -1$, which means $q_2^{(k)} < 0$, then we have

$$\begin{aligned} \frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} &= \frac{-2(p_2^{(k-1)} + r_2^{(k-1)}) + (n_k + m_k + 1)(p_3^{(k-1)} + r_3^{(k-1)})}{-2q_2^{(k-1)} + (n_k + m_k + 1)q_3^{(k-1)}} \\ &= \frac{p_3^{(k-1)} + r_3^{(k-1)}}{q_3^{(k-1)}} - \frac{2}{q_3^{(k-1)}(-2q_2^{(k)} + (n_k + m_k + 1)q_3^{(k-1)})} \end{aligned}$$

$$< \frac{p_3^{(k-1)} + r_3^{(k-1)}}{q_3^{(k-1)}} < 2$$

from Lemma 3.2 and (5) of Lemma 3.4.

(II) Suppose that $q_2^{(k-1)} < 0$ and $\frac{p_3^{(k-1)}+r_3^{(k-1)}}{q_3^{(k-1)}} < 2$ for $k \geq 2$, then we have the following.

(i) If $\varepsilon_k = +1$, which means $q_2^{(k)} > 0$, then we have

$$\frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} = \frac{p_2^{(k-1)} + r_2^{(k-1)} + p_3^{(k-1)} + r_3^{(k-1)}}{q_2^{(k-1)} + q_3^{(k-1)}} .$$

Since

$$\Psi_{\Delta_{k-1}}(0, 1) = \left(\frac{p_2^{(k-1)} + p_3^{(k-1)}}{q_2^{(k-1)} + q_3^{(k-1)}}, \frac{r_2^{(k-1)} + r_3^{(k-1)}}{q_2^{(k-1)} + q_3^{(k-1)}} \right) \neq (1, 1) ,$$

then we obtain

$$\frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} < 2 .$$

(ii) If $\varepsilon_k = -1$, which means $q_2^{(k)} < 0$, then we have

$$\begin{aligned} \frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} &= \frac{p_3^{(k-1)} + r_3^{(k-1)}}{q_3^{(k-1)}} - \frac{2}{q_3^{(k)}(-2q_2^{(k)} + (n_k + m_k + 1)q_3^{(k-1)})} \\ &< \frac{p_3^{(k-1)} + r_3^{(k-1)}}{q_3^{(k-1)}} < 2 \end{aligned}$$

from Lemma 3.2 and (5) of Lemma 3.4. This is the assertion of this lemma. □

LEMMA 3.6. *For any sequence* $((\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \ldots, (\varepsilon_k, n_k, m_k))$, $\varepsilon_i = \pm 1, n_i, m_i \geq 1,\ 1 \leq i \leq k$, *we see that*

(i) $S^k(\Delta_k) = \mathbb{X}$,

(ii)

$$|\det D\Psi_{\Delta_k}(x, y)| = \frac{1}{(q_1^{(k)}x + q_2^{(k)}y + q_3^{(k)})^3} .$$

PROOF. It is an easy consequence of induction and calculation, respectively (see also F. Schweiger [9], Proposition 2 for (ii)). □

From the above lemmas, we can show that $R(S)$ is the set of Rényi cylinders.

LEMMA 3.7. *If $\Delta_k \in R(S)$, then*

$$\sup_{(x,y)\in\mathbb{X}} |\det D\Psi_{\Delta_k}(x,y)| \leq 5^3 \inf_{(x,y)\in\mathbb{X}} |\det D\Psi_{\Delta_k}(x,y)|\,.$$

PROOF. (Case 1) For $\Delta_k = \langle(\varepsilon_1, n_1, m_1), \dots, (\varepsilon_k, n_k, m_k)\rangle$, assume that $(\varepsilon_k, n_k, m_k) \neq (\pm 1, 1, 1)$, then we see that

$$\frac{\sup_{(x,y)\in\mathbb{X}} |\det\, D\Psi_{\Delta_k}(x,y)|}{\inf_{(x,y)\in\mathbb{X}} |\det\, D\Psi_{\Delta_k}(x,y)|} = \begin{cases} \left(\dfrac{2q_1^{(k)} + q_3^{(k)}}{q_3^{(k)}}\right)^3 & \text{if } q_1^{(k)} > 0 \\ \left(\dfrac{q_3^{(k)}}{2q_1^{(k)} + q_3^{(k)}}\right)^3 & \text{if } q_1^{(k)} < 0\,. \end{cases}$$

(i) If $q_1^{(k)} > 0$ and $\varepsilon_k = -1$ then we see that

$$0 < q_1^{(k)} = -q_1^{(k-1)} - q_3^{(k-1)}\,.$$

This is the contradiction to Lemma 3.2. Then it implies $\varepsilon_k = +1$ for $q_1^{(k)} > 0$. So we have

$$q_3^{(k)} - q_1^{(k)} = (n_k + m_k - 3)(q_1^{(k-1)} + q_3^{(k-1)}) + q_3^{(k-1)} > 0\,.$$

From this fact, we obtain

$$\frac{2q_1^{(k)} + q_3^{(k)}}{q_3^{(k)}} < \frac{3q_3^{(k)}}{q_3^{(k)}} = 3\,.$$

(ii) If $q_1^{(k)} < 0$ and $\varepsilon_k = +1$ then we see that

$$0 > q_1^{(k)} = q_1^{(k-1)} + q_3^{(k-1)}\,.$$

This is the contradiction to Lemma 3.2. Then it implies that $\varepsilon_k = -1$ for $q_1^{(k)} < 0$. So we see that

$$\begin{aligned} \frac{q_3^{(k)}}{2q_1^{(k)} + q_3^{(k)}} &= \frac{(n_k + m_k + 2)q_1^{(k-1)} + (n_k + m_k + 1)q_3^{(k-1)}}{(n_k + m_k)q_1^{(k-1)} + (n_k + m_k - 1)q_3^{(k-1)}} \\ &= 1 + \frac{2q_1^{(k-1)} + 2q_3^{(k-1)}}{(n_k + m_k - 2)(q_1^{(k-1)} + q_3^{(k-1)}) + (2q_1^{(k-1)} + q_3^{(k-1)})} \\ &< 1 + \frac{2q_1^{(k-1)} + 2q_3^{(k-1)}}{3q_1^{(k-1)} + 2q_3^{(k-1)}}\,. \end{aligned}$$

