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Two-bridge links with strong triviality

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Abstract. In this article, we study strong triviality of two-bridge links. We prove that every (non-trivial) two-bridge link can not be strongly n-trivial for $n \geq 1$.

1. Introduction

A knot K in S^3 is called strongly n-trivial if there exist $n+1$ crossings contained in a diagram of K such that the result of any $0 < m \leq n+1$ crossing changes on these crossings is the trivial knot (Figure 1). Note that by definition, the unknotting number of strongly n-trivial knots is one. Recently, Askitas–Kalfagianni and Howards–Luecke have started the study of strongly n-trivial knots by using 3-manifold topology. See [1], [3]. In the previous paper [7], the author determined strong triviality of two-bridge knots, that is, only the trivial knot, the trefoil knot and the figure-eight knot have both two-bridge and strongly n-trivial diagrams for $n \geq 1$.

In the current paper, we continue to study strong triviality for *links*. Especially, we prove that every (non-trivial) two-bridge link can not be strongly n-trivial for $n \geq 1$.

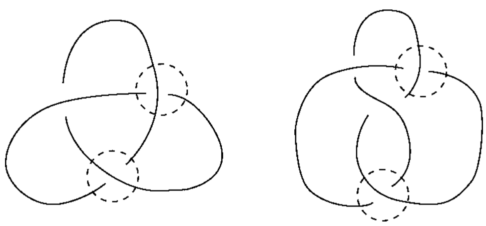

FIGURE 1. Strongly 1-trivial knots.

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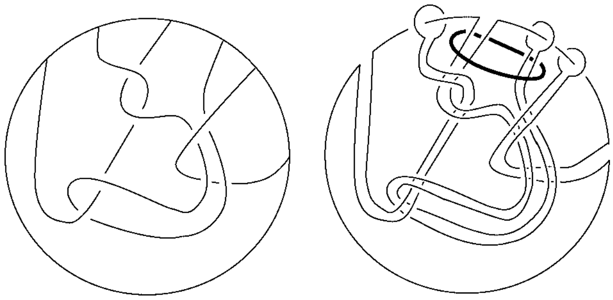

FIGURE 2. A Brunnian Suzuki graph and a strongly 2-trivial link.

2. Main result

DEFINITION 2.1. Let n be a non-negative integer. A k-component link L in S^3 is strongly n-trivial if there exist $n+1$ crossings contained in a diagram of L such that the result of any $0 < m \leq n+1$ crossing changes on these crossings is the k-component trivial link.

We remark that for a strongly n-trivial k-component link L, any crossing change in Definition 2.1 must be done on one component of L and the linking number of any 2-component in L is zero. Moreover Vassiliev invariants for L of orders $\leq n$ coincide with that for the k-component trivial link. As in [1], [3], we can construct strongly n-trivial knots for any n via "Brunnian Suzuki graphs" as follows. A Brunnian Suzuki graph G consists of one circle C and edges E such that if we remove any non-empty subset of E from G, the resulting graph is trivial. By creating hooks along E as indicated in Figure 2, we have a strongly n-trivial knot K, where $n = \sharp|E| - 1$ ($\sharp|\cdot|$ denotes the cardinality). Take a disk D such that (i) ∂D surrounds the ends of one edge in E, (ii) D does not intersect other edges in E, (iii) ∂D is sufficiently close to C, as indicated by a bold circle in Figure 2. Then we have a strongly n-trivial 2-component link $K \cup \partial D$.

Let $S(\alpha, \beta)$ be a two-bridge link whose two-fold branched cover is the lens space $L(\alpha, \beta)$, where α is an even integer and β is an integer coprime to α. Note that $S(\alpha, \beta)$ is a 2-component link and $S(0, 1)$ is the trivial link.

Our theorem is then the following.

THEOREM 2.2. *If a two-bridge link $S(\alpha, \beta)$ is non-trivial, then $S(\alpha, \beta)$ can not be strongly n-trivial for every $n \geq 1$.*

REMARK 2.3. (i) Two-bridge links with unlinking number one was determined by Kohn in [4]. He proved that a two-bridge link $S(\alpha, \beta)$ has unlinking number one if and only if $\alpha = 2r^2$ and $\beta = 2rs \pm 1$ for some coprime integers r and s. Actually, the proof of Theorem 2.2 is simiar to that of Theorem 1 in [4].

