Non-left-orderability of cyclic branched covers of pretzel knots $P(3, -3, -2k-1)$

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(Communicated by Kenji FUKAYA, M.J.A., Nov. 14, 2022)

Abstract: We prove the non-left-orderability of the fundamental group of the n-th fold cyclic branched cover of the pretzel knot $P(3, -3, -2k-1)$ for all integers k and $n \geq 1$. These 3-manifolds are L-spaces discovered by Issa and Turner.

Key words: Branched cover; non-left-orderable group; Pretzel knot.

1. Introduction. A nontrivial group G is called left-orderable, if and only if it admits a total ordering $\leq$ which is invariant under left multiplication, that is, $g \leq h$ if and only if $fg \leq fh$. An L-space is a rational homology 3-sphere with minimal Heegaard Floer homology, that is, rank $\widehat{HF} = |H_1(Y)|$. The following equivalence relationship was conjectured by Boyer, Gordon and Watson.

Conjecture 1.1 [2]. *An irreducible rational homology* 3*-sphere is an L-space if and only if its fundamental group is not left-orderable.*

Issa and Turner [5] studied a family of pretzel knots named $P(3, -3, -2k-1)$. They defined a family of two-fold quasi-alternating knots $L(k_1, k_2, \ldots, k_n)$, as shown in Fig. 1, and constructed the homeomorphism

$$\Sigma_n(P(3, -3, -2k-1)) \cong \Sigma_2(L(-k, -k, \ldots, -k)).$$

Because two-fold quasi-alternating links are Khovanov homology thin [8], the double branched covers of them are L-spaces [7]. In this way, they proved that all n-th fold cyclic branched covers $\Sigma_n(P(3, -3, -2k-1))$ are L-spaces.

We prove the following result which is consistent with Conjecture 1.1.

Theorem 1.2. *For any integers $k_1, k_2, \ldots, k_n$, the double branched cover of $L(k_1, k_2, \ldots, k_n)$ has a non-left-orderable fundamental group.*

As a corollary, the fundamental group of the n-th fold cyclic branched cover of the pretzel knot $P(3, -3, -2k-1)$ is not left-orderable.

In Section 2, we derive a presentation of the fundamental group of double branched cover of $L(k_1, k_2, \ldots, k_n)$.

In Section 3, we prove our main result.

2. The Brunner's presentation. A tool to compute the fundamental group of the double branched cover of an unsplittable link is the Brunner's presentation [3]. One could also use the Wirtinger's presentation to derive an equivalent form, but the computation would be longer. Historically, Ito [6] used the coarse Brunner's presentation, a generalized version of Brunner's presentation, to prove the non-left-orderability of double branched covers of unsplittable alternating links. Abchir and Sabak [1] proved the non-left-orderability of double branched covers of certain kinds of quasi-alternating links in a similar way.

Consider the checkerboard coloring of the knot diagram D in Fig. 1. Let G and $\tilde{G}$ be the decomposition graph as shown in Fig. 2 and the connectivity graph as shown in Fig. 3.

In the Brunner's presentation of the fundamental group $\pi_1(\Sigma_2(L(k_1, k_2, \ldots, k_n)))$. Let e_i, f_i, g_i, b_i $(1 \leq i \leq n)$ be the edge generators, and a_i, c_i $(1 \leq i \leq n)$ be the region generators. Then the local edge relations are $e_i = a_i^{k_i}$, $b_i = c_i^{-1}$, $f_i = (a_i^{-1}c_i)^{-2}$ and $g_i = c_{i-1}^{-1}a_i$ $(1 \leq i \leq n)$. And the global cycle relations are $f_i^{-1}e_ig_i = 1$ and $f_i^{-1}b_ig_{i+1} = 1$ $(1 \leq i \leq n)$. Here the subscripts are considered modulo n.

By simplification, we get the following presentation of $\pi_1(\Sigma_2(L(k_1, k_2, \ldots, k_n)))$:

$$\langle a_i, b_i | a_{i+1} = b_i^{-1}a_ib_ia_i, a_i^{k_i} = b_ia_ib_ib_{i-1}^{-1}\rangle,$$

where $i = 1, \ldots, n$ and we view subscripts modulo n.

2020 Mathematics Subject Classification. Primary 57M05, 57M12.

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doi: 10.3792/pjaa.98.017

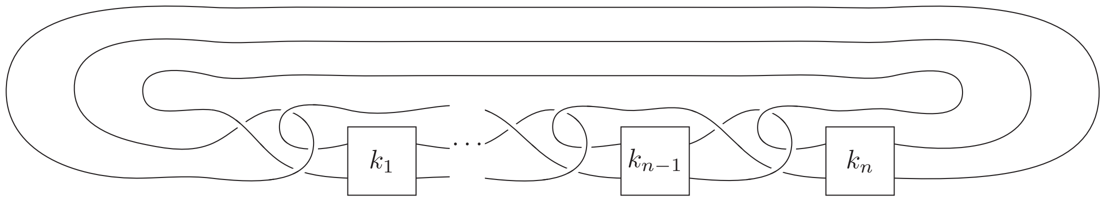

Fig. 1. The link $L(k_1, k_2, \dots, k_n)$ in [5].