From Lemma 3.2, we obtain

$$\frac{2q_1^{(k-1)} + 2q_3^{(k-1)}}{3q_1^{(k-1)} + 2q_3^{(k-1)}} < 2\,.$$

(Case 2) For $\Delta_k = \langle(\varepsilon_1, n_1, m_1), \ldots, (\varepsilon_k, n_k, m_k)\rangle$, assume $(\varepsilon_k, n_k, m_k) = (+1, 1, 1)$ and $(\varepsilon_{k-1}, n_{k-1}, m_{k-1}) \neq (+1, 1, 1)$, then we see the following two cases:

(i) If $\varepsilon_{k-1} = +1$ and $n_{k-1} + m_{k-1} \geq 3$, then we have

$$(q_1^{(k-1)}, q_3^{(k-1)}) = (q_1^{(k-2)} + q_3^{(k-2)}, (n_{k-1} + m_{k-1} - 2)q_1^{(k-2)} + (n_{k-1} + m_{k-1} - 1)q_3^{(k-2)}) .$$

(ii) If $\varepsilon_{k-1} = -1$, then we have

$$(q_1^{(k-1)}, q_3^{(k-1)}) = (-q_1^{(k-2)} - q_3^{(k-2)}, (n_{k-1} + m_{k-1} + 2)q_1^{(k-2)} + (n_{k-1} + m_{k-1} + 1)q_3^{(k-2)}) .$$

Thus we see that $q_1^{(k)} = q_1^{(k-1)} + q_3^{(k-1)} > 0$, $q_3^{(k)} = q_3^{(k-1)} > 0$ and $q_1^{(k-1)} < q_3^{(k-1)}$ for both cases. So we have

$$\frac{\sup_{(x,y)\in\mathbb{X}} |\det\, D\Psi_{\Delta_k}(x, y)|}{\inf_{(x,y)\in\mathbb{X}} \left|\det\, D\Psi_{\Delta_k}(x, y)\right|} = \left(\frac{2q_1^{(k)} + q_3^{(k)}}{q_3^{(k)}}\right)^3 = \left(\frac{2q_1^{(k-1)} + 3q_3^{(k-1)}}{q_3^{(k-1)}}\right)^3 < 5^3 .$$

(Case 3) For $\Delta_k = \langle(\varepsilon_1, n_1, m_1), \ldots, (\varepsilon_k, n_k, m_k)\rangle$, assume $(\varepsilon_k, n_k, m_k) = (-1, 1, 1)$ and $(\varepsilon_{k-1}, n_{k-1}, m_{k-1}) \neq (-1, 1, 1)$, then we see the following two cases:

(i) If $\varepsilon_{k-1} = -1$ and $n_{k-1} + m_{k-1} \geq 3$, then we have

$$(q_1^{(k-1)}, q_3^{(k-1)}) = (-q_1^{(k-2)} - q_3^{(k-2)}, (n_{k-1}+m_{k-1}+2)q_1^{(k-2)} + (n_{k-1}+m_{k-1}+1)q_3^{(k-2)})$$

$$\text{and} \quad (q_1^{(k)}, q_3^{(k)}) = (-q_1^{(k-1)} - q_3^{(k-1)}, 4q_1^{(k-1)} + 3q_3^{(k-1)}) .$$

Since $q_1^{(k-1)} < 0$ and $q_1^{(k)} < 0$, we obtain

$$\begin{aligned}
&\frac{\sup_{(x,y)\in\mathbb{X}} |\det\, D\Psi_{\Delta_k}(x, y)|}{\inf_{(x,y)\in\mathbb{X}} |\det\, D\Psi_{\Delta_k}(x, y)|} \\
&\quad = \left(\frac{q_3^{(k)}}{2q_1^{(k)} + q_3^{(k)}}\right)^3 = \left(\frac{4q_1^{(k-1)} + 3q_3^{(k-1)}}{2q_1^{(k-1)} + q_3^{(k-1)}}\right)^3 \\
&\quad = \left(2 + \frac{q_3^{(k-1)}}{2q_1^{(k-1)} + q_3^{(k-1)}}\right)^3 \\
&\quad = \left(2 + \frac{(n_{k-1} + m_{k-1} + 2)q_1^{(k-2)} + (n_{k-1} + m_{k-1} + 1)q_3^{(k-2)}}{(n_{k-1} + m_{k-1})q_1^{(k-2)} + (n_{k-1} + m_{k-1} - 1)q_3^{(k-2)}}\right)^3 \\
&\quad < \left(3 + \frac{2q_1^{(k-2)} + 2q_3^{(k-2)}}{3q_1^{(k-2)} + 2q_3^{(k-2)}}\right)^3 .
\end{aligned}$$

From Lemma 3.2, we obtain

$$\frac{2q_1^{(k-2)} + 2q_3^{(k-2)}}{3q_1^{(k-2)} + 2q_3^{(k-2)}} < 2\,.$$

(ii) If $\varepsilon_{k-1} = +1$, then we have

$$(q_1^{(k-1)}, q_3^{(k-1)}) = (q_1^{(k-2)} + q_3^{(k-2)}, (n_{k-1} + m_{k-1} - 2)q_1^{(k-2)} + (n_{k-1} + m_{k-1} - 1)q_3^{(k-2)})$$

$$\text{and} \quad (q_1^{(k)}, q_3^{(k)}) = (-q_1^{(k-1)} - q_3^{(k-1)}, 4q_1^{(k-1)} + 3q_3^{(k-1)})\,.$$

Since $q_1^{(k-1)} > 0$ and $q_1^{(k)} < 0$, we obtain

$$\begin{aligned}
\frac{\sup_{(x,y)\in\mathbb{X}} |\det D\Psi_{\Delta_k}(x, y)|}{\inf_{(x,y)\in\mathbb{X}} |\det D\Psi_{\Delta_k}(x, y)|} &= \left(\frac{q_3^{(k)}}{2q_1^{(k)} + q_3^{(k)}}\right)^3 = \left(\frac{4q_1^{(k-1)} + 3q_3^{(k-1)}}{2q_1^{(k-1)} + q_3^{(k-1)}}\right)^3 \\
&< \left(2 + \frac{q_3^{(k-1)}}{2q_1^{(k-1)} + q_3^{(k-1)}}\right)^3 \\
&< 3^3\,.
\end{aligned}$$

Then we complete this lemma. □

3.3. Weak Bernoulli properties. Now we will show that the modified negative slope algorithm is weak Bernoulli by verifying Yuri's conditions. From Lemma 3.6 (i) and Lemma 3.7, they imply that the modified negative slope algorithm satisfies (C.2) and (C.3) of Yuri's conditions. We check other conditions of Yuri's conditions as follows.

