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Homogenization on Disconnected Selfsimilar Fractal Sets in R

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Abstract. We construct diffusion processes on disconnected selfsimilar fractal sets as limits of suitably scaled random walks. We also consider homogenization problems on disconnected fractal sets. We treat environments whose means are not only finite but infinite.

1. Introduction

Homogenizations of diffusion processes were investigated by many authors. They discussed that Brownian scaled diffusion process in a random environment α converges to a constant time change of Brownian motion, i.e., $\varepsilon X(t/\varepsilon^2, \alpha) \to B(at)$ as $\varepsilon \to 0$ (see [2] and [11]).

Homogenization problems on fractal sets were firstly discussed by Kumagai and Kusuoka [10]. They studied homogenization on nested fractal sets by using the convergence of Dirichlet forms combined with some uniform estimate of the heat kernels.

In this paper, we consider homogenization problems on disconnected selfsimilar fractal sets in **R**. Brownian motion on such fractal sets was introduced by Fujita [4]. Fujita obtained the growth order of eigenvalues of the generators of the Brownian motion on the fractal sets and by this the "spectral dimensions" are determined. He also studied estimates of transition probability densities for generalized one-dimensional diffusion processes with the generators in [5].

First we construct Brownian motion on disconnected fractal sets in **R** as limits of suitably scaled random walks. This gives the "random walk dimension" of disconnected fractal sets. Then we have the same relationship among three dimensions, the spectral dimension, the random walk dimension and the Hausdorff dimension, holds for these disconnected fractal sets (see (2.5) below) as for nested fractal sets.

Next we put some randomness on pre-fractal sets and consider random walks on the pre-fractal sets in these random environments. Our aim of this paper is to show limit theorems of scaled random walks on disconnected fractal sets in random environments. In the case where the random environment has a finite mean, which corresponds to the case in which Kumagai and Kusuoka studied for nested fractal sets, we show that the sequence of scaled random walks in the random environment by the same rate as Brownian motion on this disconnected

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fractal set converges to a constant time change of Brownian motion. Further, we consider limit theorems in the case where the random environment has no mean. In order to prove these, we use a limit theorem of Stone [12] for one-dimensional diffusion processes by a similar manner to that of Kawazu-Kesten [8], who proved similar results for the case in **R**.

In Section 2, we introduce disconnected fractal sets and corresponding pre-fractal sets. We also construct Brownian motion on a fractal sets as limits of suitably scaled random walks. In Section 3, we consider the homogenization problems and our main theorems are given. Proofs of these theorems are given in Section 4.

2. Selfsimilar disconnected fractals in R

Let $r > 1$ and $\varphi = \{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ be a family of r-similitudes on [0, 1], i.e., $\varphi_i(x) = r^{-1}x + b_i$, $0 \leq b_i$ and $r^{-1} + b_i \leq 1$ for $i = 1, 2, \cdots, N$. We assume that $\varphi_1(x) = r^{-1}x$ and $\varphi_N(x) = r^{-1}x + (1 - r^{-1})$. Then it is well-known that there exists a unique compact set $\tilde{C} \subset [0, 1]$ such that

$$\tilde{C} = \bigcup_{i=1}^{N} \varphi_i(\tilde{C}) .$$

We assume

ASSUMPTION 1 (Strong separating condition). If $i \neq j$, then $\varphi_i([0, 1]) \cap \varphi_j([0, 1]) = \emptyset$.

Without loss of generality, we can assume that $r^{-1} + b_i < b_{i+1}$ for $i = 1, 2, \cdots, N - 1$. Then $\tilde{C}$ is a selfsimilar disconnected fractal set on [0, 1] associated with φ. Assumption 1 implies that $N < r$ and that $\tilde{C}$ satisfies the open set condition. Let $\tilde{m}$ be a unique measure such that $\tilde{m}(\tilde{C}) = 2$ and $\tilde{m}(A) = \sum_{i=1}^{N} N^{-1}\tilde{m}(\varphi_i^{-1}(A))$ for any Borel set $A \subset [0, 1]$. Set $\tilde{F}_0 = \{0, 1\}$ and define $\tilde{F}_{n+1} = \bigcup_{i=1}^{N} \varphi_i(\tilde{F}_n)$ inductively. Let $\tilde{F}_\infty = \bigcup_{n=0}^{\infty} \tilde{F}_n$. Then $\tilde{C}$ coincides with $Cl(\tilde{F}_\infty)$.

