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On the Existence of a Non-trivial Solution for the p-Laplacian Equation with a Jumping Nonlinearity

Mieko TANAKA

Tokyo University of Science

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Abstract. We consider the existence of a non-trivial weak solution for the equation

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where f satisfies $f(x,u) = au_+^{p-1} - bu_-^{p-1} + o(|u|^{p-1})$ $(p > 1)$ at 0 or ∞. By using Morse theory and calculating the critical groups, we show the existence of a non-trivial weak solution to the equation under mild auxiliary conditions.

1. Introduction and the Main result

In this paper, we consider the equation

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain, Δ_p denotes the p-Laplacian defined by $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ $(p > 1)$. We will treat $f \in C(\overline{\Omega} \times \mathbf{R})$ satisfying

$$f(x,u) = au_+^{p-1} - bu_-^{p-1} + o(|u|^{p-1})$$

as $|u| \to 0$ or $|u| \to \infty$, where $u_\pm := \max\{\pm u, 0\}$. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (P) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx = \int_\Omega f(x,u)\varphi\, dx$$

holds for any $\varphi \in W_0^{1,p}(\Omega)$.

The equation (P) in the case of $f(x,u) = au_+^{p-1} - bu_-^{p-1}$ has been considered by Fučík [7] $(p = 2)$ and many authors (cf. [4], [3], [5]). The set Σ_p of points $(a,b) \in \mathbf{R}^2$ for which

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the equation

$$-\Delta_p u = a u_+^{p-1} - b u_-^{p-1}, \quad u \in W_0^{1,p}(\Omega) \tag{1}$$

has a non-trivial weak solution is called the Fučík spectrum of the p-Laplacian on $W_0^{1,p}(\Omega)$ ($1 < p < \infty$) ([3]). In the case of $a = b = \lambda \in \mathbf{R}$, the equation (1) reads $-\Delta_p u = \lambda |u|^{p-2} u$. Hence (λ, λ) belongs to Σ_p if and only if λ is an *eigenvalue* of $-\Delta_p$, i.e., there exists a non-zero weak solution $u \in W_0^{1,p}(\Omega)$ to $-\Delta_p u = \lambda |u|^{p-2} u$. The set of all eigenvalues of $-\Delta_p$ is, as usual, denoted by $\sigma(-\Delta_p)$. It is well known that the first eigenvalue λ_1 of $-\Delta_p$ is positive, simple, and has a positive eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$. Therefore, Σ_p contains the lines $\{\lambda_1\} \times \mathbf{R}$ and $\mathbf{R} \times \{\lambda_1\}$. Moreover, by using the Mountain Pass Theorem, Cuesta–de Figueiredo–Gossez [3] showed that there exists a continuous curve C_2 contained in Σ_p, which passes through (λ_2, λ_2), where λ_2 is the second eigenvalue of $-\Delta_p$. The curve C_2 is described as

$$C_2 := \{ (s + c(s), c(s)) \mid s \geq 0 \} \cup \{ (c(s), s + c(s)) \mid s \geq 0 \},$$

where $c(s)$ is a continuous, strictly decreasing function on $\mathbf{R}^+$ with the property $c(0) = \lambda_2$ and $\lim_{s \to \infty} c(s) = \lambda_1$.

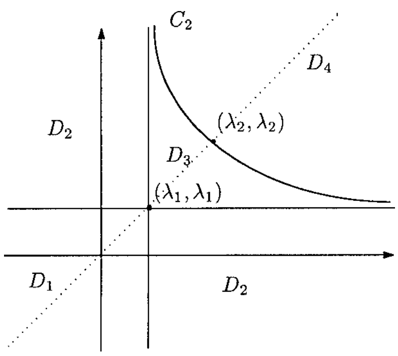
It is obvious that critical points of the following functional correspond to weak solutions of (1):

$$I_{(a,b)}(u) := \int_\Omega |\nabla u|^p \, dx - a \int_\Omega u_+^p \, dx - b \int_\Omega u_-^p \, dx, \quad u \in W_0^{1,p}(\Omega). \tag{2}$$

To explain our results of this paper, recall the notation of the critical groups of an isolated critical point at u of a C^1-class functional I. The critical groups at u are defined as

$$C_*(I, u) := H_*(U \cap I^c, U \setminus \{u\} \cap I^c),$$

where $c = I(u)$, $I^c := \{ u\, ;\ I(u) \leq c \}$, U is a neighborhood of u containing no other critical points, and $H_*(\cdot, \cdot)$ are the relative singular homology groups with a coefficient group G.