LEMMA 3.8. (C.1) *For any $(x, y) \neq (x', y') \in \mathbb{X}$, there exists $n \geq 0$ such that $S^n(x, y)$ and $S^n(x', y')$ are not the same element in a partition of $\mathbb{X}$.*

PROOF. It is easy to see that

$$\begin{aligned}
\Psi_{\Delta_k}(0, 0) &= \left(\frac{p_3^{(k)}}{q_3^{(k)}}, \frac{r_3^{(k)}}{q_3^{(k)}}\right), \\
\Psi_{\Delta_k}(1, 0) &= \left(\frac{p_1^{(k)} + p_3^{(k)}}{q_1^{(k)} + q_3^{(k)}}, \frac{r_1^{(k)} + r_3^{(k)}}{q_1^{(k)} + q_3^{(k)}}\right), \\
\Psi_{\Delta_k}(0, 1) &= \left(\frac{p_2^{(k)} + p_3^{(k)}}{q_2^{(k)} + q_3^{(k)}}, \frac{r_2^{(k)} + r_3^{(k)}}{q_2^{(k)} + q_3^{(k)}}\right) \\
\text{and} \quad \Psi_{\Delta_k}(0, 0) &= \left(\frac{p_1^{(k)} + p_2^{(k)} + p_3^{(k)}}{q_1^{(k)} + q_2^{(k)} + q_3^{(k)}}, \frac{r_1^{(k)} + r_2^{(k)} + r_3^{(k)}}{q_1^{(k)} + q_2^{(k)} + q_3^{(k)}}\right).
\end{aligned}$$

Then we show that the diameter of Δ_k is bounded above by the distance between the point $\Psi_{\Delta_k}(0,1)$ and the point $\Psi_{\Delta_k}(1,0)$ as follows. Let l be the line that passes the point $\Psi_{\Delta_k}(0,1)$ and the point $\Psi_{\Delta_k}(1,0)$. Then we see that

$$l : (q_1^{(k)} + q_3^{(k)})(x+y) - ((p_1^{(k)} + p_3^{(k)}) + (r_1^{(k)} + r_3^{(k)})) = 0\,.$$

Let $d(k,x,y)$ be the distance between the point $\Psi_{\Delta_k}(0,1)$ and the point $\Psi_{\Delta_k}(1,0)$, $h_1(k,x,y)$ be the distance between the line l and the point $\Psi_{\Delta_k}(0,0)$ and $h_2(k,x,y)$ be the distance between the line l and the point $\Psi_{\Delta_k}(1,1)$. Then we have

$$d(k,x,y) = \sqrt{\left(\frac{p_1^{(k)} + p_3^{(k)}}{q_1^{(k)} + q_3^{(k)}} - \frac{p_2^{(k)} + p_3^{(k)}}{q_2^{(k)} + q_3^{(k)}}\right)^2 + \left(\frac{r_1^{(k)} + r_3^{(k)}}{q_1^{(k)} + q_3^{(k)}} - \frac{r_2^{(k)} + r_3^{(k)}}{q_2^{(k)} + q_3^{(k)}}\right)^2}\,,$$

$$h_1(k,x,y) = \frac{\left|(q_1^{(k)} + q_3^{(k)})\dfrac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} - (p_1^{(k)} + p_3^{(k)} + r_1^{(k)} + r_3^{(k)})\right|}{\sqrt{2}(q_1^{(k)} + q_3^{(k)})} \quad \text{and}$$

$$h_2(k,x,y) = \frac{\left|(q_1^{(k)} + q_3^{(k)})\dfrac{p_1^{(k)} + p_2^{(k)} + p_3^{(k)} + r_1^{(k)} + r_2^{(k)} + r_3^{(k)}}{q_1^{(k)} + q_2^{(k)} + q_3^{(k)}} - (p_1^{(k)} + p_3^{(k)} + r_1^{(k)} + r_3^{(k)})\right|}{\sqrt{2}(q_1^{(k)} + q_3^{(k)})}\,.$$

From Lemma 3.2 and (3), (4) and (5) of Lemma 3.4, we obtain

$$d(k,x,y) = \frac{\sqrt{2}}{q_1^{(k)} + q_3^{(k)}}\,,$$

$$h_1(k,x,y) = \frac{1}{\sqrt{2}q_3^{(k)}(q_1^{(k)} + q_3^{(k)})}$$

$$\text{and} \quad h_2(k,x,y) = \frac{1}{\sqrt{2}(q_1^{(k)} + q_3^{(k)})(q_1^{(k)} + q_2^{(k)} + q_3^{(k)})}\,.$$

These imply that the diameter of Δ_k is bounded above by $d(k,x,y)$. Next we show that $d(k,x,y)$ is monotone decreasing.

