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# On the Moduli Space of Pointed Algebraic Curves of Low Genus II —Rationality—

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**Abstract.** We show that the moduli space $\mathcal{M}_{g,1}^N$ of pointed algebraic curves of genus $g$ with a given numerical semigroup $N$ is an irreducible *rational* variety if $N$ is generated by less than five elements for low genus ($g \leq 6$) except one case. As a corollary to this result, we get a computational proof of the rationality of the moduli space $\mathcal{M}_{g,1}$ of pointed algebraic curves of genus $g$ for $1 \leq g \leq 3$. If $g \leq 5$, we also have that $\mathcal{M}_{g,1}^N$ is an irreducible rational variety for any semigroup $N$ except two cases. It is known that such a moduli space $\mathcal{M}_{g,1}^N$ is non-empty for $g \leq 7$.

## 1. Introduction

The purpose of this note is to show that the set of all pointed algebraic curves of genus $g$ with a given Weierstrass gap sequence at the point is parametrized with a finite number of parameters in a generically one-to-one manner if $g$ is small. This note is the second part of the preceding paper [12]. So before stating the main result, let us briefly review the results obtained in [12].

Let $\mathbf{N}_0 := \{0, 1, 2, \ldots\}$ be the additive semigroup of nonnegative integers and $N \subset \mathbf{N}_0$ a subsemigroup of $\mathbf{N}_0$. We call $N$ *a numerical semigroup of genus* $g$ if the complement $\mathbf{N}_0 - N = \{j_1, j_2, \ldots, j_g\}$ $(j_1 < j_2 < \cdots < j_g)$ consists of $g$ elements.

Consider a couple $(X, P)$ of a nonsingular projective curve $X$ of genus $g$ over the complex number field $\mathbf{C}$ and a point $P \in X$. The coarse moduli space of all isomorphism classes of such couples is denoted by $\mathcal{M}_{g,1}$. Set $N_P := \{n \in \mathbf{N}_0 \mid$ there exists a rational function $f$ on $X$ such that $f$ is regular on $X - \{P\}$ and has a pole of order exactly $n$ at $P$ $\} \subset \mathbf{N}_0$. Then $N_P$ is a numerical semigroup of genus $g$ whose complement in $\mathbf{N}_0$ is the Weierstrass gap sequence at $P$.

For a given numerical semigroup $N$ of genus $g$, let $\mathcal{M}_{g,1}^N$ be the moduli space of all isomorphism classes of pointed algebraic curves $(X, P)$ with $N_P = N$. Then $\mathcal{M}_{g,1}^N$ is a subscheme of $\mathcal{M}_{g,1}$ and we have a direct sum decomposition of $\mathcal{M}_{g,1}$: $\mathcal{M}_{g,1} = \bigcup_N \mathcal{M}_{g,1}^N$,

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where $N$ runs through all possible numerical semigroups of genus $g$. We note that $\mathcal{M}_{g,1}^N$ is known to be non-empty for any numerical semigroup $N$ of genus $g \leq 7$ by [9].

In the first part [12], we calculated explicitly the defining equations of the projectivized moduli space $\overline{\mathcal{M}_{g,1}^N}$ in the ambient weighted projective space for any numerical semigroup $N$ of genus $g \leq 5$ except a few cases. We also computed the dimension and showed the irreducibility of $\mathcal{M}_{g,1}^N$ utilizing these defining equations.

Our main result in this note is that the moduli space $\mathcal{M}_{g,1}^N$ is *an irreducible rational variety* if $N$ is generated by less than five generators for $g \leq 6$ except $N = N(6)_5$ (see Table 2 in Appendix A for the notation of the semigroup). From this, we get several corollaries such as:

(1) a computational proof of the rationality of the moduli space $\mathcal{M}_{g,1}$ of pointed algebraic curves of genus $g$ for $1 \leq g \leq 3$,

(2) the rationality of $\mathcal{M}_{g,1}^N$ for any semigroup $N$ of genus $1 \leq g \leq 5$ except two cases,

(3) an interesting example of deformation of a monomial curve whose negative base space consists of two irreducible components.

We note that in [12], we observed that many of these $\overline{\mathcal{M}_{g,1}^N}$ are isomorphic to weighted projective spaces so that we claimed that $\mathcal{M}_{g,1}^N$ is unirational in these cases ([12; Corollary 3.4]), which is not good enough since they are actually rational. So we improve this weak corollary to the stronger Corollary 3.5 in this note.

We also note that the moduli space $\mathcal{M}_g$ of all algebraic curves of genus $g$ is known to be rational for $g \leq 6$ ([8]), and $\mathcal{M}_{4,1}$ is proved to be rational in [3], which are much deeper than our computational result.

We used the computer algebra system SINGULAR [6] for the computations of the defining ideal **Js** of the negative base space of the miniversal deformation of a monomial curve of $N$ and the primary decomposition of **Js**. We also used the computer algebra system Magma [2] for auxiliary computations.

## 2. Preliminaries

We first review the relation of deformation of monomial curves and $\mathcal{M}_{g,1}^N$ (Pinkham's theory) as needed later. A similar summary is found in [12; sec. 2]. For more details, see [13; Cap. IV].

Let $N \subset \mathbf{N}_0$ be a numerical semigroup of genus $g$. We simply call a numerical semigroup a semigroup for short from now on. Then $N$ is finitely generated as semigroup and is expressed as $N = \langle a_1, a_2, \ldots, a_m \rangle$ ($a_i \in \mathbf{N}_0$). The monomial ring $B_N$ of $N$ is defined as $B_N := \mathbf{C}[t^{a_1}, t^{a_2}, \ldots, t^{a_m}] \subset \mathbf{C}[t]$ where $\mathbf{C}[t]$ is the polynomial ring of 1 variable over $\mathbf{C}$, and $C_N := \operatorname{Spec} B_N$ is the monomial curve of $N$.

Given a semigroup $N$ of genus $g$, there always exists an algebraic miniversal deformation $\Phi : \mathcal{X} \to S$ of the monomial curve $C_N$. Namely,

$$\Phi : \mathcal{X} = \operatorname{Spec} \mathbf{C}[x_1, x_2, \dots, x_n, t_1, \dots, t_r]/\mathbf{Es} \to S = \operatorname{Spec} \mathbf{C}[t_1, t_2, \dots, t_r]/\mathbf{Is}$$

is a flat morphism such that $\Phi^{-1}(P) \cong C_N$, $P$ is the origin of $S$. **Es** and **Is** are ideals of the polynomial rings, and the morphism $\Phi$ is induced by the natural projection. Let $T^1(C_N)$ be the $\mathbf{C}$-vector space of isomorphism classes of the first order deformations of $C_N$. Then $T^1(C_N)$ is naturally isomorphic to the Zariski tangent space of $S$ at $P$ and $r = \dim_{\mathbf{C}} T^1(C_N)$. The 1-dimensional algebraic torus $\mathbf{C}^\times$ acts naturally on the monomial curve $C_N$, which induces a natural algebraic $\mathbf{C}^\times$-action on $\mathcal{X}$ and $S$ such that $\Phi$ is $\mathbf{C}^\times$-equivariant. We may suppose $\mathbf{C}^\times$ acts on $\mathcal{X}$ and $S$ by $\alpha \circ x_i := \alpha^{a_i} x_i$ and $\alpha \circ t_i := \alpha^{-e_i} t_i$ ($a_i, e_i \in \mathbf{Z}, \alpha \in \mathbf{C}^\times$).