Next, we define an unbounded fractal set on $[0, \infty)$. Let $F_0 = \bigcup_{n=0}^{\infty} \varphi_1^{-n}(\tilde{F}_n)$, $F_n = \varphi_1^n(F_0)$, $F_\infty = \bigcup_{n=0}^{\infty} F_n$ and $C = Cl(F_\infty)$. Then C is an unbounded selfsimilar disconnected fractal set on $[0, \infty)$ associated with φ and F_0 is called a pre-fractal set. We denote points of any F_0 and F_n by $\{0 = a_0, a_1, a_2, \cdots\}$ and $\{0 = a_0^{(n)}, a_1^{(n)}, a_2^{(n)}, \cdots\}$ in the order from left, respectively. We define a measure m on C by $m(A) = N^n\tilde{m}(\varphi_1^n(A))$ for any Borel set $A \subset [0, r^n]$. $m(x)$ is called an infinitely extended selfsimilar measure.

For the infinitely extended selfsimilar measure m, we consider a generalized one-dimensional diffusion process $\{X(t), t \geq 0\}$ with a generator

$$\frac{d}{dm(x)}\frac{d}{dx} . \tag{2.1}$$

This process $\{X(t)\}$ can be regarded as a Brownian motion on C. In order to see this, we construct $\{X(t)\}$ from a limit of suitably scaled random walks on a pre-fractal set F_0 by a similar manner developed by Stone [12]. Let

$$2h = \min_{1 \le i \le 2N-1} (a_i - a_{i-1}) . \tag{2.2}$$

We define a random walk $\{R_{hj}, j = 0, 1, 2, \cdots\}$ starting from 0 on a pre-fractal set F_0 whose jumps occur at integral multiples of h as follows:

For $i \in \mathbf{N}$,

$$P\{R_{h(j+1)} = a_{i-1} | R_{hj} = a_i\} = \frac{h}{a_i - a_{i-1}} ,$$

$$P\{R_{h(j+1)} = a_i | R_{hj} = a_i\} = 1 - \left\{ \frac{h}{a_i - a_{i-1}} + \frac{h}{a_{i+1} - a_i} \right\} ,$$

$$P\{R_{h(j+1)} = a_{i+1} | R_{hj} = a_i\} = \frac{h}{a_{i+1} - a_i} ,$$

and at the origin,

$$P\{R_{h(j+1)} = a_1 | R_{hj} = 0\} = \frac{h}{a_1} ,$$

$$P\{R_{h(j+1)} = 0 | R_{hj} = 0\} = 1 - \frac{h}{a_1} .$$

Let σ_i be the waiting time of the random walk at the state a_i. Then from the definitions above, we see that $h^{-1}\sigma_i$ has a geometrical distribution with the mean

$$\left(\frac{h}{a_i - a_{i-1}} + \frac{h}{a_{i+1} - a_i} \right)^{-1} \quad \text{and} \quad \frac{a_1}{h}$$

corresponding to $i \neq 0$ and $i = 0$, respectively.

We next consider the suitable scaling. For simplicity we consider a triadic Cantor set, and then $h = 1/2$. For any $m, n, k \in \mathbf{N}$, there exist $k_0, k_1, k_2 \in \mathbf{N}$ such that

$$a_{k-1}^{(n)} = a_{k_0}^{(n+m)} , \quad a_k^{(n)} = a_{k_1}^{(n+m)} \quad \text{and} \quad a_{k+1}^{(n)} = a_{k_2}^{(n+m)} .$$

In the same manner as that of $\{R_{j/2}\}$, we define random walks $\{R_{j/2}^{(n)}\}$ and $\{R_{j/2}^{(n+m)}\}$ on F_n and F_{n+m}, respectively. Then we have the following relation:

$$\begin{aligned} &6^m E[\min\{j : R_{(i+j)/2}^{(n)} \text{ hits } a_{k-1}^{(n)} \text{ or } a_{k+1}^{(n)}\} \mid R_{i/2}^{(n)} = a_k^{(n)}] \\ &\quad = E[\min\{j : R_{(i+j)/2}^{(n+m)} \text{ hits } a_{k_0}^{(n+m)} \text{ or } a_{k_2}^{(n+m)}\} \mid R_{i/2}^{(n+m)} = a_{k_1}^{(n+m)}] . \end{aligned} \tag{2.3}$$