(i) If $q_1^{(k-1)} > 0$, then by Lemma 3.2, we see that

$$q_1^{(k)} + q_3^{(k)} = \begin{cases} (n_k + m_k - 1)q_1^{(k-1)} + (n_k + m_k)q_3^{(k-1)} & \text{if } \varepsilon_k = +1 \\ (n_k + m_k + 1)q_1^{(k-1)} + (n_k + m_k)q_3^{(k-1)} & \text{if } \varepsilon_k = -1 \end{cases}$$

$$> q_1^{(k-1)} + q_3^{(k-1)} \quad \text{for } \varepsilon_k = \pm 1\,.$$

(ii) If $q_1^{(k-1)} < 0$, then by Lemma 3.2, we see that

$$q_1^{(k)} + q_3^{(k)} = \begin{cases} (n_k + m_k - 1)(q_1^{(k-1)} + q_3^{(k-1)}) + q_3^{(k-1)} & \text{if } \varepsilon_k = +1 \\ (n_k + m_k - 1)(q_1^{(k-1)} + q_3^{(k-1)}) + (2q_1^{(k-1)} + q_3^{(k-1)}) & \text{if } \varepsilon_k = -1 \end{cases}$$

$$> q_1^{(k-1)} + q_3^{(k-1)} \quad \text{for } \varepsilon_k = \pm 1 .$$

These complete the proof of this lemma. □

LEMMA 3.9. (C.4) *We have*

$$\sum_{k=1}^{\infty} \lambda(\mathbf{D}_k) < \infty$$

where $\lambda(A)$ denotes the two dimensional Lebesgue measure of a Borel set A of $\mathbb{R}^2$.

PROOF. It is easy to see that

$$\mathcal{D}_k = \{\langle \underbrace{(+1,1,1),\ldots,(+1,1,1)}_{k \text{ times}} \rangle, \langle \underbrace{(-1,1,1),\ldots,(-1,1,1)}_{k \text{ times}} \rangle\} .$$

Then we see that

$$\langle \underbrace{(-1,1,1),\ldots,(-1,1,1)}_{k \text{ times}} \rangle = \left\{ (x,y) \,|\, 2 - \frac{k+1}{k} x \le y < 1,\ \frac{k}{k+1} - \frac{k}{k+1} x \le y < 1 \right\} .$$

From Lemma 4.5 of [7], we obtain

$$\lambda(\mathbf{D}_k) = \frac{2}{(k+1)(2k+1)} .$$

This is the assertion of this lemma. □

Then we obtain the following theorem by Theorem 1 of [10].

THEOREM 3.10. *There exists an absolutely continuous invariant probability measure μ for S and (S, μ) is exact.*

PROOF. We see that the modified negative slope algorithm satisfies (C.1)–(C.4) of Yuri's conditions. Hence we complete the proof of Theorem 3.10 by Theorem 1 of [10]. □

REMARK 3.11. The exactness implies not only ergodicity but also mixing of all degrees. In [5], they showed the explicit form of the density function $\frac{d\mu}{d\lambda}$, which we will see in §4, and its ergodicity.

Next we show the following theorem.

THEOREM 3.12 (Rohlin's entropy formula). *The entropy $h_\mu(S)$ of $(\mathbb{X}, S, \mu)$ is given by*

$$h_\mu(S) = \int_{\mathbb{X}} \log |\det DS| \, d\mu .$$

In the following, we show (C.5)–(C.8) of Yuri's conditions, which imply this theorem.

LEMMA 3.13 (C.5).

$$\sum_{l=0}^{\infty} \sum_{\Delta_l \in \mathcal{D}_l} \left(\sup_{(x,y)\in\left(\cup_{j=1}^{k} \mathbf{B}_j\right)} |\det D\Psi_{\Delta_l}(x, y)| \right) < \infty .$$

PROOF. It is easy to see that

$$\det D\Psi_{\Delta_l}(x, y) = \frac{1}{(-lx - ly + 2l + 1)^3}$$

for $\Delta_l = \langle(-1, 1, 1), \ldots, (-1, 1, 1)\rangle$. Then we complete this lemma from Lemma 4.7 of [7]. □

LEMMA 3.14 (C.6).

$$\sharp\mathcal{D}_1 = 2 .$$

PROOF. This is obvious. □

LEMMA 3.15 (C.7). *For every $\Delta_k \in \mathcal{D}_k$, we have*

$$\frac{\sup_{(x,y)\in\mathbb{X}} |\det D\Psi_{\Delta_k}(x, y)|}{\inf_{(x,y)\in\mathbb{X}} |\det D\Psi_{\Delta_k}(x, y)|} = \mathcal{O}(k^3) .$$

PROOF. This follows from Lemma 3.13 and Lemma 4.9 of [7]. □

LEMMA 3.16 (C.8). *The function* $\log |\det DS|$ *is integrable with respect to* λ.

PROOF. We can complete this lemma by Lemma 4.10 of [7]. □

Then we finish the proof of the Theorem 3.12 by Theorem 2 of [10].
We show that the modified negative slope algorithm is weak Bernoulli in the following.

THEOREM 3.17. *The modified negative slope algorithm with the absolutely continuous invariant probability measure μ is weak Bernoulli.*

To prove this theorem, we show (C.4)* and (C.9) of Yuri's conditions.

LEMMA 3.18 (C.4)*.

$$\sum_{k=1}^{\infty} \lambda(\mathbf{D}_k) \cdot \log k < \infty .$$

PROOF. Since we have $\lambda(\mathbf{D}_k) = \frac{2}{(k+1)(2k+1)}$ from the proof of Lemma 3.9. This is the assertion of this lemma. □

LEMMA 3.19 (C.9). *If* $\langle(\varepsilon_1, n_1, m_1), \dots, (\varepsilon_k, n_k, m_k)\rangle \in \mathcal{D}_k^c$ *and* $\langle(\varepsilon_2, n_2, m_2), \dots, (\varepsilon_k, n_k, m_k)\rangle \in \mathcal{D}_{k-1}$, *then we have* $\langle(\varepsilon_1, n_1, m_1)\rangle \in \mathbf{B}_1$, *that is*, $(\varepsilon_1, n_1, m_1) \neq (\pm 1, 1, 1)$.

PROOF. It is easy to see from the definitions of $\mathcal{D}_k$ and $\mathcal{B}_k$. □

Since S satisfies (C.1)–(C.9) with (C.4)*, it implies the assertion of Theorem 3.17 by Theorem 3 of [10].

4. Absolutely continuous invariant measure

In [5], the density function of the absolutely continuous invariant probability measure of the modified negative slope algorithm was given by

$$\frac{d\mu}{d\lambda} = \frac{1}{4\log 2}\frac{1}{(x+y)(2-x-y)}\,.$$

This was checked by Kuzmin's equation

$$f(x,y) = \sum_{\varepsilon=\pm 1, n,m \ge 1} f(\Psi_{(\varepsilon,n,m)}(x,y))|\det \Psi_{(\varepsilon,n,m)}(x,y)|$$

where $f(x,y) = \frac{1}{(x+y)(2-x-y)}$.