We will use only the part of $S$ with weight $e_i < 0$ (the negative part of the base space $S$). Namely, let $\{i_1, \dots, i_l\} \subset \{1, 2, \dots, r\}$ be the integers with $e_{i_j} \geq 0$ and set $S_- := \{t_{i_1} = \cdots = t_{i_l} = 0\} \cap S$. We restrict the miniversal deformation $\Phi$ to $S_-$ and get $\varphi : \mathcal{X}' := \Phi^{-1}(S_-) \to S_-$. Once again, we reset $\mathcal{X}' = \operatorname{Spec} \mathbf{C}[x_1, x_2, \dots, x_n, t_1, \dots, t_s]/\mathbf{Fs}$, $S_- = \operatorname{Spec} \mathbf{C}[t_1, t_2, \dots, t_s]/\mathbf{Js}$ so that $\mathbf{C}^\times$ acts on $t_i$ with weight $-e_i > 0$ and on $x_i$ with weight $a_i$. Thus $\mathbf{C}[x_1, x_2, \dots, x_n, t_1, \dots, t_s]$ and $\mathbf{C}[t_1, t_2, \dots, t_s]$ are weighted polynomial rings with weight$(x_i) = a_i$ and weight$(t_i) = -e_i > 0$ and **Fs** and **Js** are homogeneous ideals in them.

We then projectivize each fiber of $\varphi$ by adding one point. More precisely, take a set of generators of the defining ideal **Fs** of $\mathcal{X}'$:

$$\mathbf{Fs} = \langle F_i \mid 1 \leq i \leq m \rangle \subset \mathbf{C}[x_1, x_2, \dots, x_n, t_1, \dots, t_s]\,.$$

Introduce a new indeterminate $x_{n+1}$ with weight 1 and substitute $t_i x_{n+1}^{-e_i}$ in the variable $t_i$ of $F_i$. Then we get a polynomial $F_i' \in \mathbf{C}[t_1, \dots, t_s, x_1, \dots, x_n, x_{n+1}]$, which is homogeneous with respect to the variables $(x_1, \dots, x_{n+1})$. Set $\mathbf{Fs}' := \langle F_i' \rangle$ and we get a projective flat morphism $\pi : \operatorname{Proj} \mathbf{C}[x_1, \dots, x_n, x_{n+1}, t_1, \dots, t_s]/\mathbf{Fs}' \to S_-$. Then any fiber of $\pi$ is a projective algebraic curve and the equation $\{x_{n+1} = 0\}$ gives a section of $\pi$ so that $\pi$ is a flat family of projective pointed algebraic curves over $S_-$.

The fundamental theorem of Pinkham ([13; Theorem (13.9)]) states:

THEOREM 2.1. *Set $U := \{x \in S_- \mid$ the fiber $\varphi^{-1}(x)$ is smooth$\} \subset S_-$. If $U$ is not empty, then there exists an isomorphism $U/\mathbf{C}^\times \cong \mathcal{M}_{g,1}^N$. This isomorphism is given by $U \ni x \mapsto \pi^{-1}(x) \in \mathcal{M}_{g,1}^N$.*

If $U$ is non-empty, we say the monomial curve $C_N$ is *negatively smoothable*. Thus $\mathcal{M}_{g,1}^N$ is non-empty if and only if $C_N$ is negatively smoothable.

We next review the well-known fact that the weighted projective space $\mathbf{P}^n{}_{(e_0, e_1, \dots, e_n)}$ with weights $(e_0, e_1, \dots, e_n)$ is rational. We also include a proof (taken from [7]) in order to prepare some notations for later use.

PROPOSITION 2.2. $\mathbf{P}^n{}_{(e_0,e_1,\dots,e_n)}$ *is rational.*

PROOF. Let $Z := \mathbf{P}^n$ be the ordinary projective space with homogeneous coordinates $(\alpha_0, \dots, \alpha_n)$, and $(\beta_0, \dots, \beta_n)$ the homogeneous coordinates of $W := \mathbf{P}^n{}_{(e_0,e_1,\dots,e_n)}$. Consider the finite morphism $p : Z \to W$ defined by $\beta_i := \alpha_i^{e_i}$ $(0 \le i \le n)$. If the action of the group $G := \mathbf{Z}_{e_0} \times \cdots \times \mathbf{Z}_{e_n}$ on $Z$ is defined by $(g_0, \dots, g_n) \circ (\alpha_0, \dots, \alpha_n) := (\zeta_{e_0}^{g_0}\alpha_0, \dots, \zeta_{e_n}^{g_n}\alpha_n)$, where $\mathbf{Z}_{e_i} := \mathbf{Z}/e_i\mathbf{Z}$, $(g_0, \dots, g_n) \in G$ and $\zeta_{e_i}$ is the primitive $e_i$-th root of unity, then $W$ is identified with the quotient space of this action.

We consider the affine open pieces $Z_0 := \{\alpha_0 \neq 0\}(\cong \mathbf{A}^n) \subset Z$ with inhomogeneous coordinates $(\gamma_1, \dots, \gamma_n) = (\alpha_1/\alpha_0, \dots, \alpha_n/\alpha_0)$ and $W_0 := \{\beta_0 \neq 0\} \subset W$. Let $H := \mathbf{Z}_{e_1} \times \cdots \times \mathbf{Z}_{e_n}$ be the subgroup of $G$ and consider the action of $H$ on $Z_0 \cong \mathbf{A}^n$ defined by $(g_1, \dots, g_n) \circ (\gamma_1, \dots, \gamma_n) := (\zeta_{e_1}^{g_1}\gamma_1, \dots, \zeta_{e_n}^{g_n}\gamma_n)$. Then the quotient space $V_0 := Z_0/H \cong \mathbf{A}^n/H$ is isomorphic again to an affine space $\mathbf{A}^n$ with coordinates $\delta_i := \gamma_i^{e_i}$ $(1 \le i \le n)$. Finally, consider the $\mathbf{Z}_{e_0}$-action on $V_0 \cong \mathbf{A}^n$ defined by $g_0 \circ (\delta_1, \dots, \delta_n) := (\zeta_{e_0}^{-g_0e_1}\delta_1, \dots, \zeta_{e_0}^{-g_0e_n}\delta_n)$. Then the restriction map $p|_{Z_0} : Z_0 \to W_0$ is factored as $Z_0 \to V_0 \cong \mathbf{A}^n \to \mathbf{A}^n/\mathbf{Z}_{e_0} \cong W_0$.