In addition, we should introduce subsets D_1 to D_4 of $\mathbf{R}^2$ as follows (see the figure below).

$$\begin{aligned} D_1 &= \{(a,b) \mid a, b < \lambda_1\} \\ D_2 &= \{(a,b) \mid b < \lambda_1 < a\} \cup \{(a,b) \mid a < \lambda_1 < b\} \\ D_3 &= \{(a,b) \mid (a,b) \notin \overline{D_1 \cup D_2} \text{ and } (a,b) \text{ lies below the curve } C_2\} \\ D_4 &= \{(a,b) \mid (a,b) \text{ lies above the curve } C_2\} \end{aligned}$$

Then, in [5] and [3], it is shown that $D_i \cap \Sigma_p = \emptyset$ $(i = 1, 2, 3)$. Moreover, the following result is also obtained in [5] and [3] as to the critical groups of $I_{(a,b)}$ at 0:

$$\begin{cases} C_q(I_{(a,b)}, 0) = \delta_{q,0} G & \text{if } (a,b) \in D_1 \\ C_q(I_{(a,b)}, 0) = 0 & \text{if } (a,b) \in D_2 \\ C_q(I_{(a,b)}, 0) = \delta_{q,1} G & \text{if } (a,b) \in D_3 \\ C_0(I_{(a,b)}, 0) = C_1(I_{(a,b)}, 0) = 0 & \text{if } (a,b) \in D_4 \setminus \Sigma_p \end{cases} \tag{3}$$

However, if $(a,b) \in D_4 \setminus \Sigma_p$ with $a \neq b$, [5] and [8] do not tell about $C_q(I_{(a,b)}, 0)$ $(q \geq 2)$. One of our main purpose is to show that if (a,b) belongs to some region in $D_4 \setminus \Sigma_p$, then there exists some number $q \geq 2$ such that $C_q(I_{(a,b)}, 0)$ is nontrivial.

To state our result in this paper, let us recall the definition of the Perera's eigenvalues of $-\Delta_p$. For symmetric subset $A \subset W_0^{1,p}(\Omega) \setminus \{0\}$, we denote by $i_Y(A)$ the Yang index (see [11] and Appendix). Then, Perera has proved in [11] that

$$\lambda_m := \inf\left\{ \sup_{u \in A} \int_\Omega |\nabla u|^p \, dx \, ; \; A \subset S \text{ is symmetric and } i_Y(A) \geq m-1 \right\}, \tag{4}$$

where $S := \{u \in W_0^{1,p}(\Omega) \, ; \, \int_\Omega |u|^p \, dx = 1\}$, is the eigenvalue of $-\Delta_p$ such that $\lambda_m \nearrow +\infty$ as $m \to \infty$. We define

$$Q_m := \{(a,b) \in \mathbf{R}^2 \, ; \; \lambda_m < a, b < \lambda_{m+1}\}.$$

Now our result reads as follows.

PROPOSITION 1. *If $(a,b) \in \overline{Q}_m \setminus \Sigma_p$ for $m \geq 2$, then $C_m(I_{(a,b)}, 0) \neq 0$.*

REMARK 2. For $p = 2$, the above result is contained in several papers (cf. [13]).

In the case of $(a,b) \in \overline{Q}_m \setminus \Sigma_p$ with $a = b$, Perera [11] proved $C_m(I_{(a,a)}, 0) \neq 0$, and Jiang [8] showed that there exists some number q such that $C_q(I_{(a,a)}, 0) \neq 0$.

For our main result on the existence of non-trivial weak solutions to (P), we consider the following assumptions for $f \in C(\overline{\Omega} \times \mathbf{R})$ and set $F(x,u) := \int_0^u f(x,s)\, ds$;

(f_1^0) $f(x,u) = a_0 u_+^{p-1} - b_0 u_-^{p-1} + o(|u|^{p-1})$ as $|u| \to 0$,

(f_2^0) there exist some $\delta > 0$ and $\mu \in (0, p)$ such that

$$f(x,u)u > 0 \quad \text{if } 0 < |u| \leq \delta,$$

$$\mu F(x,u) \geq f(x,u)u \geq 0 \quad \text{if } |u| \leq \delta,$$

(f_1^∞) $f(x,u) = au_+^{p-1} - bu_-^{p-1} + o(|u|^{p-1})$ as $|u| \to \infty$,

(f_2^∞) there exist some numbers $R, c > 0, q < p^* - 1, \theta > p$ such that

$$|f(x,u)| \le c(1 + |u|^q) \quad \text{for any } (x,u)\,,$$
$$f(x,u)u \ge \theta F(x,u) > 0 \quad \text{if } |u| \ge R\,,$$

where $p^* = Np/(N-p)$ for $p < N$, $p^* = \infty$ for $p \ge N$.