In this section, we give the same result by a different way which is called a "natural extension method". This method was originally introduced by [8] for a class of continued fraction algorithms. Let $\overline{\mathbb{X}} = \mathbb{X} \times \{(-\infty, 0)^2 \cup (1, \infty)^2\}$. For $(x, y, z, w) \in \overline{\mathbb{X}}$, we define a map $\overline{S}$ on $\overline{\mathbb{X}}$ by

$$\overline{S}(x,y,z,w) = \begin{cases} \left(n'(x,y) - \dfrac{y}{(x+y)-1},\ m'(x,y) - \dfrac{x}{(x+y)-1},\ n'(x,y) - \dfrac{w}{(z+w)-1},\right. \\ \qquad \left. m'(x,y) - \dfrac{z}{(z+w)-1}\right) & \text{if } x+y > 1 \\ \left(\dfrac{1-y}{1-(x+y)} - n(x,y),\ \dfrac{1-x}{1-(x+y)} - m(x,y),\ \dfrac{1-w}{1-(z+w)} - n(x,y),\right. \\ \qquad \left. \dfrac{1-z}{1-(z+w)} - m(x,y)\right) & \text{if } x+y < 1\,, \end{cases}$$

where $n'(x,y) = n(x,y)+1$ and $m'(x,y) = m(x,y)+1$. Then it is easy to see that $\overline{S}$ is bijective on $\overline{\mathbb{X}}$ except for the set of four dimensional Lebesgue measure 0.

PROPOSITION 4.1. *The measure $\overline{\mu}$ defined by*

$$\frac{d\overline{\mu}}{d\overline{\lambda}} = \frac{1}{|(x+y)-(z+w)|^3}$$

is an invariant measure for $\overline{S}$, where $\overline{\lambda}$ denotes the four dimensional Lebesgue measure.

PROOF. We complete this lemma by Proposition 5.1 of [7]. □

COROLLARY 4.2. *The measure μ defined by*

$$\frac{d\mu}{d\lambda} = \frac{1}{4\log 2}\frac{1}{(x+y)(2-x-y)}$$

is an invariant probability measure for S.

PROOF. It is easy to see that the projection of $\overline{\mu}$ to $\mathbb{X}$ is an invariant measure for S. Then we have

$$\int_{(-\infty,0)\times(-\infty,0)} \frac{1}{|(x+y)-(z+w)|^3} dz dw + \int_{(1,\infty)\times(1,\infty)} \frac{1}{|(x+y)-(z+w)|^3} dz dw$$

$$= \frac{1}{(x+y)(2-x-y)}.$$

This is the assertion of this corollary. □

Then we can compute the entropy $h_\mu(S)$ explicitly from Theorem 3.12 and Corollary 4.2.

PROPOSITION 4.3.

$$h_\mu(S) = \frac{\pi^2}{8\log 2}.$$

PROOF. From Proposition 5.3 of [7] and Corollary 4.2, we complete this lemma. □

From this proposition, we obtain the exponential divergence of $q_3^{(k)}$ as $k \to \infty$.

PROPOSITION 4.4.

$$\lim_{k\to\infty} \frac{1}{k} \log q_3^{(k)} = \frac{\pi^2}{24\log 2}$$

for λ-a.e. $(x, y) \in \mathbb{X}$.

PROOF. From the Shannon-MacMillan-Breiman theorem, we have

$$-\lim_{k\to\infty} \frac{1}{k} \log \mu(\Delta_k) = \frac{\pi^2}{8\log 2} \quad \mu\text{-a.e.}$$

where Δ_k is defined by $(\varepsilon_i, n_i, m_i) = (\varepsilon_i(x, y), n_i(x, y), m_i(x, y))$ for $1 \le i \le k$. We take (x, y) so that $h(\overline{S}(x, y, z, w))\cdot|\det D(\overline{S}(x, y, z, w))|\cdot h^{-1}(x, y, z, w) = 1$ for $h(x, y, z, w) = d\overline{\mu}/d\overline{\lambda}$ holds. Then we choose a subsequence $((l_k) : k \ge 1)$ by

$$l_1 = \min\{l \ge 1 \mid (\varepsilon_l(x, y), n_l(x, y), m_l(x, y)) \ne (\pm 1, 1, 1)\}$$

and

$$l_{k+1} = \begin{cases} l_k + 1 & \text{if } (\varepsilon_{l_k}(x, y), n_{l_k}(x, y), m_{l_k}(x, y)) \ne (\pm 1, 1, 1) \\ l_k + \max\{l \ge 0 \mid (\varepsilon_{l_{k+i}}(x, y), n_{l_{k+i}}(x, y), m_{l_{k+i}}(x, y)) \\ \qquad = (+1, 1, 1) \quad \text{for } 0 \le i \le l\} + 1 & \\ & \text{if } (\varepsilon_{l_k}(x, y), n_{l_k}(x, y), m_{l_k}(x, y)) = (+1, 1, 1) \\ l_k + \max\{l \ge 0 \mid (\varepsilon_{l_{k+i}}(x, y), n_{l_{k+i}}(x, y), m_{l_{k+i}}(x, y)) \\ \qquad = (-1, 1, 1) \quad \text{for } 0 \le i \le l\} + 1 & \\ & \text{if } (\varepsilon_{l_k}(x, y), n_{l_k}(x, y), m_{l_k}(x, y)) = (-1, 1, 1) \end{cases}$$

for $k \ge 1$, which means that we choose all cylinders $\Delta_l \in R(S)$. Since Δ_l is bounded away from (0, 0) and (1, 1), there exists a constant $C_1 > 1$ such that

$$\frac{1}{C_1}\lambda(\Delta_{l_k}) < \mu(\Delta_{l_k}) < C_1\lambda(\Delta_{l_k}).$$