Now $W_0 \cong \mathbf{A}^n/\mathbf{Z}_{e_0}$ is rational. Indeed, consider the abelian subgroup

$$K := \{(m_1, \dots, m_n) \,|\, e_1m_1 + \cdots + e_nm_n \text{ is divisible by } e_0\} \subset \mathbf{Z}^n .$$

Then $K$ is also an abelian free group of rank $n$. Let $M_i := (m_{i1}, \dots, m_{in})$ $(1 \le i \le n)$ be the generators of $K$. Then the invariant subring of the Laurent polynomial ring $\mathbf{C}[\delta_1, \dots, \delta_n, \delta_1^{-1}, \dots, \delta_n^{-1}]$ under the induced $\mathbf{Z}_{e_0}$-action is given by $\mathbf{C}[\delta^{M_1}, \dots, \delta^{M_n}, \delta^{-M_1}, \dots, \delta^{-M_n}]$, where $\delta^{M_i} := \delta_1^{m_{i1}} \dots \delta_n^{m_{in}}$ (note that $m_{ij}$ may be negative). Thus the function field $\mathbf{C}(W_0)$ of $W_0$ is $\mathbf{C}(\delta^{M_1}, \dots, \delta^{M_n})$, which is rational. □

## 3. Main Result

We state and prove our main result:

THEOREM 3.1. *Let $N$ be any numerical semigroup of genus $1 \le g \le 6$ generated by less than 5 generators. Then $\mathcal{M}_{g,1}^N$ is an irreducible rational variety except $N = N(6)_5$.*

REMARK 3.2. In the exceptional case $N = N(6)_5$, we do not know if $\mathcal{M}_{g,1}^N$ is an irreducible rational variety or not, since we cannot compute the defining ideal **Js** of the negative base space $S_-$ in this case.

PROOF OF THEOREM 3.1. (i) Suppose $N$ is generated by 2 or 3 elements. Then the second cohomology group $T^2(C_N)$ of the cotangent complex of the monomial curve $C_N$ is equal to $\{0\}$ ([1; 4.2.1,4,2.2]). Thus the defining (obstruction) ideal **Js** of $S_-$ is equal to $\{0\}$.

On the other hand, $\mathcal{M}_{g,1}^N$ is non-empty for genus $g \le 7$ by [9]. Therefore $\mathcal{M}_{g,1}^N$ is a non-empty Zariski open subset of $S_-/\mathbf{C}^\times$ which is isomorphic to a weighted projective space since $\mathbf{Js} = \{0\}$. Thus $\mathcal{M}_{g,1}^N$ is an irreducible rational variety from Proposition 2.2 in this case.

(ii) Suppose $N$ is generated by 4 elements. There are 14 semigroups with 4 generators in the range $1 \le g \le 6$. Now we compute the ideal $\mathbf{Js}$ for these semigroups (for most semigroups of genus $g \le 5$, $\mathbf{Js}$ was computed in the first part [12]). The result is summarized in Appendix B. For this computation, we used the computer algebra system SINGULAR [6] and the SINGULAR library `deform.lib` for computing the miniversal deformations by Martin [10], [11]. We are very sorry that we cannot get the $\mathbf{Js}$ in the case of $N(6)_5$ for lack of memories.

We will show the irreducibility and rationality of $\mathcal{M}_{g,1}^N$ for the 14 semigroups except $N = N(6)_5$. This is done by case by case study of the generators of $\mathbf{Js}$.

First we note that if $N = N(5)_6, N(5)_{10}, N(5)_{11}, N(6)_{14}$, then our computation shows $T^2(C_N) = \{0\}$ and so $\mathcal{M}_{g,1}^N$ is a non-empty Zariski open subset of a weighted projective space. Hence we are done in these cases.

Let us study the other cases. We pick up three typical and interesting cases, namely $N = N(3)_4, N(6)_8, N(6)_{15}$, and investigate each of them in some detail.

(a) Suppose $N = N(3)_4$. In this case, $\mathbf{Js}$ is a homogeneous prime ideal so that $X := \operatorname{Proj} \mathbf{C}[A, B, \dots, K]/\mathbf{Js} \simeq S_-/\mathbf{C}^\times$ is an irreducible variety. In order to show $\mathbf{Js}$ is prime, we used the SINGULAR library `primdec.lib` [5]. Since $\mathcal{M}_{g,1}^N$ is a non-empty Zariski open subset of $X$, it remains to show the rationality of $X$.

Let $J_i$ $(1 \le i \le 6)$ be the generators of $\mathbf{Js}$ as in Appendix B. We choose the variable $D$ of $\mathbf{C}[A, \dots, D, \dots, K]$ whose weight is 2 (the smallest weight). Let $J_i'$ $(1 \le i \le 6)$ be the polynomial in $\mathbf{C}[a, b, c, e, \dots, k]$ obtained by setting $D = 1, a = A, \dots, k = K$ in $J_i$, and $\mathbf{Js}'$ be the ideal generated by them. Let $p : Z = \mathbf{P}^{10} \to W$ be as in Proposition 2.2 and set $V_0' := p^{-1}(X \cap \{D \neq 0\})/H \subset V_0 \cong \mathbf{A}^{11}$ and $W_0' := X \cap \{D \neq 0\} (\cong V_0'/\mathbf{Z}_2) \subset W_0$. Then $V_0'$ is isomorphic to $\operatorname{Spec} \mathbf{C}[a, b, c, e, \dots, k]/\mathbf{Js}'$, and $W_0'$ is isomorphic to $\operatorname{Spec} \{\mathbf{C}[a, b, c, e, \dots, k]/\mathbf{Js}'\}^{\mathbf{Z}_2}$, where $\mathbf{Z}_2 = \{1, \iota\}$ acts on $V_0$ by $\iota \circ (a, b, c, e, f, g, h, i, j, k) = (-a, -b, c, -e, f, g, h, -i, -j, k)$. Now $V_0'$ is rational. Indeed, from $J_1' = J_3' = J_4' = 0$, we can eliminate $f, b, g$ respectively, and the rest of the equations hold identically. Thus the function field $\mathbf{C}(V_0')$ of $V_0'$ is the rational function field $\mathbf{C}(a, c, e, h, i, j, k)$. Now $\mathbf{C}(W_0')$ is isomorphic to the invariant field $\mathbf{C}(a, c, e, h, i, j, k)^{\mathbf{Z}_2}$. This invariant field is $\mathbf{C}(a^2, ae, ai, aj, c, h, k)$, which is a rational function field of dimension 7.

Similar computations work for $N(4)_4$, $N(5)_4$, $N(5)_5.N(5)_9$, and $\mathcal{M}_{g,1}^N$ is an irreducible rational variety of dimension given in Table 1 for these semigroups.