Our existence result is the following.

THEOREM 3. (i) *If* (f_1^0) *with* $(a_0, b_0) \in \overline{Q}_m \setminus \Sigma_p$ $(m \ge 2)$ *and* (f_2^∞) *hold, then* (P) *has a non-trivial weak solution.*

(ii) *If* (f_2^0) *and* (f_1^∞) *with* $(a,b) \in \overline{Q}_m \setminus \Sigma_p$ *hold, then* (P) *has a non-trivial weak solution.*

(iii) *If* (f_1^0) *with* $(a_0, b_0) \in D_4 \setminus \Sigma_p$ *and* (f_1^∞) *with* $(a,b) \in D_2$ *hold, then* (P) *has at least two non-trivial weak solution.*

REMARK 4. In the cases $(a_0, b_0) \in D_i$ $(i = 1, 2, 3)$, $a_0 = b_0 \notin \sigma(-\Delta_p)$, Jiang ([8]) showed under (f_2^∞) that (P) has a non-trivial solution. Under the same assumptions as (iii) in Theorem 1, it was proved that (P) has at least one non-trivial solution in [5].

For $p = 2$, if $Q_m \neq \emptyset$, then $Q_m \cap (\mathbf{R}^2 \setminus \Sigma_2) \neq \emptyset$ (see [14]). However, for $p \neq 2$, we do not know whether $Q_m \cap \Sigma_p$ is empty or not.

2. Preliminaries

Now we recall the definition of homological linking (see [2]).

DEFINITION 5. Let D, S, A be subsets of a real Banach space X, m a nonnegative integer, and $\mathbf{K}$ a field. We say that (D, S) links A homologically in dimension m over $\mathbf{K}$, if $S \subset D$, $S \cap A = \emptyset$ and there exists a $z \in H_m(X, S; \mathbf{K})$ such that

$$z \in \operatorname{Im} i_*\,, \quad z \notin \operatorname{Im} j_*\,, \tag{5}$$

where $i_*\colon H_m(D, S; \mathbf{K}) \to H_m(X, S; \mathbf{K})$, $j_*\colon H_m(X \setminus A, S; \mathbf{K}) \to H_m(X, S; \mathbf{K})$.

The following result is proved in [2] by using $\mathbf{Z}_2$-cohomological index (see [6]).

THEOREM 6 ([2, Theorem 4.1]). *Let* $\{\lambda_n\}_n$ *be the Perera's spectrum of* $-\Delta_p$ *defined by* (4). *Let* $m \in \mathbf{N} \cup \{0\}$, $\lambda_m < \alpha \le \beta < \lambda_{m+1}$ *(where* $\lambda_0 = -\infty$*)*, $R > 0$ *and set*

$$S := \left\{ u \in W_0^{1,p}(\Omega)\,;\ R^p = \int_\Omega |\nabla u|^p\,dx < \alpha \int_\Omega |u|^p\,dx \right\},$$
$$A := \left\{ u \in W_0^{1,p}(\Omega)\,;\ \int_\Omega |\nabla u|^p\,dx \ge \beta \int_\Omega |u|^p\,dx \right\}.$$

Then there exists a compact subset C *of* $W_0^{1,p}(\Omega)$ *such that* $(S \cup C, S)$ *links* A *homologically in dimension* m *over* $\mathbf{Z}_2$.

By this homological linking, we can find a critical value.