(ii) After writing this paper, the author was informed that Tsutsumi have proved that strongly n-trivial links are *boundary links* [8]. So Theorem 2.2 is also obtained from his theorem.

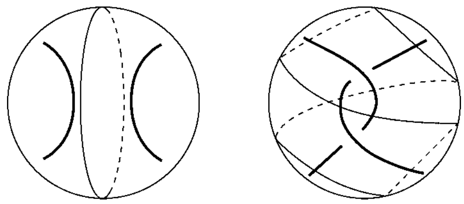

FIGURE 3.

3. Proof of Theorem 2.2

Since strongly n-trivial link is obviously strongly m-trivial for all $m \leq n$, it is sufficient to show that if $S(\alpha, \beta)$ is strongly 1-trivial, then $S(\alpha, \beta)$ is the 2-component trivial link. As in [4], [7], we will prove Theorem 2.2 via Dehn surgery techniques.

Let $N(k)$ be a regular neighbourhood of a knot k in a closed orientable 3-manifold M, with μ a meridian for $N(k)$ and let $E(k) = M - int N(k)$ be the exterior of k in M. Now let $k(\gamma)$ denote the manifold obtained by attaching a solid torus V to $E(k)$ so that a curve of slope γ on $\partial E(k)$ bounds a disk in V, where γ indicates the isotopy class of an essential simple closed curve on $\partial E(k)$. For a knot k in S^3, a slope γ is represented by an element $\frac{m}{n} \in \mathbf{Q} \cup \{\frac{1}{0}\}$, where $\gamma = m\mu_k + n\lambda_k$ in $H_1(\partial E(k), \mathbf{Z})$, (μ_k, λ_k) is a preferred meridian-longitude pair of k and m, n are coprime integers. We shall say that $k(\gamma)$ is the result of γ-Dehn surgery on k. Dehn surgery on a 2-component link $k \cup k'$ is also defined in the same way and denoted by $k \cup k'(\gamma, \gamma')$ for slopes γ and γ' of k and k', respectively. For two slopes γ and δ in $\partial E(k)$, let $\Delta(\gamma, \delta)$ be their minimal geometric intersection number.

The next well-known lemma was first observed by Montesinos in [5].

LEMMA 3.1. *Let K and K' be knots in S^3 and let M_K and $M_{K'}$ be the two-fold branched covering spaces of S^3 along K and K', respectively. Suppose the result of a crossing change on K is K'. Then $M_{K'}$ is obtained by γ-Dehn surgery on some knot k in M_K, with $\Delta(\gamma, \mu) = 2$.*

PROOF. We regard the crossing change as a replacement of sufficiently small 2-string trivial tangles $(B, t) \hookrightarrow (S^3, K)$ and $(B, t') \hookrightarrow (S^3, K')$. Then the two-fold branched covering spaces of the tangles along the 2-strings are solid tori V and V', respectively. Therefore $M_{K'}$ equals the result of some γ-Dehn surgery on a knot in M_K. Let D and D' be essential meridian disks for (B, t) and (B, t'), respectively (see Figure 3). Then in M_K, the preimages of each component of ∂D and $\partial D'$ essentially intersects each other at 2 points on $\partial V = \partial V'$. Hence we have $\Delta(\gamma, \mu) = 2$. □

A *torus knot* in a lens space is a knot isotopic onto a Heegaard torus of the lens space. For the proof of Theorem 2.2, we need the following theorem essentially obtained in [6].

THEOREM 3.2. *Let k_T be a torus knot in $S^2 \times S^1$. Suppose that $k_T(\gamma)$ is homeomorphic to $S^2 \times S^1$ for some slope γ with $\Delta(\gamma, \mu) = 2$. Then k_T is the trivial knot, that is, k_T bounds a disk in $S^2 \times S^1$.*

PROOF. We use the notation in [6, p. 1568] and suppose that k_T is a"(m, n)-torus knot" $C_{m,n}$ in $L(0, 1) = S^2 \times S^1$, where m and n are coprime integers. In this case, remark that $C_{m,n}$ is trivial if and only if $n = 0$. Then by [6, Lemma 7], if $k_T(\gamma)$ is a lens space, then $k_T(\gamma)$ is orientation preserving homeomorphic to $L(-2n^2, 2bn^2 + 2mn \pm 1)$ for some b. (In [6, p. 1568], $r = 0, s = 1, a = -1, d = 2$). Therefore by our assumption, n must be zero. This completes the proof. □

The next easy lemma depends on a specific property of $S^2 \times S^1$.