(b) Suppose $N = N(6)_8$. In this case the ideal $\mathbf{Js}$ is not prime and is an intersection of two homogeneous prime ideals; $\mathbf{Js} = \mathbf{p}_1 \cap \mathbf{p}_2$ (for the generators of $\mathbf{p}_i$, see Appendix B). Thus $S_- = Y_1 \cup Y_2$, where $Y_i := V(\mathbf{p}_i)$ (the affine variety defined by $\mathbf{p}_i$). Both of the components

$Y_i$ are 12-dimensional and $Y_1 \cap Y_2$ is 11-dimensional. On the first component $Y_1$, any fiber of $\varphi$ is singular. Indeed, it is clear that $Y_1 \simeq \mathbf{A}^{12}$ by eliminating the variables $O, M, K$ from the defining equations of $Y_1$. Restrict $\varphi$ to $Y_1 \simeq \mathbf{A}^{12}$ and consider the generic fiber $\varphi^{-1}(\eta)$ ($\eta \in \mathbf{A}^{12}$ is the generic point) defined over the rational function field $k := \mathbf{C}(A, \dots, N)$ ($K, M$ omitted). From the generators of the defining ideal **Fs** of the total space $\mathcal{X}'$, we find that $\varphi^{-1}(\eta)$ is defined by the following 6 equations $j_1, \dots, j_6$ in the affine space $\mathbf{A}^4(k)$ with variables $(x, y, z, w)$. In the following, a monomial $x^a y^b z^c w^d$ is written as xaybzcwd which is the output format of SINGULAR.

```
j_1=(-GJ)*x2y2+(J)*y4+(-HJ)*x2y+(-BJ-GJN)*xy2+(GJ2-JL)*y3-xz2+(GJL2+HJL
      -IJ)*x2+(BJL-DJ-HJN)*xy+(-CJ-GJ2L+2GJN2+HJ2)*y2+(-FJ)*xz+(-2N)*z2+w2+
      (DJL+GJL2N+HJLN-IJN)*x+(BJLN+2CJL+DJN-EJ-GJ2L2-2HJ2L+2HJN2)*y+(-2FJN)*z
      +(AJ)*w+(-BJL2N-CJL2-DJLN+EJL+GJ2L3-2GJL2N2+HJ2L2-2HJLN2+2IJN2);
j_2=(G)*x2y2-y4+(H)*x2y+(B+GN)*xy2+(-GJ+L)*y3+(-GL2-HL+I)*x2+(-BL+D+HN)*xy
      +(C+GJL-2GN2-HJ)*y2+(F)*xz+zw+(-DL-GL2N-HLN+IN)*x+(-BLN-2CL-DN+E+GJL2
      +2HJL-2HN2)*y+(AJ+2FN)*z+(A)*w+(A2J+BL2N+CL2+DLN-EL-GJL3+2GL2N2-HJL2
      +2HLN2-2IN2);
j_3=-x2z+(-N)*xz+(J)*yz+yw+(AJ)*y+(-JL+2N2)*z+(-L)*w+(-AJL);
j_4=-xy3+(B)*x2y+(G+N)*y3+(D)*x2+(-BN+C)*xy+(-GL+H)*y2+z2+(-BLN-CL-2DN+E+IJ)
      *x+(-CN-GL2-2HL+I)*y+(A+FJ)*z+(F)*w+(AFJ+BLN2+CLN+DN2-EN+GL3+HL2-IJN
      -IL);
j_5=yz-xw+(-AJ)*x+(-L)*z+(N)*w+(AJN);
j_6=-x3+(J)*xy+y2+(-JL+3N2)*x+(-JN-2L)*y+(JLN+L2-2N3)
```

Then the jacobian matrix $m_1$ of the defining equations of $\varphi^{-1}(\eta)$ with respect to $(x, y, z, w)$ is a $6 \times 4$ matrix whose $(i, j)$-th entry $m1[i, j]$ is given by:

```
m1[1,1]=(-2GJ)*xy2+(-2HJ)*xy+(-BJ-GJN)*y2-z2+(2GJL2+2HJL-2IJ)*x+(BJL-DJ-HJN)
        *y+(-FJ)*z+(DJL+GJL2N+HJLN-IJN);
m1[1,2]=(-2GJ)*x2y+(4J)*y3+(-HJ)*x2+(-2BJ-2GJN)*xy+(3GJ2-3JL)*y2+(BJL-DJ-HJN)
        *x+(-2CJ-2GJ2L+4GJN2+2HJ2)*y+(BJLN+2CJL+DJN-EJ-GJ2L2-2HJ2L+2HJN2);
m1[1,3]=-2*xz+(-FJ)*x+(-4N)*z+(-2FJN);
m1[1,4]=2*w+(AJ);
m1[2,1]=(2G)*xy2+(2H)*xy+(B+GN)*y2+(-2GL2-2HL+2I)*x+(-BL+D+HN)*y+(F)*z+
        (-DL-GL2N-HLN+IN);
m1[2,2]=(2G)*x2y-4*y3+(H)*x2+(2B+2GN)*xy+(-3GJ+3L)*y2+(-BL+D+HN)*x
        +(2C+2GJL-4GN2-2HJ)*y+(-BLN-2CL-DN+E+GJL2+2HJL-2HN2);
m1[2,3]=(F)*x+w+(AJ+2FN);
m1[2,4]=z+(A);
m1[3,1]=-2*xz+(-N)*z;
m1[3,2]=(J)*z+w+(AJ);
m1[3,3]=-x2+(-N)*x+(J)*y+(-JL+2N2);
m1[3,4]=y+(-L);
m1[4,1]=-y3+(2B)*xy+(2D)*x+(-BN+C)*y+(-BLN-CL-2DN+E+IJ);
m1[4,2]=-3*xy2+(B)*x2+(3G+3N)*y2+(-BN+C)*x+(-2GL+2H)*y+(-CN-GL2-2HL+I);
m1[4,3]=2*z+(A+FJ);
m1[4,4]=(F);
m1[5,1]=-w+(-AJ);
m1[5,2]=z;
m1[5,3]=y+(-L);
m1[5,4]=-x+(N);
m1[6,1]=-3*x2+(J)*y+(-JL+3N2);
m1[6,2]=(J)*x+2*y+(-JN-2L);
```

```
m1[6,3]=0;
m1[6,4]=0
```

Take a $k$-rational point $P := (N, L, 0, -AJ) \in k^4$ of $\varphi^{-1}(\eta)$. Then setting $x = N$, $y = L$, $z = 0$, $w = -AJ$, the jacobian matrix at $P$ is:

$$m_1(P) = \begin{pmatrix} -3IJN & -EJ + JL^3 & -3FJN & -AJ \\ 3IN & E - L^3 & 3FN & A \\ 0 & 0 & 0 & 0 \\ E + IJ - L^3 & I & A + FJ & F \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the rank of $m_1(P)$ is 2 and so the generic fiber is singular at $P$.