On the other hand, there exists a constant $C_2 > 1$ such that

$$\frac{1}{C_2(q_3^{(l)})^3} < \lambda(\Delta_l) < \frac{C_2}{(q_3^{(l)})^3}$$

whenever $\Delta_l \in R(S)$, see Lemma 3.7. Hence we obtain

$$\lim_{k\to\infty} \frac{1}{l_k} \log q_3^{(l_k)} = \frac{\pi^2}{24 \log 2}$$

for μ-a.e. $(x, y) \in \mathbb{X}$. It is clear that $q_3^{(k)} = q_3^{(k-1)}$ if $(\varepsilon_k(x, y), n_k(x, y), m_k(x, y)) = (+1, 1, 1)$ and $2q_1^{(k)} + q_3^{(k)} = 2q_1^{(k-1)} + q_3^{(k-1)}$ if $(\varepsilon_k(x, y), n_k(x, y), m_k(x, y)) = (-1, 1, 1)$. Since the indicator function of $\langle(\pm 1, 1, 1)\rangle$ is obviously integrable with respect to μ,

$$\lim_{k\to\infty} \frac{l_k - l_{k-1}}{l_k} = 0$$

for μ-a.e. $(x, y) \in \mathbb{X}$. Hence we have

$$\lim_{l\to\infty} \frac{1}{l} \log q_3^{(l)} = \frac{\pi^2}{24 \log 2}$$

for μ-a.e, or equivalently λ-a.e, $(x, y) \in \mathbb{X}$. □

5. Characterization of periodic points of the modified negative slope algorithm

In the previous section, we define $\overline{S}$, the natural extension of the modified negative slope algorithm, on $\overline{\mathbb{X}} = \mathbb{X} \times \{(-\infty, 0)^2 \cup (1, \infty)^2\}$. In this section, we show the following theorem.

THEOREM 5.1. *Suppose iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ does not stop. Then the sequence $(S^k(x, y) : k \geq 0)$ is purely periodic if and only if x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \overline{\mathbb{X}}$ where x^* denotes the algebraic conjugate of x.*

5.1. Necessary part of Theorem 5.1. We show two lemmas to prove the necessary condition of Theorem 5.1.

LEMMA 5.2. *Suppose iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ does not stop. Then, x and y are in the same quadratic extension of $\mathbb{Q}$ if the sequence $(S^k(x, y) : k \geq 0)$ is purely periodic.*

PROOF. Suppose the sequence $(S^k(x, y) : k \geq 0)$ is purely periodic for $(x, y) \in \mathbb{X}$, then there exists $l > 0$ such that $S^l(x, y) = (x, y)$. From Lemma 3.4, we see that

$$x + y = \frac{(p_2^{(l)} + r_2^{(l)})(x + y) + (p_3^{(l)} + r_3^{(l)})}{q_2^{(l)}(x + y) + q_3^{(l)}}.$$

Then we have the following quadratic equation with respect to $(x + y)$.

$$q_2^{(l)}(x + y)^2 + (q_3^{(l)} - p_2^{(l)} - r_2^{(l)})(x + y) - (p_3^{(l)} + r_3^{(l)}) = 0.$$

Here, we put a function $g(z)$ as follows.

$$g(z) = q_2^{(l)} z^2 + (q_3^{(l)} - p_2^{(l)} - r_2^{(l)})z - (p_3^{(l)} + r_3^{(l)}).$$

Then it is clear that $g(z) = 0$ has a root in $0 < z < 2$.

From Lemma 3.2, Lemma 3.3 and Lemma 3.5, we see the following.

(I) If $q_2^{(l)} > 0$, we see that

$$\begin{aligned} g(0) &= -p_3^{(l)} - r_3^{(l)} < 0 \quad \text{and} \\ g(2) &= 4q_2^{(l)} + 2q_3^{(l)} - 2(p_2^{(l)} + p_3^{(l)}) - (p_3^{(l)} + r_3^{(l)}) \\ &= 2q_2^{(l)}\left(2 - \frac{p_2^{(l)} + r_2^{(l)}}{q_2^{(l)}}\right) + q_3^{(l)}\left(2 - \frac{p_3^{(l)} + r_3^{(l)}}{q_3^{(l)}}\right) > 0. \end{aligned}$$

(II) If $q_2^{(l)} < 0$, we see that

$$\begin{aligned} g(0) &= -p_3^{(l)} - r_3^{(l)} < 0 \quad \text{and} \\ g(2) &= 2q_2^{(l)}\left(2 - \frac{p_2^{(l)} + r_2^{(l)}}{q_2^{(l)}}\right) + q_3^{(l)}\left(2 - \frac{p_3^{(l)} + r_3^{(l)}}{q_3^{(l)}}\right). \end{aligned}$$

From (5) of Lemma 3.4, we have

$$g(2) = 2q_2^{(l)}\left\{2 - \left(\frac{p_3^{(l)} + r_3^{(l)}}{q_3^{(l)}} + \frac{1}{q_2^{(l)}q_3^{(l)}}\right)\right\} + q_3^{(l)}\left(2 - \frac{p_3^{(l)} + r_3^{(l)}}{q_3^{(l)}}\right)$$

$$= \frac{1}{q_3^{(l)}}\left\{(2q_3^{(l)} - (p_3^{(l)} + r_3^{(l)}))(2q_2^{(l)} + q_3^{(l)}) - 2\right\}.$$

We see that $2q_3^{(l)} - (p_3^{(l)} + r_3^{(l)})$ and $2q_2^{(l)} + q_3^{(l)}$ are positive integers. So, if $2q_2^{(l)} + q_3^{(l)} = 1$, then we have

$$\begin{aligned}\Psi_{\Delta_l}(1,1) &= \left(\frac{p_1^{(l)} + p_2^{(l)} + p_3^{(l)}}{q_1^{(l)} + q_2^{(l)} + q_3^{(l)}}, \frac{r_1^{(l)} + r_2^{(l)} + r_3^{(l)}}{q_1^{(l)} + q_2^{(l)} + q_3^{(l)}}\right)\\ &= (p_1^{(l)} + p_2^{(l)} + p_3^{(l)}, r_1^{(l)} + r_2^{(l)} + r_3^{(l)})\\ &\in \mathbb{X}.\end{aligned}$$

Thus we see

$$0 \le p_1^{(l)} + p_2^{(l)} + p_3^{(l)} \le 1.$$

From Lemma 3.1, we have

$$-1 \le 2p_2^{(l)} + p_3^{(l)} \le 0.$$

This is the contradiction to Lemma 3.3. Then we have

$$(2q_3^{(l)} - (p_3^{(l)} + r_3^{(l)}))(2q_2^{(l)} + q_3^{(l)}) \ge 2.$$

Note that if $x + y \in \mathbb{Q}$ for $(x, y) \in \mathbb{X}$, then $(S^k(x, y) : k \ge 0)$ is not periodic from Proposition 2.3. Since we assume that $(S^k(x, y) : k \ge 0)$ is purely periodic for $(x, y) \in \mathbb{X}$, we obtain $g(2) \ne 0$. This implies that $g(2) > 0$.