LEMMA 3.3. *A torus knot in $S^2 \times S^1$ represents the trivial element in the fundamental group of $S^2 \times S^1$, which is isomorphic to* $\mathbf{Z}$, *if and only if it is the trivial knot.*

PROPOSITION 3.4. *If a two-bridge link $S(\alpha, \beta)$ is strongly 1-trivial, then there is a 2-component link $k_1 \cup k_2$ in $S^2 \times S^1$ such that $k_1(\gamma_1)$ and $k_2(\gamma_2)$ are homeomorphic to $S^2 \times S^1$ and $k_1 \cup k_2(\gamma_1, \gamma_2)$ is homeomorphic to $L(\alpha, \beta)$, where each slope γ_i satisfies $\Delta(\gamma_i, \mu) = 2$. Moreover each k_i is the trivial knot in the original $S^2 \times S^1$ and a torus knot in $k_j(\gamma_j) = S^2 \times S^1$ $(i \neq j)$.*

PROOF. By definition, there are two crossings in a diagram of $S(\alpha, \beta)$ such that both each crossing change and simultaneous crossing changes yield the 2-component trivial link. The two-fold cover of the 2-component trivial link in S^3 is $S^2 \times S^1$. Therefore by Lemma 3.1 and its proof, there exists a 2-component link $k_1 \cup k_2$ in $S^2 \times S^1$ such that $k_1(\gamma_1) = k_2(\gamma_2) = S^2 \times S^1$ and $k_1 \cup k_2(\gamma_1, \gamma_2) = L(\alpha, \beta)$, where $\Delta(\gamma_i, \mu) = 2$ $(i = 1, 2)$. Moreover, since $\Delta(\gamma_i, \mu) = 2$, by Culler–Gordon–Luecke–Shalen's cyclic surgery theorem in [2], the exterior spaces $E(k_1)$ and $E(k_2)$ are Seifert fibred manifolds, respectively. But as in [6, Lemma 4] this implies k_1 and k_2 are fibers for some Seifert fibrations of both $S^2 \times S^1$ and $L(\alpha, \beta)$. Moreover, it is well-known that any fiber for a Seifert fibration of a lens space is isotopic to some torus knot (for example, see [6]). By Theorem 3.2, it follows that k_1 and k_2 are trivial in the original $S^2 \times S^1$, respectively. This completes the proof of Proposition 3.4. □

We are ready to prove Theorem 2.2.

PROOF OF THEOREM 2.2. Suppose a two-bridge link $S(\alpha, \beta)$ is strongly 1-trivial. Then there is a 2-component link $k_1 \cup k_2$ in $S^2 \times S^1$ which satisfies the properties in the statement of Proposition 3.4. Since k_2 is trivial, k_1 in $k_2(\gamma_2) = S^2 \times S^1$ may be considered as a full-twisted k_1 along some disk spanned by k_2 in the original $S^2 \times S^1$. Therefore notice that the full-twisted k_1 is also homotopic to zero. By our assumption and Lemma 3.3, we conclude that the full-twisted k_1 is the trivial knot in $S^2 \times S^1$. It is well-known that Dehn surgery on the trivial knot in $S^2 \times S^1$ is either a non-trivial connected sums of two lens spaces or $S^2 \times S^1$

itself. Hence it follows that $k_1 \cup k_2(\gamma_1, \gamma_2) = L(\alpha, \beta)$ is equal to $S^2 \times S^1$. So we have $\alpha = 0$ and $\beta = 1$ and $S(\alpha, \beta) = S(0, 1)$ is the 2-component trivial link in S^3.

This completes the proof of Theorem 2.2. $\square$

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