THEOREM 7 ([2, Theorem 3.2]). *Let (D, S) link A homologically in dimension m over $\mathbf{K}$, and let $z \in H_m(X, S; \mathbf{K})$ satisfy* (5)*. Assume that $I \in C^1(X, \mathbf{R})$ satisfies* (PS) *condition and*

$$-\infty < \inf_A I\,, \quad \sup_D I < +\infty\,, \quad \sup_S I < \inf_A I\,.$$

And define

$$c := \inf \left\{ \begin{array}{ll} b \in \mathbf{R}\,; & S \subset I^b \text{ and } z \text{ belongs to the image of the homomorphism} \\ & H_m(I^b, S; \mathbf{K}) \to H_m(X, S; \mathbf{K}) \text{ induced by inclusion} \end{array} \right\},$$

where $I^b := \{u \in X\,;\, I(u) \le b\}$. Then c is a critical value of I such that $\inf_A I \le c \le \sup_D I$. Moreover, if each element of $K_c := \{u \in X\,;\, I'(u) = 0, I(u) = c\}$ is isolated in the critical set of I, then there exists $u \in K_c$ with $C_m(I, u) \neq 0$.

Now we prove Proposition 1 which is the main tool for the proof of (i) and (ii) in Theorem 3.

PROOF OF PROPOSITION 1. For every $(a_0, b_0) \in \mathbf{R}^2 \setminus \Sigma_p$, according to [5, Proposition 1.2.], if (a, b) is sufficiently close to (a_0, b_0), then $C_*(I_{(a,b)}, 0) \cong C_*(I_{(a_0,b_0)}, 0)$ holds. Hence, it suffices to show that $C_m(I_{(a,b)}, 0) \neq 0$ for $(a, b) \in Q_m \setminus \Sigma_p$ because Σ_p is closed.

Therefore, let $(a, b) \in Q_m \setminus \Sigma_p$. Choose α, β satisfying $\lambda_m < \alpha < \min\{a, b\} \le \max\{a, b\} < \beta < \lambda_{m+1}$ and $R > 0$. Define

$$S := \left\{ u \in W_0^{1,p}(\Omega)\,;\; R^p = \int_\Omega |\nabla u|^p\, dx < \alpha \int_\Omega |u|^p\, dx \right\},$$
$$A := \left\{ u \in W_0^{1,p}(\Omega)\,;\; \int_\Omega |\nabla u|^p\, dx \ge \beta \int_\Omega |u|^p\, dx \right\}.$$

By Theorem 6, there exists some compact subset C of $W_0^{1,p}(\Omega)$ such that $(S \cup C, S)$ links A homologically in dimension m over $\mathbf{Z}_2$.

In the case of $b \le a$, we have for any $u \in S$

$$\begin{aligned} I_{(a,b)}(u) &= \textstyle\int_\Omega |\nabla u|^p\, dx - b \int_\Omega |u|^p\, dx - (a - b) \int_\Omega u_+^p\, dx \\ &\le (1 - b/\alpha) R^p\,. \end{aligned}$$

Similarly, if $a \le b$, then $I_{(a,b)}(u) \le (1 - a/\alpha)R^p$ for any $u \in S$. Therefore $\sup_S I_{(a,b)} \le (1 - \min\{a, b\}/\alpha)R^p < 0$. On the other hand, for $u \in A$,

$$\begin{aligned} I_{(a,b)}(u) &\ge \beta \textstyle\int_\Omega |u|^p\, dx - a \int_\Omega u_+^p\, dx - b \int_\Omega u_-^p\, dx \\ &\ge (\beta - a) \textstyle\int_\Omega u_+^p\, dx + (\beta - b) \int_\Omega u_-^p\, dx \ge 0\,. \end{aligned}$$

Thus $\inf_A I_{(a,b)} = 0$ and hence $\sup_S I_{(a,b)} < \inf_A I_{(a,b)}$. Since C is compact and $I_{(a,b)}$ is C^1-class, $\max_C I_{(a,b)} < +\infty$ and so $\sup_{S \cup C} I_{(a,b)} < +\infty$.

It follows from Theorem 7 that $C_m(I_{(a,b)}, 0) \neq 0$ because 0 is the only critical point of $I_{(a,b)}$. ∎

3. Proof of Theorem 3

Now we prove Theorem 3. Define $X := W_0^{1,p}(\Omega)$ and

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx$$

for $u \in X$, where $F(x,u) = \int_0^u f(x,s)\,ds$. Then J is a functional of class C^1, and critical points of J are weak solutions of the equation (P). Moreover, it is well known that J satisfies (PS) condition if (f_1^∞) with $(a,b) \notin \Sigma_p$ or (f_2^∞) holds (see [12], [9]). If there exists a constant c such that $J^c := \{u \in X\,;\ J(u) \le c\}$ contains no critical value of J, then we define $C_*(J,\infty) := H_*(X, J^c)$, which denotes the critical groups of J at infinity.