Thus, if $(S^k(x, y) : k \ge 0)$ is purely periodic for $(x, y) \in \mathbb{X}$, then $x + y$ is a quadratic irrational number and the algebraic conjugate of $x + y$ satisfies $(x + y)^* < 0$ if $q_2^{(l)} > 0$ or $(x + y)^* > 2$ if $q_2^{(l)} < 0$ from (I) and (II). Furthermore, from Lemma 3.1 and Lemma 3.4, we have

$$x = \frac{p_2^{(k)}(x + y) + p_3^{(k)}}{q_2^{(k)}(x + y) + q_3^{(k)} - 1},$$

$$y = \frac{r_2^{(k)}(x + y) + r_3^{(k)}}{q_2^{(k)}(x + y) + q_3^{(k)} + 1}.$$

This is the assertion of this lemma. □

LEMMA 5.3. *Let* $\Gamma = \{(z, w) \mid z + w < 0,\ z + w > 2\}$. *Suppose iteration by the modified negative slope algorithm* S *of* $(x, y) \in \mathbb{X}$ *does not stop. Then, for* $(x, y, z, w) \in \mathbb{X} \times \Gamma$, *there exists* $k_0 \in \mathbb{N}$ *s.t. for* $k > k_0$, $\overline{S}^k(x, y, z, w) \in \overline{\mathbb{X}}$.

PROOF. Suppose that $(z, w) \in \{(-\infty, 0)^2 \cup (1, \infty)^2\}$, $(z', w') \in \Gamma \backslash \{(-\infty, 0)^2 \cup (1, \infty)^2\}$ and $z + w = z' + w'$ (see Fig. 1).

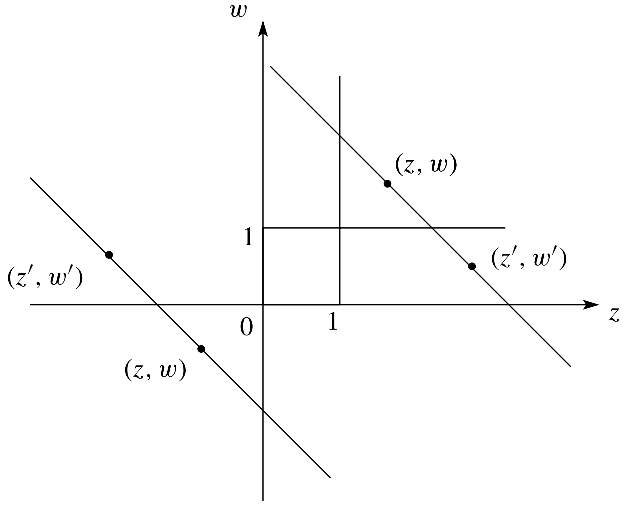

FIGURE 1

Then we have

$$
\begin{aligned}
&|z_1 - z_1'| + |w_1 - w_1'| \\
&\quad = \begin{cases} \left| \dfrac{1 - w}{1 - (z + w)} - \dfrac{1 - w'}{1 - (z' + w')} \right| + \left| \dfrac{1 - z}{1 - (z + w)} - \dfrac{1 - z'}{1 - (z' + w')} \right| & \text{if } \varepsilon_1 = +1 \\ \left| \dfrac{w'}{(w' + z') - 1} - \dfrac{w}{(z + w) - 1} \right| + \left| \dfrac{z'}{(w' + z') - 1} - \dfrac{z}{(z + w) - 1} \right| & \text{if } \varepsilon_1 = -1 \end{cases} \\
&\quad = \begin{cases} \dfrac{1}{|1 - (z + w)|} (|w' - w| + |z' - z|) & \text{if } \varepsilon_1 = +1 \\ \dfrac{1}{|(z + w) - 1|} (|w - w'| + |z - z'|) & \text{if } \varepsilon_1 = -1 \end{cases} \\
&\quad < |z - z'| + |w - w'| \,.
\end{aligned}
$$

Here, we write k-th iteration by $\overline{S}$ of $(z, w) \in \Gamma$ as (z_k, w_k) for simplicity. By the simple calculation, we see that $z_k + w_k = z_k' + w_k'$ for $k \geq 1$. Then, if iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ does not stop, there exists $C > 1$ s.t.

$$|z_k - z'_k| + |w_k - w'_k| < \frac{1}{C^k}(|z - z'| + |w - w'|)$$

for $k \geq 1$. Since $\overline{\mathbb{X}}$ is $\overline{S}$-invariant, there exists $k_0 \in \mathbb{N}$ s.t. for $k > k_0$, we have

$$(z'_k, w'_k) \in (-\infty, 0)^2 \cup (1, \infty)^2 .$$

Note that the sequence $((z_k, w_k) : k \geq 1)$ does not converge to the boundary of $\overline{\mathbb{X}}$ if the sequence $(S^k(x, y) : k \geq 1)$ does not stop at any finite k. We write the image by $\overline{S}$ of (z, w) as $\overline{S}(z, w)$ for simplicity.