For each of disconnected fractal sets, we put

$$R(t) = R_{hj}, \quad hj \le t < h(j+1) .$$

From the relation (2.3), we should set $3^{-n}R(6^n t)$ and then expect to have a non-trivial limiting process as $n \to \infty$. Let $D = D([0,\infty), \mathbf{R})$ be the space of $\mathbf{R}$-valued right continuous functions with left limits and vanishing at 0 with the Skorohod topology. Then we have the following convergence of space-time scaled random walks to the generalized diffusion process $\{X(t), t \geq 0\}$ with the generator (2.1) in D.

THEOREM 1. *The space-time scaled random walk*

$$\{r^{-n} R\left((rN)^n t\right), t \geq 0\}$$

converges weakly in D to the process $\{X(t)\}$ *starting from* 0 *with the generator* (2.1) *as* $n \to \infty$.

REMARK 1. In the case of a triadic Cantor set, $r = 3$, $N = 2$ and $rN = 6$.

This theorem can be proved in a similar way to (i) of Theorem 3 below. The process $\{X(t)\}$ satisfies the following scaling relation:

$$\{rX(t), t \geq 0\} \stackrel{d}{=} \{X((rN)t), t \geq 0\}. \tag{2.4}$$

This implies that the random walk dimension d_w of C is given by

$$d_w = \log(rN)/\log r.$$

The growth order of the eigenvalues of (2.1) obtained in [4] implied the spectral dimension $d_s = 2\log N/\log(rN)$. By Assumption 1, the Hausdorff dimension $d_f = \log N/\log r$, therefore the following relationship is also satisfied in the case of selfsimilar disconnected fractal sets:

$$d_s d_w = 2d_f. \tag{2.5}$$

3. Homogenization problem

Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}$ be a sequence of independent and identically distributed random variables with values in $(0,\infty)$. For $x \in [0,\infty)$ and $k = 0, 1, 2, \cdots$, we set

$$\begin{aligned}
&I(0) = 0 \quad \text{and} \quad I(x) = a_l + k, \quad a_l + k < x \leq \{a_l + (k+1)\} \wedge a_{l+1}, \\
&J_0(0) = 0 \quad \text{and} \quad J_0(l) = \sum_{i=1}^{l} \lfloor a_i - a_{i-1} \rfloor, \qquad (3.1)\\
&J_1(x) = J_0(l) + k, \quad a_l + k < x \leq \{a_l + (k+1)\} \wedge a_{l+1},
\end{aligned}$$

where $\lfloor x \rfloor$ denotes the function of rounding up to the integer. For a given α, we set

$$S(x) = \begin{cases} 0, & x \leq 0, \\ S(I(x)) + \alpha_{J_1(x)+1}(x - I(x)), & x > 0. \end{cases}$$

Denote by $\mathcal{A}$ the σ-field generated by α. For a given $\mathcal{A}$, we define a birth and death process $\{Y(t), t \geq 0\}$ starting from 0 on a pre-fractal set F_0 in the random environment $\mathcal{A}$ by using this $S(x)$ as follows:

For $i \in \mathbf{N}$,

$$
\begin{aligned}
&P\{Y(t+h) = a_{i-1} | Y(t) = a_i, \mathcal{A}\} = \frac{h}{S(a_i) - S(a_{i-1})} + o(h)\,, \\
&P\{Y(t+h) = a_i | Y(t) = a_i, \mathcal{A}\} = 1 - \left\{ \frac{h}{S(a_i) - S(a_{i-1})} + \frac{h}{S(a_{i+1}) - S(a_i)} \right\} + o(h)\,, \\
&P\{Y(t+h) = a_{i+1} | Y(t) = a_i, \mathcal{A}\} = \frac{h}{S(a_{i+1}) - S(a_i)} + o(h)\,, \\
&P\{Y(t+h) = a_j | Y(t) = a_i, \mathcal{A}\} = o(h), \quad \text{for } j \notin \{i-1, i, i+1\}\,,
\end{aligned}
$$