PROOF OF (I). Jiang [8] has shown that if (f_2^∞) holds, then $C_q(J,\infty) = 0$ for any $q \in \mathbf{Z}$. Moreover, it follows from the homotopy invariance that $C_q(I_{(a_0,b_0)}, 0) \cong C_q(J,0)$ for any $q \in \mathbf{Z}$ under the assumption (f_1^0) with $(a_0,b_0) \notin \Sigma_p$ (see [8]).

By Proposition 1,

$$C_m(J,\infty) = 0 \neq C_m(I_{(a_0,b_0)}, 0) \cong C_m(J,0)\,.$$

Hence J must have a non-trivial critical point (cf. [1]).

PROOF OF (II). It is known that $C_q(J,0) = 0$ for any $q \in \mathbf{Z}$ by the assumption (f_2^0) (see [9]). By the homotopy invariance, it follows from (f_1^∞) with $(a,b) \notin \Sigma_p$ that $C_q(J,\infty) \cong C_q(I_{(a,b)}, 0)$ for any $q \in \mathbf{Z}$ (see [8]).

Therefore $C_m(J,0) \ncong C_m(J,\infty)$ by Proposition 1. This yields that J has a non-trivial critical point.

PROOF OF (III). Let

$$f_\pm(x,u) := \begin{cases} f(x,u) & \text{if} \quad \pm u \ge 0\,, \\ 0 & \text{if} \quad \pm u < 0\,, \end{cases} \qquad F_\pm(x,u) := \int_0^u f_\pm(x,s)\,ds$$

and

$$J_\pm(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F_\pm(x,u)\,dx\,.$$

If $(a,b) \in D_2$, then $b < \lambda_1 < a$ or $a < \lambda_1 < b$ holds. Thus, we treat each case.

THE CASE OF $b < \lambda_1 < a$. It is easily seen that J_- is bounded below and coercive. Therefore there exists a global minimum point w of J_-. From Lemma 2.2. in [5], w is a critical point and a local minimum point of J. Hence

$$C_q(J,w) \cong C_q(J_-,w) = \delta_{q,0}G \quad \text{for any } q \in \mathbf{Z}\,. \tag{6}$$

Moreover, it follows from (f_1^0) with $(a_0, b_0) \in D_4 \setminus \Sigma_p$ that $J_-(w) = \min_X J_- < 0 = J_-(0)$, and so $w \neq 0$.

We assume that J has no critical points other than 0 and w. Because w is a critical point of J_-, w is a negative function and $J_-(w) = J(w)$. Let $c := J(w) = J_-(w) < J_-(0) = 0$. Then, for $\varepsilon > 0$ sufficiently small with $c + \varepsilon < 0$,

$$\begin{cases} H_q(X, J^{c+\varepsilon}) \cong C_q(J, 0) \quad \text{for any } q \in \mathbf{Z}, \\ H_q(J^{c+\varepsilon}, J^{c-\varepsilon}) \cong C_q(J, w) = \delta_{q,0}G \quad \text{for any } q \in \mathbf{Z}, \\ \text{and} \quad H_q(X, J^{c-\varepsilon}) \cong C_q(J, \infty) \cong C_q(I_{(a,b)}, 0) = 0 \quad \text{for any } q \in \mathbf{Z}. \end{cases} \tag{7}$$

Set $B_n(A, B) := \operatorname{rank} H_n(A, B)$ and $P(t, A, B) := \sum_{n=0}^{\infty} B_n(A, B)t^n$. It follows from the exact sequence for $(X, J^{c+\varepsilon}, J^{c-\varepsilon})$ that for $t \geq 0$

$$P(t, X, J^{c+\varepsilon}) + P(t, J^{c+\varepsilon}, J^{c-\varepsilon}) = P(t, X, J^{c-\varepsilon}) + (1 + t)Q(t),$$

where $Q(t) = \sum_{n=0}^{\infty} R_{n+1}t^n$, $R_n := \operatorname{rank} \partial_{n*}$ and

$$\partial_{n*} \colon H_n(X, J^{c+\varepsilon}) \to H_{n-1}(J^{c+\varepsilon}, J^{c-\varepsilon})$$