(I) For $w < 0$, we see that

$$\overline{S}(0, w) = \begin{cases} \left(1 - n\, ,\ \dfrac{1}{1 - w} - m\right) & \text{if } \varepsilon = +1 \\ \left(n + \dfrac{1}{w - 1}\, ,\ 1 + m\right) & \text{if } \varepsilon = -1 . \end{cases}$$

(II) For $w > 1$, we see that

$$\overline{S}(1, w) = \begin{cases} \left(\dfrac{1}{w} - (1 + n)\, ,\ -m\right) & \text{if } \varepsilon = +1 \\ \left(n\, ,\ (1 + m) - \dfrac{1}{w}\right) & \text{if } \varepsilon = -1 . \end{cases}$$

From Remark 2.4, we see that $((z_k, w_k) : k \geq 1)$ does not converge to the boundary of $\overline{\mathbb{X}}$ if the sequence $(S^k(x, y) : k \geq 1)$ does not stop at any finite k. □

Now we can complete the necessary part of Theorem 5.1.

PROOF (necessary part of Theorem 5.1). Suppose the sequence $(S^k(x, y) : k \geq 0)$ is purely periodic for $(x, y) \in \mathbb{X}$. Then we see that x and y are in the same quadratic extension of $\mathbb{Q}$ from Lemma 5.2. It is easy to see that $(\overline{S}^k(x, y, x^*, y^*) : k \geq 0)$ is purely periodic if $(S^k(x, y) : k \geq 0)$ is purely periodic, where x^* is the algebraic conjugate of x (see Remark 3.5 of [6] for details). Therefore we see that there exists $N > 0$ such that $\overline{S}^N(x, y, x^*, y^*) \in \overline{\mathbb{X}}$ from Lemma 5.3. Since $\overline{\mathbb{X}}$ is $\overline{S}$-invariant, we obtain $(x, y, x^*, y^*) \in \overline{\mathbb{X}}$. □

5.2. Sufficient part of Theorem 5.1. We show the sufficient part of Theorem 5.1 in this subsection. Suppose x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \overline{\mathbb{X}}$. Then we show that the cardinality of $(x, y, x^*, y^*) \in \overline{\mathbb{X}}$ is finite and the orbit of (x, y, x^*, y^*) by $\overline{S}$ is purely periodic. We prepare some lemmas to prove the sufficient condition of Theorem 5.1.

LEMMA 5.4 (Ishimura-Ito [6]). *If α' is equivalent to a quadratic irrational number α with respect to modular transformations, then the discriminant of α' and α are equal.*

See Lemma 3.8 of [6] for the proof of this lemma, Definition 3.6 of [6] for the definition of "discriminant" and Definition 3.7. of [6] for the definition of "equivalent".

LEMMA 5.5 (Ishimura-Ito [6]). *The cardinality of quadratic equations* $ax^2+bx+c = 0$ *with fixed discriminant where* $a, b, c \in \mathbb{Z}$, $GCM(a, b, c) = 1$, $ac < 0$ *is finite.*

See Lemma 3.9 of [6] for the proof of this lemma.

LEMMA 5.6. *Assume that* α *and* β *are in the same quadratic extension of* $\mathbb{Q}$ *and* $(\alpha, \beta, \alpha^*, \beta^*) \in \overline{\mathbb{X}}$, *then* $D_{\alpha+\beta}$ *is greater than* D_α *and* D_β, *where* D_α *is the discriminant of* α.

PROOF. If $(\alpha, \beta, \alpha^*, \beta^*) \in \mathbb{X} \times (-\infty, 0)^2$, we obtain the assertion of this lemma from Lemma 3.10 of [6]. If $(\alpha, \beta, \alpha^*, \beta^*) \in \mathbb{X} \times (1, \infty)^2$, we see that

$$(\alpha, \alpha^*) = \left(\frac{-b - c\sqrt{\theta}}{a}, \frac{-b + c\sqrt{\theta}}{a}\right), \quad a, c > 0\,, \quad GCM(a, b, c) = 1$$

$$(\beta, \beta^*) = \left(\frac{-q - r\sqrt{\theta}}{p}, \frac{-q + r\sqrt{\theta}}{p}\right), \quad p, r > 0\,, \quad GCM(p, q, r) = 1$$

where θ does not contain square numbers as factors. By the same calculations as Lemma 3.10 of [6], we complete this lemma. □

We give the last lemma to complete Theorem 5.1.

LEMMA 5.7. *Suppose iteration by* S *of* $(x, y) \in \mathbb{X}$ *does not stop. Then the sequence* $(\overline{S}^k(x, y, x^*, y^*) : k \geq 0)$ *is purely periodic if* x *and* y *are in the same quadratic extension of* $\mathbb{Q}$ *and* $(x, y, x^*, y^*) \in \overline{\mathbb{X}}$, *where* x^* *denotes the algebraic conjugate of* x.

PROOF. If x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \overline{\mathbb{X}}$, then we see that $x + y$ is equivalent to $x_k + y_k$, $k \geq 1$ with respect to S from (3) and (8) of Lemma 3.4. It implies that D_{x+y} is equal to $D_{x_k+y_k}$ for all $k \geq 1$ by Lemma 5.4. From Lemma 5.6, D_{x_k} and D_{y_k} are bounded above by D_{x+y} for all $k \geq 1$. This implies that the cardinality of $\{S^k(x, y) \mid (x, y) \in \mathbb{X},\ k \geq 0\}$ is finite from Lemma 5.5. Since $\overline{\mathbb{X}}$ is $\overline{S}$-invariant, there exists $l \geq 1$ s.t. for any $k > l$,

$$\overline{S}^k(x, y, x^*, y^*) = \overline{S}^{k+l}(x, y, x^*, y^*)\,.$$

Since $\overline{S}$ is bijective on $\overline{\mathbb{X}}$, we see that

$$\overline{S}^{k-1}(x, y, x^*, y^*) = \overline{S}^{k+l-1}(x, y, x^*, y^*)\,.$$

By induction, we get

$$(x, y, x^*, y^*) = \overline{S}^l(x, y, x^*, y^*)\,.$$

This completes this lemma and the proof of Theorem 5.1. □

Then we have the following corollary of Theorem 5.1.

COROLLARY 5.8. *Suppose iteration by the modified negative slope algorithm S of $(x, y) \in \mathbb{X}$ does not stop. Then x and y are in the same quadratic extension of $\mathbb{Q}$ if and only if the sequence $(S^k(x, y) : k \geq 0)$ is eventually periodic.*

See Corollary 3.12. of [6] for the proof.

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