and at the origin,

$$
\begin{aligned}
&P\{Y(t+h) = a_1 | Y(t) = 0, \mathcal{A}\} = \frac{h}{S(a_1)} + o(h)\,, \\
&P\{Y(t+h) = 0 | Y(t) = 0, \mathcal{A}\} = 1 - \frac{h}{S(a_1)} + o(h)\,, \\
&P\{Y(t+h) = a_j | Y(t) = 0, \mathcal{A}\} = o(h), \quad \text{for } j \in \{2, 3, 4, \cdots\}\,,
\end{aligned}
$$

as $h \to 0$. Then we have the following.

THEOREM 2. *Let $\{X(t), t \geq 0\}$ be the diffusion process on C with the generator* (2.1).

(i) *If $E[\alpha_1] = a < \infty$, then the process $\{r^{-n} Y(a(rN)^n t), t \geq 0\}$ converges weakly in D to the process $\{X(t)\}$ as $n \to \infty$.*

(ii) *If there exists a slowly varying function $L_1(\cdot)$ such that*

$$
\frac{1}{nL_1(n)} \sum_{i=1}^{n} \alpha_i \to 1
$$

in probability, then the process $\{r^{-n} Y((rN)^n L_1(r^n) t), t \geq 0\}$ converges weakly in D to the process $\{X(t)\}$ as $n \to \infty$.

(iii) *If α_i's belong to the domain of attraction of a one-sided positive stable distribution of index $\alpha \in (0, 1)$, i.e., there exists a slowly varying function $L_2(\cdot)$ such that*

$$
\frac{1}{nL_2(n)} \sum_{i=1}^{n} \alpha_i
$$

converges in law to a one-sided positive stable distribution of index α. Then the process $\{r^{-n} Y(\{r^{1/\alpha} N\}^n L_2(r^n) t), t \geq 0\}$ converges weakly in D to the process $\{\tilde{X}(t)\}$ as $n \to \infty$. This $\{\tilde{X}(t)\}$ has the following semi-selfsimilar property,

$$
\{\tilde{X}(t), t \geq 0\} \stackrel{\mathrm{d}}{=} \{r^{-1} \tilde{X}(r^{1/\alpha} N t), t \geq 0\}\,, \tag{3.2}
$$

where $\stackrel{\mathrm{d}}{=}$ *denotes the equality in all joint distributions.*

We will prove this theorem in the next section.

Next, we consider homogenization problems for random walks. For any $\theta > 0$, we set $\alpha_{i,\theta} = \alpha_i \vee \theta$. From $\alpha_{i,\theta}$'s, we define $S_\theta(x)$ as

$$S_\theta(x) = \begin{cases} 0\,, & x \le 0\,, \\ S_\theta(I(x)) + \alpha_{J_1(x)+1,\theta}(x - I(x))\,, & x > 0\,. \end{cases} \tag{3.3}$$

Then we have $S_\theta(x) \to S(x)$ as $\theta \to 0$. Let $\theta_n > 0$, $n \in \mathbf{N}$ satisfying $\theta_n \to 0$ as $n \to \infty$ and

$$h_n = h\theta_n \tag{3.4}$$

(see (2.2) for the definition of h). For $n \in \mathbf{N}$, we define a random walk $\{W_n(t), t \ge 0\}$ starting from 0 on a pre-fractal set F_0 in the random environment $\mathcal{A}$ whose jumps occur at integral multiples of h_n as follows: For $i \in \mathbf{N}$ and $j = 0, 1, 2, \cdots$,

$$\begin{aligned} &P\{W_n(h_n(j+1)) = a_{i-1} | W_n(h_n j) = a_i, \mathcal{A}\} = \frac{h_n}{S_{\theta_n}(a_i) - S_{\theta_n}(a_{i-1})}\,, \\ &P\{W_n(h_n(j+1)) = a_i | W_n(h_n j) = a_i, \mathcal{A}\} \\ &\qquad = 1 - \left\{ \frac{h_n}{S_{\theta_n}(a_i) - S_{\theta_n}(a_{i-1})} + \frac{h_n}{S_{\theta_n}(a_{i+1}) - S_{\theta_n}(a_i)} \right\}, \\ &P\{W_n(h_n(j+1)) = a_{i+1} | W_n(h_n j) = a_i, \mathcal{A}\} = \frac{h_n}{S_{\theta_n}(a_{i+1}) - S_{\theta_n}(a_i)}\,, \end{aligned}$$