denotes the boundary homomorphism (see [10]). From (7), we obtain

$$1 + P(t, X, J^{c+\varepsilon}) = (1 + t)Q(t). \tag{8}$$

Taking $t = 0$ in the equation (8), we have $Q(0) = 1$ because $P(0, X, J^{c+\varepsilon}) = 0$. Moreover, taking $t = -1$ in the equation (8), we have also $P(-1, X, J^{c+\varepsilon}) = -1$, and so there exists some number $q_0 \geq 2$ such that $C_{q_0}(J, 0) \neq 0$ because $C_q(J, 0) \cong C_q(I_{(a_0,b_0)}, 0) = 0$ for $q = 0, 1$. It yields that there exists some $n_0 \geq 1$ such that $R_{n_0+1} \neq 0$. Indeed, if $R_{n+1} = 0$ for all $n \geq 1$, then for $t \geq 0$,

$$1 + \left(\operatorname{rank} C_{q_0}(J, 0)\right) t^{q_0} \leq 1 + P(t, X, J^{c+\varepsilon}) = (1 + t)R_1.$$

This shows a contradiction.

Now from $R_{n_0+1} \neq 0$,

$$0 \neq H_{n_0}(J^{c+\varepsilon}, J^{c-\varepsilon}) \cong C_{n_0}(J, w).$$

It contradicts to (6). Thus J has a critical point other than 0 and w.

THE CASE OF $a < \lambda_1 < b$. It is easily seen that J_+ is bounded below and coercive. By using J_+ instead of J_-, we can prove that J has at least two non-trivial critical points, similarly as in the case of $b < \lambda_1 < a$.

APPENDIX. Following Perera [11], let (X, A), $A \subset X$, be a pair of closed symmetric subsets of a Banach space. $C(X, A)$ denotes the singular chain complex with $\mathbf{Z}_2$ coefficients, and $T_\sharp$ the chain map of $C(X, A)$ induced by the antipodal map $T \colon x \mapsto -x$. We say that a q-chain c is symmetric if $T_\sharp(c) = c$, which holds if and only if $c = c' + T_\sharp(c')$ for some q-chain c'. The symmetric q-chains form a subgroup $C_q(X, A; T)$ of $C_q(X, A)$. Then the boundary operator ∂_p maps $C_q(X, A; T)$ into $C_{q-1}(X, A; T)$, hence these subgroups form a subcomplex $C(X, A; T)$.

Define homomorphisms $\nu\colon Z_q(X; T) \to \mathbf{Z}_2$ inductively in q by

$$\nu(z) = \begin{cases} \mathrm{In}\,(c) & \text{for } q = 0\,, \\ \nu(\partial c) & \text{for } q \geq 1\,, \end{cases}$$

if $z = c + T_\sharp(c)$, where the index $\mathrm{In}\,(c)$ of a 0-chain $c = \sum_i n_i\sigma_i$ is defined by $\mathrm{In}\,(c) = \sum_i n_i \pmod 2$.

Then we can define the homomorphism

$$\nu_*\colon H_q(X; T) \to \mathbf{Z}_2 \quad \text{by } \nu_*([z]) = \nu(z)\,,$$

and the Yang index of X by

$$i_Y(X) = \inf\{l \geq -1\,;\ \nu_* H_{l+1}(X; T) = 0\}$$

taking $\inf \emptyset = \infty$.

Now we consider the spectrum of $-\Delta_p$. The eigenvalues of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \ \Omega\,, \\ u = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{9}$$

are critical values of the following C^1 functional

$$I(u) = \int_\Omega |\nabla u|^p\,dx\,, \quad \text{for } u \in S := \{\, u \in X\,;\ \int_\Omega |u|^p\,dx = 1\,\}\,,$$

which satisfies the (PS) condition.

Let $\mathcal{A}$ be the class of closed symmetric subsets of S, and set

$$\mathcal{F}_l := \{\, A \in \mathcal{A}\,;\ i_Y(A) \geq l-1\,\}\,,$$
$$\lambda_l := \inf_{A \in \mathcal{F}_l} \sup_{u \in A} I(u)\,.$$

Then λ_l is an eigenvalues of (9) and $\{\lambda_l\}$ satisfies $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_l \leq \cdots$, and $\lambda_l \nearrow +\infty$ as $l \to \infty$ ([12, Proposition 3.1.]).

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Present Address:
DEPARTMENT OF MATHEMATICS,
TOKYO UNIVERSITY OF SCIENCE,
WAKAMIYA-CHO, SHINJYUKU-KU, TOKYO, 162–0827 JAPAN.
e-mail: tanaka@ma.kagu.tus.ac.jp