On the other hand, the projectivization $X_2 := \operatorname{Proj} \mathbf{C}[A, B, \ldots, O]/\mathbf{p}_2$ of the second component $Y_2$ is rational. Indeed, since weight($M$) = 1, by setting $M = 1$ in the generators of $\mathbf{p}_2$, we get a set of inhomogeneous generators $l_i$ $(1 \le i \le 5)$ which defines the affine part $X_2' := X_2 \cap \{M \neq 0\} \subset \{M \neq 0\} \simeq \mathbf{A}^{14}$, where

```
l_1=F;
l_2=2GLO-GO2+AO+HO-I;
l_3=L3O-IJL+2IN2-IK-EO;
l_4=4GJL2-8GLN2+4GN2O-GJO2+2AJL+2HJL+4GKL-2L3-4AN2-4HN2+AJO+HJO-2GKO-IJ+2AK
    +2HK+2E;
l_5=GO4-8GIJL+16GIN2-4AL2O-4HL2O-2ALO2-2HLO2-AO3-HO3-8GIK+4IL2-8EGO+2ILO+IO2
```

Setting $F = 0$, we may assume $X_2'$ is defined by $l_i = 0$ $(2 \le i \le 5)$ in $\mathbf{A}^{13}$. From $l_2 = 0$, $l_4 = 0$, we eliminate $I$, $E$ respectively. Then $l_3 = 0$ and $l_5 = 0$ holds identically. Thus the affine part $X_2'$ is isomorphic to $\mathbf{A}^{11}$. Hence $X_2$ is rational.

The generic fiber of $\varphi$ over $X_2$ is smooth. This is guaranteed by [9]. To make sure, we can take a random point from $X_2$, say $P := (1, 0, 0, 0, 12, 0, 2, 1, 0, 1, -1, 1, 2, 1, 0) \in X_2$. Then by a direct computation of the rank of the jacobian matrix, we find that $\varphi^{-1}(P)$ is smooth. Thus $\mathcal{M}_{6,1}^{N(6)_8}$ is a non-empty Zariski open subset of $X_2$, which shows that $\mathcal{M}_{6,1}^{N(6)_8}$ is an irreducible rational variety of dimension 11.

(c) Suppose $N = N(6)_{15}$. In this case, we cannot show that the ideal $\mathbf{Js}$ is prime since the generators of $\mathbf{Js}$ are too complicated. Instead, we proceed as follows. Let $W := \operatorname{Proj} \mathbf{C}[A, B, \ldots, P]$ be the ambient weighted projective space of $X := \operatorname{Proj} \mathbf{C}[A, B, \ldots, P]/\mathbf{Js}$, $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ (see Appendix B for the explicit form of $J_i$). Since weight$(O) = 1$, by setting $O = 1$ in $J_i$, we get the inhomogeneous generators $j_i$ $(1 \le i \le 6)$ of the defining ideal $\mathbf{js}$ of $X' := X \cap \{O \neq 0\}$ in $\{O \neq 0\} \simeq \mathbf{A}^{15}$ :

```
j_1 = -G2N-2GMN-2GN3-IN-JK-KL-M2N-2MN3-N5;
j_2 = -DGK-DKN2-FK-G2K-G2P-2GKN2-2GMP-2GN2P-IP-K2N-KN4-KNP-M2P-2MN2P-N4P;
j_3 = -AK+DKN+G2M-G2N2+2GKN+2GM2-2GN4+IM-IN2+KMN+KN3+M3+M2N2-MN4-N6;
j_4 = -AN+DN2+2GN2-JM+JN2-LM+LN2+MN2+N4;
j_5 = DGN+DN3+FN+G^2N+2GN3-JP+KN2-LP+N5+N2P;
j_6 = -AP-DGM-DMN2+DNP-FM-G2M-2GMN2+2GNP+JNP-KMN+LNP-MN4+N3P.
```

From $j_1 = 0, j_4 = 0, j_5 = 0$, we eliminate $I, A, F$ respectively (note that we need the variable $N$ in the denominator). Then the remaining three equations hold identically. Thus $X'$ is isomorphic to $\mathbf{A}^{12}$ if restricted to the open subset $\{N \neq 0\} \subset \mathbf{A}^{15}$. Set $X'' := X' \cap \{N \neq 0\} \subset \operatorname{Spec} \mathbf{C}[A, B, \dots, P, 1/N]$ ($O$ omitted). Let $\overline{X''}$ be the closure of $X''$ in $\mathbf{A}^{15}$. Then $\overline{X''}$ is defined by the saturation ideal **ks** $:=$ (**js** $: N^{\infty}$) of the ideal **js** with respect to $N$. Using the weighted degree reverse lex order $(A, B, \dots, P)$ ($O$ omitted) with weights $(9, 4, 2, 7, 8, 13, 6, 3, 12, 5, 9, 5, 6, 2, 10)$, we can calculate the saturation ideal **ks** by using "sat(**js**,N)" command of the SINGULAR library `elim.lib`. Then we get **ks** $=$ **js**. Thus we have $\overline{X''} = X'$.

Next we show the projective closure $\overline{X'}$ of $X'$ in $W$ is equal to $X$. Indeed, the defining equations of $\overline{X'}$ in $W$ can be obtained as the homogenization of the Groebner basis of the ideal **js** with respect to a degree monomial order ([4; 8.4 Th. 4]). Now with respect to the same weighted degree reverse lex order as above, the generators $j_1, \dots, j_6$ happen to be the Groebner basis of **js** (this is not always the case). The homogenization of $j_1, \dots, j_6$ are equal to $J_1, \dots, J_6$. Thus we find $\overline{X'} = X$. Hence $X$ is an irreducible rational variety (if we give the reduced structure).

The existence of a smooth fiber of $\varphi$ is guaranteed by [9]. Or if we take a point $R := (1, -1, 1, -1, -1, 10, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1) \in X$, then we can check the fiber $\varphi^{-1}(P)$ is smooth by calculating the rank of the jacobian matrix. Thus $\mathcal{M}_{6,1}^{N(6)_{15}}$ is a non-empty Zariski open subset of $X$, which shows that $\mathcal{M}_{6,1}^{N(6)_{15}}$ is an irreducible rational variety of dimension 12.

The same argument works for $N = N(6)_6, N(6)_{13}$. In these 3 cases, we do not know if **Js** is prime (**Js** may have an embedded component). We just showed that $\mathcal{M}_{g,1}^{N}$ with reduced structure is an irreducible rational variety. $\square$

We discuss some corollaries which are deduced from Theorem 3.1 and also give several remarks.

Let $\mathcal{M}_{g,1}$ be the moduli space of pointed algebraic curves of genus $g$ and $N$ be the semigroup of ordinary points, namely $N = \langle g+1, g+2, \dots, 2g+1 \rangle = \{0, g+1, g+2, \dots\}$. Then $\mathcal{M}_{g,1}$ is an irreducible variety of dimension $3g - 2$ and $\mathcal{M}_{g,1}^{N}$ is a dense subscheme of $\mathcal{M}_{g,1}$ ([13; (14.11)(5)]). Thus the rationality of $\mathcal{M}_{g,1}^{N}$ is equivalent to that of $\mathcal{M}_{g,1}$. From Theorem 3.1, we have:

COROLLARY 3.3. *$\mathcal{M}_{g,1}$ is rational for $1 \leq g \leq 3$.*

REMARK 3.4. In [3], the rationality of $\mathcal{M}_{4,1}$ is proved in a geometrical way. So the next question is if $\mathcal{M}_{5,1}$ is rational or not.