and at the origin,

$$\begin{aligned} &P\{W_n(h_n(j+1)) = a_1 | W_n(h_n j) = 0, \mathcal{A}\} = \frac{h_n}{S_{\theta_n}(a_1)}\,, \\ &P\{W_n(h_n(j+1)) = 0 | W_n(h_n j) = 0, \mathcal{A}\} = 1 - \frac{h_n}{S_{\theta_n}(a_1)}\,. \end{aligned}$$

We set

$$W_n(t) = W_n(h_n j)\,, \quad h_n j \le t < h_n(j+1)\,. \tag{3.5}$$

Then we have the following.

THEOREM 3. *Let* $\{X(t), t \ge 0\}$ *be the diffusion process on* C *with the generator* (2.1).

(i) *If* $E[\alpha_1] = a < \infty$, *then the process* $\{r^{-n} W_n(a(rN)^n t), t \ge 0\}$ *converges weakly in* D *to the process* $\{X(t)\}$ *as* $n \to \infty$.

(ii) *If there exists a slowly varying function* $L_1(\cdot)$ *such that*

$$\frac{1}{nL_1(n)} \sum_{i=1}^{n} \alpha_i \to 1$$

in probability, then the process $\{r^{-n}W_n((rN)^nL_1(r^n)t), t \geq 0\}$ *converges weakly in* D *to the process* $\{X(t)\}$ *as* $n \to \infty$.

(iii) *If* α_i*'s belong to the domain of attraction of a one-sided positive stable distribution of index* $\alpha \in (0, 1)$, *then the process* $\{r^{-n}W_n((r^{1/\alpha}N)^nL_2(r^n)t), t \geq 0\}$ *converges weakly in* D *to the semi-selfsimilar process* $\{\tilde{X}(t)\}$ *as* $n \to \infty$, *where* $L_2(x)$ *and* $\tilde{X}(t)$ *were appeared in* (iii) *of Theorem* 2.

REMARK 2. If there exists $\theta_1 > 0$ such that $\alpha_i > \theta_1$ for any $i \in \mathbf{N}$, then we can take a constant time unit $h_1 = h\theta_1$ instead of h_n in the same manner as that in Theorem 1.

4. Proof of theorems

4.1. Proof of Theorem 2. We can construct the birth and death process $\{Y(t), t \geq 0\}$ on F_0 as a generalized one-dimensional diffusion process with the generator

$$\frac{d}{dm_0(x)}\frac{d}{dS(x)},$$

where $m_0(x) = 0$ for $x < 0$ and $m_0(x) = i$ for $a_{i-1} \leq x < a_i$. $\{Y(t)\}$ is realized by a space-time scaled Brownian motion on $\mathbf{R}$ as follows:

Let $\Omega = \{\omega \in C[0, \infty) : \omega(0) = 0\}$ and let P be the Wiener measure on Ω. We denote by $B(t)$ the value of a function $\omega \in \Omega$ at time t. We define $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}$ on another probability space $(\Omega_1, \mathcal{F}_1, P_1)$ and consider the homogenization problem on $(\Omega \times \Omega_1, P \otimes P_1)$. For a fixed α, we set

$$\begin{aligned} l(t, x) &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon, x+\varepsilon]}(B_s)ds, \\ M_0(x) &= m_0 \circ S^{-1}(x), \\ A_0(t) &= \int_{\mathbf{R}} l(t, x)M_0(dx). \end{aligned}$$

Then two processes $\{Y(t)\}$ and $\{S^{-1}(B(A_0^{-1}(t)))\}$ are equivalent in law.

PROOF OF (i) OF THEOREM 2. From the definition of $J_1(x)$ and $N < r$ (that is induced by Assumption 1), we have

$$|J_1(r^n) - r^n| \leq (N-1)\sum_{k=0}^{n-1} r^k \leq N^n.$$

Hence, for any $x \in [0, r^m]$ we have

$$|J_1(r^nx) - r^nx| \leq N^{m+n}. \tag{4.1}$$

The definition of $S(x)$ implies

$$\left| S(I(x)) - \sum_{i=1}^{J_1(x)} \alpha_i \right| \leq \sum_{\{i:\ a_i - I(a_i) < 1\}} \alpha_i \,. \tag{4.2}$$

By (4.1), (4.2) and $E[\alpha_1] = a < \infty$, we see that

$$\frac{S(r^n x)}{ar^n} \to x \tag{4.3}$$

uniformly on any compact set almost surely with respect to P_1. Set

$$Y^{(n)}(t) = r^{-n} Y(a(rN)^n t)\,, \quad n = 1, 2, 3, \cdots \,. \tag{4.4}$$

Then the generator of $\{Y^{(n)}(t)\}$ is given by

$$\frac{d}{dm_n(x)} \frac{d}{dS_n(x)}\,,$$

where $m_n(x) = N^{-n} m_0(r^n x)$ and $S_n(x) = (ar^n)^{-1} S(r^n x)$. Put $Z^{(n)}(t) = S_n(Y^{(n)}(t))$, then the generator of $\{Z^{(n)}(t)\}$ is given by

$$\frac{d}{dM_n(x)} \frac{d}{dx}\,,$$

where $M_n(x) = m_n \circ S_n^{-1}(x)$. Letting

$$A_n(t) = \int_{\mathbf{R}} l(t, x) M_n(dx)\,,$$

$$A(t) = \int_{\mathbf{R}} l(t, x) m(dx)\,,$$

we have

LEMMA 1. *$\{B(A_n^{-1}(t)), t \geq 0\}$ converges to $\{B(A^{-1}(t)), t \geq 0\}$ in the J_1-topology in D as $n \to \infty$.*

PROOF. It is enough to check that for P_1-almost every $\omega_1 \in \Omega$, the measure M_n and m satisfy conditions (i)–(viii) of Theorem 1 in [12]. In our case, $k_n(x) \equiv 0$, $a_0 = 0$ (reflecting boundary) and $b_0 = \infty$, and we see that (4.3) implies conditions (i)–(iii) and (v)–(viii). Hence we have only to check condition (iv). Assume that y_n is the increasing point of $M_n (n = 1, 2, 3, \cdots)$ and that y_n converges to y_0. Then there exists $a_j^{(n)} \in F_n$ such that $y_n = S_n(a_j^{(n)})$ and $a_j^{(n)}$ converges to $c \in C$, therefore $S_n(a_j^{(n)})$ converges to c which is the increasing point of m. □

Lemma 1 and (4.3) imply that $\{S_n^{-1}(B(A_n^{-1}(t)))\}$ converges to $\{B(A^{-1}(t))\}$ in the J_1-topology in D as $n \to \infty$ almost surely, which completes the proof of (i) of Theorem 2. □

PROOF OF (ii) OF THEOREM 2. Set $S_n(x) = \{r^n L_1(r^n)\}^{-1} S(r^n x)$. Then $S_n(x)$ converges to x uniformly on any compact sets in probability with respect to P_1. Therefore, we can prove (ii) of Theorem 2 by a repetition of the argument of proof of (i). □

PROOF OF (iii) OF THEOREM 2. Set $S_n(x) = \{r^{n/\alpha} L_2(r^n)\}^{-1} S(r^n x)$. Then noting $\alpha_i > 0$ for any $i \in \mathbf{N}$ and $0 < \alpha < 1$, by a similar argument of proof of (i), the distribution of $\{S_n(x)\}$ converges in law to a increasing α-stable process $\{\xi(x)\}$ with $0 < \alpha < 1$. By the Skorohod realization theorem of almost sure convergence, there exist a probability space $(\hat{\Omega}_1, \hat{\mathcal{F}}_1, \hat{P}_1)$ and D-valued random variables $\hat{S}_n$ and $\hat{\xi}$ such that

(i) $\hat{S}_n$ converges to $\hat{\xi}$ in D almost surely with respect to $\hat{P}_1$ as $n \to \infty$,

(ii) the distributions of $\hat{S}_n$ and $\hat{\xi}$ are equal to those of S_n and ξ, respectively.