In the range $1 \leq g \leq 5$, we have:

COROLLARY 3.5. *$\mathcal{M}_{g,1}^{N}$ is an irreducible rational variety for any semigroup of genus $1 \leq g \leq 5$ except $N(5)_8$ and $N(5)_{12}$.*

REMARK 3.6. In the case of $N = N(6)_8$, we observed that the negative base space $S_-$ of the miniversal deformation of the monomial curve $C_N$ consists of 2 irreducible components. On one component, $C_N$ is (negatively) smoothable and on the other component, it is not. Up to genus 6, this is the only semigroup whose negative base space is reducible as far as we can calculate.

REMARK 3.7. Our main ingredient of the proof of Theorem 3.1 is the computation of the obstruction ideal **Js** by means of SINGULAR. The computation of **Js** for a semigroup $N$ with 5 (or more) generators seems far beyond the reach of our computational ability.

## Appendix A

In this appendix, we summarize the information of $\mathcal{M}_{g,1}^N$ obtained so far for any numerical semigroup $N$ of genus $1 \le g \le 6$ in the following 2 tables. In the notation $N = N(g)_i = \langle a_1, \dots, a_n \rangle$, $g$ is the genus, $i$ is the ID number and $\{a_1, \dots, a_n\}$ are the generators of $N$ as semigroup. We set $\overline{\mathcal{M}_{g,1}^N} := S_-/\mathbf{C}^\times = \operatorname{Proj} \mathbf{C}[A, B, \dots]/\mathbf{Js}$.

TABLE 1. dimension, structure, irreducibility and rationality of $\overline{\mathcal{M}_{g,1}^N}$ for $1 \le g \le 5$

| semigroup = ⟨generators⟩ | $\dim \mathcal{M}_{g,1}^N$ | structure of $\overline{\mathcal{M}_{g,1}^N}$ | irreducibility | rationality |
|---|---|---|---|---|
| $N(1)_1 = \langle 2, 3 \rangle$ | 1 | $\mathbf{P}^1_{(4,6)}$ | ○ | ○ |
| $N(2)_1 = \langle 2, 5 \rangle$ | 3 | $\mathbf{P}^3_{(4,6,8,10)}$ | ○ | ○ |
| $N(2)_2 = \langle 3, 4, 5 \rangle$ | 4 | $\mathbf{P}^4_{(5,2,3,6,4)}$ | ○ | ○ |
| $N(3)_1 = \langle 2, 7 \rangle$ | 5 | $\mathbf{P}^5_{(4,6,8,10,12,14)}$ | ○ | ○ |
| $N(3)_2 = \langle 3, 5, 7 \rangle$ | 6 | $\mathbf{P}^6_{(7,1,4,3,6,9,5)}$ | ○ | ○ |
| $N(3)_3 = \langle 3, 4 \rangle$ | 5 | $\mathbf{P}^5_{(2,5,8,6,9,12)}$ | ○ | ○ |
| $N(3)_4 = \langle 4, 5, 6, 7 \rangle$ | 7 | $\operatorname{Proj} \mathbf{C}[A, \dots, K]/\mathbf{Js}$ | ○ | ○ |
| $N(4)_1 = \langle 2, 9 \rangle$ | 7 | $\mathbf{P}^7_{(4,6,8,10,12,14,16,18)}$ | ○ | ○ |
| $N(4)_2 = \langle 3, 7, 8 \rangle$ | 8 | $\mathbf{P}^8_{(4,7,3,6,9,2,5,8,6)}$ | ○ | ○ |
| $N(4)_3 = \langle 3, 5 \rangle$ | 7 | $\mathbf{P}^7_{(1,4,7,10,6,9,12,15)}$ | ○ | ○ |
| $N(4)_4 = \langle 4, 6, 7, 9 \rangle$ | 9 | $\operatorname{Proj} \mathbf{C}[A, \dots, M]/\mathbf{Js}$ | ○ | ○ |
| $N(4)_5 = \langle 4, 5, 7 \rangle$ | 8 | $\mathbf{P}^8_{(3,1,5,2,6,10,4,8,7)}$ | ○ | ○ |
| $N(4)_6 = \langle 4, 5, 6 \rangle$ | 7 | $\mathbf{P}^7_{(10,2,6,3,7,4,8,12)}$ | ○ | ○ |
| $N(4)_7 = \langle 5, 6, 7, 8, 9 \rangle$ | 10 | ? | ○ | ○ |
| $N(5)_1 = \langle 2, 11 \rangle$ | 9 | $\mathbf{P}^9_{(4,6,8,10,12,14,16,18,20,22)}$ | ○ | ○ |
| $N(5)_2 = \langle 3, 8, 10 \rangle$ | 10 | $\mathbf{P}^{10}_{(5,8,3,6,9,12,1,4,7,10,6)}$ | ○ | ○ |
| $N(5)_3 = \langle 3, 7, 11 \rangle$ | 9 | $\mathbf{P}^9_{(11,2,5,8,3,6,9,12,15,7)}$ | ○ | ○ |
| $N(5)_4 = \langle 4, 7, 9, 10 \rangle$ | 11 | $\operatorname{Proj} \mathbf{C}[A, \dots, O]/\mathbf{Js}$ | ○ | ○ |
| $N(5)_5 = \langle 4, 6, 9, 11 \rangle$ | 10 | $\operatorname{Proj} \mathbf{C}[A, \dots, N]/\mathbf{Js}$ | ○ | ○ |
| $N(5)_6 = \langle 4, 6, 7 \rangle$ | 9 | $\mathbf{P}^9_{(4,8,2,6,10,14,1,5,8,12)}$ | ○ | ○ |
| $N(5)_7 = \langle 4, 5, 11 \rangle$ | 9 | $\mathbf{P}^9_{(6,11,2,3,7,4,8,12,5,10)}$ | ○ | ○ |
| $N(5)_8 = \langle 5, 7, 8, 9, 11 \rangle$ | 12 | ? | ? | ? |
| $N(5)_9 = \langle 5, 6, 8, 9 \rangle$ | 11 | $\operatorname{Proj} \mathbf{C}[A, \dots, O]/\mathbf{Js}$ | ○ | ○ |
| $N(5)_{10} = \langle 5, 6, 7, 9 \rangle$ | 10 | $\mathbf{P}^{10}_{(2,3,4,9,7,1,8,6,5,5,3)}$ | ○ | ○ |
| $N(5)_{11} = \langle 5, 6, 7, 8 \rangle$ | 9 | $\mathbf{P}^9_{(8,7,2,3,4,5,10,6,9,4)}$ | ○ | ○ |
| $N(5)_{12} = \langle 6, 7, 8, 9, 10, 11 \rangle$ | 13 | ? | ○ | ? |