We set

$$\hat{M}_n(x) = m_n \circ \hat{S}_n^{-1}(x)\,,$$

$$\hat{M}(x) = m \circ \hat{\xi}^{-1}(x)\,,$$

$$\hat{A}_n(t) = \int_{\mathbf{R}} l(t, x) \hat{M}_n(dx)\,,$$

$$\hat{A}(t) = \int_{\mathbf{R}} l(t, x) \hat{M}(dx)\,.$$

Then we have the following lemma.

LEMMA 2. *$\{B(\hat{A}_n^{-1}(t)), t \geq 0\}$ converges to $\{B(\hat{A}^{-1}(t)), t \geq 0\}$ in the J_1-topology in D almost surely with respect to $\hat{P}_1$ as $n \to \infty$.*

From Lemma 2 and the Skorohod realization theorem, we see that $\{\hat{S}_n^{-1}(B(\hat{A}_n^{-1}(t)))\}$ converges to $\{\hat{\xi}^{-1}(B(\hat{A}^{-1}(t)))\}$ in the J_1-topology in D almost surely with respect to $\hat{P}_1$. Set

$$\hat{X}^{\xi}(t) = \hat{\xi}^{-1}(B(A^{-1}(t)))\,, \quad t \geq 0\,, \tag{4.5}$$

then we have the following scaling property:

LEMMA 3. *$\{\hat{X}^{\xi}, t \geq 0\}$ has the semi-selfsimilar property such that*

$$\{\hat{X}^{\xi}(t), t \geq 0\} \stackrel{\mathrm{d}}{=} \{r^{-n} \hat{X}^{\xi}((r^{1/\alpha} N)^n t), t \geq 0\}\,,$$

for any $n \in \mathbf{Z}$.

PROOF. Since $\hat{\xi}$ is an α-stable process, we have that $\{\hat{\xi}(x)\} \stackrel{\mathrm{d}}{=} \{r^{-n/\alpha} \hat{\xi}(r^n x)\}$ and $\{\hat{M}(x)\} \stackrel{\mathrm{d}}{=} \{N^{-n} \hat{M}(r^{n/\alpha} x)\}$ for any $n \in \mathbf{Z}$. Therefore, in the same way as that of Lemma 3 in [8] we have the assertion. □

Hence, these arguments complete the proof of (iii) of Theorem 2. □

4.2. Proof of Theorem 3. First we construct a random walk $\{W_n(t), t \geq 0\}$ on F_0 from a birth and death process as follows:

Let $\{Y_{\theta_n}(t)\}$ be a generalized one-dimensional diffusion process with a generator

$$\frac{d}{dm_0(x)}\frac{d}{dS_{\theta_n}(x)} .$$

For $\{Y_{\theta_n}(t)\}$, we define $\tau_{n,j}$ and $T_{n,j}$ inductively as

$$\begin{aligned} &T_{n,0} = 0 , \\ &T_{n,j+1} = T_{n,j} + \tau_{n,j+1} , \\ &\tau_{n,j+1} = \inf\{t > 0 : Y_{\theta_n}(T_{n,j} + t) \neq Y_{\theta_n}(T_{n,j})\} . \end{aligned} \tag{4.6}$$

For $a_i \in F_0$, we set

$$w_{n,i} = \left(\frac{1}{S_{\theta_n}(a_{i+1}) - S_{\theta_n}(a_i)} + \frac{1}{S_{\theta_n}(a_i) - S_{\theta_n}(a_{i-1})}\right) h_n .$$

Then we have $w_{n,i} \in (0, 1]$. Let $\theta_{n,i}$ be the unique solution of $1 - w_{n,i} = \exp\left\{-\frac{w_{n,i}\theta_{n,i}}{h_n}\right\}$. In the case where $Y_{\theta_n}(T_j) = a_i$, using this $\theta_{n,i}$, we set

$$\sigma_{n,j+1} = \begin{cases} h_n & \text{if } w_{n,i} = 1 , \\ mh_n & \text{if } 0 < w_{n,i} < 1 \quad \text{and } (m-1)\theta_{n,i} \leq \tau_{n,j+1} < m\theta_{n,i} , \end{cases} \tag{4.7}$$

and

$$\begin{aligned} &U_{n,0} = 0 , \\ &U_{n,j+1} = U_{n,j} + \sigma_{n,j+1} . \end{aligned} \tag{4.8}$$