TABLE 2. dimension, structure, irreducibility and rationality of $\overline{\mathcal{M}_{g,1}^N}$ for $g = 6$

| semigroup= ⟨generators⟩ | dim $\mathcal{M}_{g,1}^N$ | structure of $\overline{\mathcal{M}_{g,1}^N}$ | irreducibility | rationality |
|---|---|---|---|---|
| $N(6)_1 = \langle 2, 13\rangle$ | 11 | $\mathbf{P}^{11}_{(4,6,8,10,12,14,16,18,20,22,24,26)}$ | ○ | ○ |
| $N(6)_2 = \langle 3, 10, 11\rangle$ | 12 | $\mathbf{P}^{12}_{(4,7,10,2,5,8,11,3,6,9,12,6,9)}$ | ○ | ○ |
| $N(6)_3 = \langle 3, 8, 13\rangle$ | 11 | $\mathbf{P}^{11}_{(13,8,1,4,7,10,6,9,12,15,18,3)}$ | ○ | ○ |
| $N(6)_4 = \langle 3, 7\rangle$ | 10 | $\mathbf{P}^{10}_{(2,5,8,11,14,6,9,12,15,18,21)}$ | ○ | ○ |
| $N(6)_5 = \langle 4, 9, 10, 11\rangle$ | ? | ? | ? | ? |
| $N(6)_6 = \langle 4, 7, 10, 13\rangle$ | 12 | Proj $\mathbf{C}[A, \dots, P]/\mathbf{Js}$ | ○ | ○ |
| $N(6)_7 = \langle 4, 7, 9\rangle$ | 11 | $\mathbf{P}^{11}_{(4,8,12,1,5,3,7,2,6,10,14,9)}$ | ○ | ○ |
| $N(6)_8 = \langle 4, 6, 11, 13\rangle$ | 11 | Proj $\mathbf{C}[A, \dots, O]/\mathbf{p_2}$ | ○ | ○ |
| $N(6)_9 = \langle 4, 6, 9\rangle$ | 10 | $\mathbf{P}^{10}_{(3,2,6,8,12,4,8,12,10,14,18)}$ | ○ | ○ |
| $N(6)_{10} = \langle 4, 5\rangle$ | 10 | $\mathbf{P}^{10}_{(2,6,10,3,7,11,15,8,12,16,20)}$ | ○ | ○ |
| $N(6)_{11} = \langle 5, 8, 9, 11, 12\rangle$ | ? | ? | ? | ? |
| $N(6)_{12} = \langle 5, 7, 9, 11, 13\rangle$ | ? | ? | ? | ? |
| $N(6)_{13} = \langle 5, 7, 8, 11\rangle$ | 12 | Proj $\mathbf{C}[A, \dots, P]/\mathbf{Js}$ | ○ | ○ |
| $N(6)_{14} = \langle 5, 7, 8, 9\rangle$ | 11 | $\mathbf{P}^{11}_{(6,8,7,9,1,10,2,3,4,6,11,5)}$ | ○ | ○ |
| $N(6)_{15} = \langle 5, 6, 9, 13\rangle$ | 12 | Proj $\mathbf{C}[A, \dots, P]/\mathbf{Js}$ | ○ | ○ |
| $N(6)_{16} = \langle 5, 6, 8\rangle$ | 11 | $\mathbf{P}^{11}_{(2,4,5,10,4,7,12,3,8,10,1,6)}$ | ○ | ○ |
| $N(6)_{17} = \langle 5, 6, 7\rangle$ | 11 | $\mathbf{P}^{10}_{(14,3,8,4,9,10,15,6,2,7,5)}$ | ○ | ○ |
| $N(6)_{18} = \langle 6, 8, 9, 10, 11, 13\rangle$ | ? | ? | ? | ? |
| $N(6)_{19} = \langle 6, 7, 9, 10, 11\rangle$ | ? | ? | ? | ? |
| $N(6)_{20} = \langle 6, 7, 8, 10, 11\rangle$ | ? | ? | ? | ? |
| $N(6)_{21} = \langle 6, 7, 8, 9, 11\rangle$ | ? | ? | ? | ? |
| $N(6)_{22} = \langle 6, 7, 8, 9, 10\rangle$ | ? | ? | ? | ? |
| $N(6)_{23} = \langle 7, 8, 9, 10, 11, 12, 13\rangle$ | 16 | ? | ○ | ? |

## Appendix B

We list the generators of the defining ideal **Js** of $S_-$ for the semigroup $N$ with 4 generators and $T^2(C_N) \neq \{0\}$ in the range $1 \leq g \leq 6$ (except $N = N(6)_5$). A monomial $A^a B^b \dots$ is written as $AaBb\dots$.

• $\overline{\mathcal{M}_{3,1}^{N(3)_4}}$ := Proj $\mathbf{C}[A, B, \dots, K]/\mathbf{Js}$, where $\mathbf{C}[A, B, \dots, K]$ is a weighted graded polynomial ring of 11 variables with weights (7, 5, 6, 2, 3, 8, 6, 4, 5, 3, 4) and $\mathbf{Js} = \langle J_1, \dots, J_6\rangle$ is a homogeneous prime ideal generated by

```
J_1=-AC-FI-2H2I+AEJ+CHJ+HIK-CJK-EHJ2+DIJ2+EJ2K;
J_2=BC-CI+GI+FJ-AK+CDJ-BEJ+2H2J-IJ2+DIK-JK2-DEJ2-DJ3;
J_3=-BI+I2+AJ-DIJ-HJ2+J2K;
J_4=GI-AK-EIJ-IJ2+DIK+HJK-JK2;
J_5=GJ-BK+IK-EJ2-J3;
J_6=CG+FK-CEJ-CJ2+CDK-BEK+2H2K+EIK-HK2-DEJK-DJ2K;
```

• $\overline{\mathcal{M}_{4,1}^{N(4)_4}}$ := Proj $\mathbf{C}[A, B, \ldots, M]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, M]$ is a weighted graded polynomial ring of 13 variables with weights $(5, 7, 3, 6, 4, 6, 5, 1, 2, 8, 8, 4, 10)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ is a homogeneous prime ideal generated by

```
J_1=-BJ-GM+3EFG+BFI-FHJ-CJL+CIM-3CEFI+F2HI+CFIL;
J_2=-BC-AG-CFH+ACI-C2L;
J_3=-BD+GK-DFH-CIK-CDL;
J_4=-AD-CK;
J_5=-AJ+CM-3CEF+AFI;
J_6=-JK-DM+3DEF+FIK;
```

• $\overline{\mathcal{M}_{5,1}^{N(5)_4}}$ := Proj $\mathbf{C}[A, B, \ldots, O]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, O]$ is a weighted graded polynomial ring of 15 variables with weights $(1, 3, 6, 10, 7, 2, 5, 9, 7, 4, 8, 6, 4, 5, 8)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ is a homogeneous prime ideal generated by

```
J_1=-IK+HL+IO+FIL+BL2-IM2;
J_2=EL-IL+KN-NO+AJK-FLN+M2N-AJO-AFJL+AJM2;
J_3=-EI+I2-HN-AHJ-BLN-ABJL;
J_4=DN+EO-IO+ADJ-BEN+2BIN+CMN-ABEJ+2ABIJ+ACJM;
J_5=-DL+KO-O2-BIL-CLM-BKN-FLO+M2O+BNO-ABJK+BFLN
     -BM2N+ABJO+ABFJL-ABJM2;
J_6=DI-HO+BI2+CIM+BHN-BLO+ABHJ+B2LN+AB2JL;
```