If $w_{n,i} \in (0, 1)$, then the value of $h_n^{-1}\sigma_{n,j}$ is geometrically distributed with the mean $w_{n,i}^{-1}$. Then the random walk $\{W_n(t)\}$ is expressed as

$$W_n(t) = Y_{\theta_n}(T_{n,j}) , \quad U_{n,j} \leq t < U_{n,j+1} . \tag{4.9}$$

In the case where the state space is F_n, we take $h_n \searrow 0$ as (3.4) and set for $n \in \mathbf{N}$

$$Y_{\theta_n}^{(n)}(t) = \begin{cases} r^{-n} Y_{\theta_n}(a(rN)^n t) & \text{in case (i)} , \\ r^{-n} Y_{\theta_n}((rN)^n L_1(r^n) t) & \text{in case (ii)} , \\ r^{-n} Y_{\theta_n}((r^{1/\alpha}N)^n L_2(r^n) t) & \text{in case (iii)} . \end{cases}$$

We set

$$S_{\theta_n}^{(n)}(x) = \begin{cases} (ar^n)^{-1} S_{\theta_n}(r^n x) & \text{in case (i)} , \\ \{r^n L_1(r^n)\}^{-1} S_{\theta_n}(r^n x) & \text{in case (ii)} , \\ \{r^{n/\alpha} L_2(r^n)\}^{-1} S_{\theta_n}(r^n x) & \text{in case (iii)} . \end{cases}$$

Then the generator of $\{Y^{(n)}_{\theta_n}(t)\}$ is given by

$$\frac{d}{dm_n(x)}\frac{d}{dS^{(n)}_{\theta_n}(x)}\,.$$

We see that

$$\begin{aligned} &S^{(n)}_{\theta_n} \to x \quad \text{almost surely} && \text{in case (i)}\,,\\ &S^{(n)}_{\theta_n} \to x \quad \text{in probability} && \text{in case (ii)}\,,\\ &S^{(n)}_{\theta_n} \to \xi(x) \quad \text{in law} && \text{in case (iii)}\,, \end{aligned}$$

as $n \to \infty$. Hence, the birth and death process $\{Y^{(n)}_{\theta_n}(t)\}$ also converges weakly to each of the limit processes in Theorem 2 respectively as $n \to \infty$.

For the scaled birth and death process $\{Y^{(n)}_{h_n}(t)\}$, we define $T^{(n)}_{n,j}$ and $\tau^{(n)}_{n,j}$ in the same way as (4.6). We set that

$$h^{(n)}_n = \begin{cases} \{a(rN)^n\}^{-1}h_n & \text{in case (i)}\,,\\ \{(rN)^n L_1(r^n)\}^{-1}h_n & \text{in case (ii)}\,,\\ \{(r^{1/\alpha}N)^n L_2(r^n)\}^{-1}h_n & \text{in case (iii)}\,. \end{cases}$$

Using h_n, we define $\sigma^{(n)}_{n,j}$ and $U^{(n)}_{n,j}$ in the same way as (4.7) and (4.8), respectively. Then by a similar argument to that in Section 4 in [12], we have the following lemma.

LEMMA 4. *For any fixed $t > 0$ and any $\varepsilon > 0$,*

$$P\left\{\sup_{\substack{1\le l\le k\\ T^{(n)}_{n,k}\le t}} |T^{(n)}_{n,l} - U^{(n)}_{n,l}| > \varepsilon\right\} \le \frac{14}{\varepsilon^2}(h^{(n)}_n)^2\,. \tag{4.10}$$

Since $\sum_{n=1}^{\infty}(h^{(n)}_n)^2 < \infty$, we have

PROPOSITION 1. *For any $t > 0$, $\max_{\substack{1\le l\le k\\ T^{(n)}_{n,k}\le t}} |T^{(n)}_{n,l} - U^{(n)}_{n,l}|$ converges to 0 uniformly almost surely as $n \to \infty$.*

This and Theorem 2 imply Theorem 3. □

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