• $\overline{\mathcal{M}_{5,1}^{N(5)_5}}$ := Proj $\mathbf{C}[A, B, \ldots, N]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, N]$ is a weighted graded polynomial ring of 14 variables with weights $(4, 8, 4, 8, 2, 6, 10, 14, 6, 9, 12, 3, 1, 7)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ is a homogeneous prime ideal generated by

```
J_1=-BJ+HL;
J_2=HI+BK;
J_3=HM-BN;
J_4=-IJ-KL-HM+BN;
J_5=KM+IN;
J_6=-JM+LN;
```

• $\overline{\mathcal{M}_{5,1}^{N(5)_9}}$ := Proj $\mathbf{C}[A, B, \ldots, O]/\mathbf{J_5}$, where $\mathbf{C}[A, B, \ldots, O]$ is a weighted graded polynomial ring of 15 variables with weights $(3, 9, 1, 2, 4, 10, 4, 5, 6, 5, 6, 7, 7, 2, 8)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ is a homogeneous prime ideal generated by

```
J_1=-2HI+IJ-GL;
J_2=-2BH+BJ+LM+2AHK-AJK;
J_3=-BG-IM+AGK;
J_4=-2FH+FJ-LO-2H3+3H2J-3HJ2+J3+2EHK-EJK+G2L+2CGH2
     -2CGHJ+1/2CGJ2-2CDHM+CDJM -1/2C2G2H+1/4C2G2J;
J_5=-FG+IO-GH2-G2I+GHJ-GJ2+EGK+CG2H-1/2CG2J-CDGM
     -1/4C2G3;
J_6=FM+BO+H2M+GIM-HJM+J2M-EKM-AKO-CGHM+1/2CGJM
    +CDM2+1/4C2G2M;
```

• $\overline{\mathcal{M}_{6,1}^{N(6)_6}}$ := Proj $\mathbf{C}[A, B, \ldots, K]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, K]$ is a weighted graded polynomial ring of 11 variables with weights $(7, 5, 6, 2, 3, 8, 6, 4, 5, 3, 4)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ is a homogeneous ideal. The equations $J_1, \ldots, J_6$ are too lengthy to write down here so that we omit them (for instance $J_1$ consists of 228 terms).

• $\overline{\mathcal{M}_{6,1}^{N(6)_8}}$ := Proj $\mathbf{C}[A, B, \ldots, O]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, O]$ is a weighted graded polynomial ring of 15 variables with weights $(11, 8, 12, 14, 18, 9, 4, 10, 16, 2, 8, 6, 1, 4, 6)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6 \rangle$ is a homogeneous *non-prime* ideal generated by

```
J_1=AK+EM+AJL+HKM-2AN2+HJLM+2GKLM-L3M-2HMN2-GKMO+2GJL2M-4GLMN2-GJLMO+2GMN2O
J_2=IK+EO+IJL-2IN2-L3O
```

```
J_3=-FK-FJL+2FN2
J_4=FM
J_5=IM-AO-HMO-2GLMO+GMO2
J_6=FO
```

$\mathbf{Js} = \mathbf{p}_1 \cap \mathbf{p}_2$, where $\mathbf{p}_i$ is a homogeneous prime ideal. $\mathbf{p}_1 = \langle O, M, JL - 2N^2 + K\rangle$ and $\mathbf{p}_2 = \langle k_1, \ldots, k_5\rangle$ with

```
k_1=F,
k_2=2GLMO-GMO2+HMO-IM+AO,
k_3=L3O-IJL+2IN2-IK-EO,
k_4=4GJL2M-8GLMN2+4GMN2O-GJMO2+2HJLM+4GKLM-2L3M-4HMN2+HJMO-2GKMO+2AJL-IJM
   +2HKM-4AN2+AJO+2AK+2EM,
k_5=GMO4-8GIJLM+16GIMN2-4HL2MO-2HLMO2-HMO3-8GIKM+4IL2M-4AL2O
    -8EGMO+2ILMO-2ALO2+IMO2-AO3
```

• $\overline{\mathcal{M}_{6,1}^{N(6)_{13}}} :=$ Proj $\mathbf{C}[A, B, \ldots, P]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, P]$ is a weighted graded polynomial ring of 16 variables with weights $(11, 3, 1, 6, 2, 7, 4, 9, 14, 5, 10, 8, 8, 4, 2, 5)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6\rangle$ is a homogeneous ideal generated by

```
J_1=KM-IN-BHNO-FJNO+FNOP-J2NO2+2JNO2P-NO2P2-B2GNO2-BEJNO2+BJNO3+BENO2P-BNO3P
J_2=-AM-IP+EMNP-BHOP-FJOP-MNOP+FOP2-J2O2P+2JO2P2-O2P3-CEMNO2+CMNO3-B2GO2P
    -BEJO2P+BJO3P+BEO2P2-BO3P2
J_3=AN+KP-EN2P+N2OP+CEN2O2-CN2O3
J_4=-LN-MN+KO+BNP-EN2O+N2O2
J_5=-LM-M2+IO+BMP-EMNO+BHO2+FJO2+MNO2-FO2P+J2O3-2JO3P+O3P2+B2GO3+BEJO3-BJO4
    -BEO3P+BO4P
J_6=-AO-LP-MP+BP2-CENO3+CNO4
```

• $\overline{\mathcal{M}_{6,1}^{N(6)_{15}}} :=$ Proj $\mathbf{C}[A, B, \ldots, P]/\mathbf{Js}$, where $\mathbf{C}[A, B, \ldots, P]$ is a weighted graded polynomial ring of 16 variables with weights $(9, 4, 2, 7, 8, 13, 6, 3, 12, 5, 9, 5, 6, 2, 1, 10)$ and $\mathbf{Js} = \langle J_1, \ldots, J_6\rangle$ is a homogeneous ideal generated by

```
J_1=-JK-KL-IN-G2N-2GMN-M2N-2GN3O2-2MN3O2-N5O4
J_2=-FK-IP-DGK-G2P-2GMP-M2P-G2KO-KNOP-K2NO2-DKN2O2-2GN2O2P-2MN2O2P-2GKN2O3
    -N4O4P-KN4O5
J_3=-AK+IM+G2M+2GM2+M3+DKN+2GKNO+KMNO-IN2O2-G2N2O2+M2N2O2+KN3O3-2GN4O4-MN4O4
    -N6O6
J_4=-JM-LM-AN+DN2+2GN2O+MN2O+JN2O2+LN2O2+N4O3
J_5=FN-JP-LP+DGN+G2NO+N2OP+KN2O2+DN3O2+2GN3O3+N5O5
J_6=-FM-AP-DGM+DNP-G2MO+2GNOP-KMNO2+JNO2P+LNO2P-DMN2O2-2GMN2O3+N3O3P-MN4O5
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