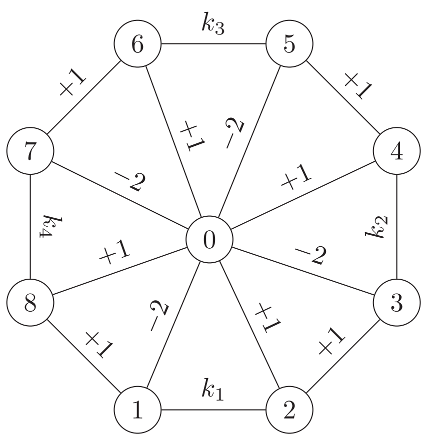

Fig. 2. Decomposition graph G for $n = 4$.

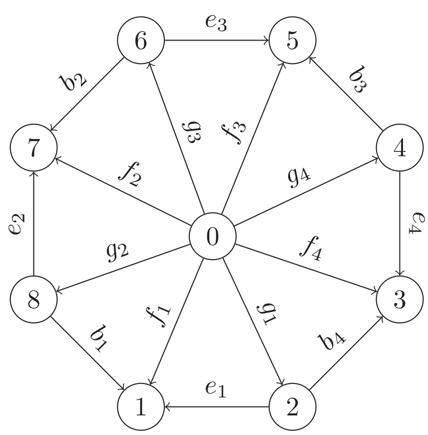

Fig. 3. Connectivity graph $\tilde{G}$ for $n = 4$.

3. The non-left-orderability. We are going to prove our non-left-orderability result. Let $\leq$ be any total preorder on the fundamental group $\pi_1(\Sigma_2(L(k_1, k_2, \dots, k_n)))$ which is invariant under left multiplication. Then we have the following lemmas.

Lemma 3.1. *If*

$$\forall m.(b_i^{-1} a_{i+1}^{m} \geq 1),$$

then

$$\forall m_2 \exists m_1.(a_{i+1}^{-m_1} b_i^{-1} a_i^{m_2} \geq 1).$$

Proof. By taking $m = 0$ in the assumption, we have $b_i^{-1} \geq 1$.

For the sake of contradiction, suppose that

$$\neg \forall m_2 \exists m_1.(a_{i+1}^{-m_1} b_i^{-1} a_i^{m_2} \geq 1)$$

which is equivalent to

$$\exists m_2 \forall m_1.\neg(a_{i+1}^{-m_1} b_i^{-1} a_i^{m_2} \geq 1).$$

By the strong connectedness of $\leq$, for any integers m_1, m_2, we have

$$\neg(a_{i+1}^{-m_1} b_i^{-1} a_i^{m_2} \geq 1) \Longrightarrow (a_{i+1}^{-m_1} b_i^{-1} a_i^{m_2} \leq 1).$$

Hence

$$\forall m_1.(a_{i+1}^{-m_1} b_i^{-1} a_i^{m_2} \leq 1)$$

for some integer m_2.

By taking $m_1 = 0$, we have

$$b_i^{-1} a_i^{m_2} \leq 1.$$

Since

$$a_{i+1} = b_i^{-1} a_i b_i a_i,$$

we have

$$(a_{i+1} a_i^{-1})^{m_2} b_i^{-1} = b_i^{-1} a_i^{m_2}.$$

Hence we have $a_i^{m_2} \leq 1$ and $(a_{i+1} a_i^{-1})^{m_2} \leq 1$, which implies $a_{i+1}^{m_2} \leq 1$. By assumption, we have

$$\forall m.(a_{i+1}^{-m} a_i^{m_2} \leq 1).$$

So we have

$$\forall (m \text{ has same sign as } m_2).(a_{i+1}^{-m} a_i^{m_2} a_{i+1}^{m} \leq 1).$$

If $m_2 \geq 0$, then

$$\begin{aligned}(a_{i+1} a_i^{-1})^{m_2} \\ = a_{i+1}^{m_2} (a_{i+1}^{-m_2+1} a_i^{-1} a_{i+1}^{m_2-1}) \cdots (a_{i+1}^{-1} a_i^{-1} a_{i+1}) a_i^{-1}.\end{aligned}$$

If $m_2 < 0$, then

$$
\begin{aligned}
&(a_{i+1}a_i^{-1})^{m_2}\\
&\quad = a_{i+1}^{m_2}(a_{i+1}^{-m_2}a_ia_{i+1}^{m_2})\cdots(a_{i+1}^2a_ia_{i+1}^{-2})(a_{i+1}a_ia_{i+1}^{-1}).
\end{aligned}
$$

Therefore, we have

$$a_{i+1}^{-m_2}(a_{i+1}a_i^{-1})^{m_2} \geq 1.$$

So we have

$$
\begin{aligned}
&a_{i+1}^{-m_2}b_i^{-1}a_i^{m_2}\\
&\quad = a_{i+1}^{-m_2}(a_{i+1}a_i^{-1})^{m_2}b_i^{-1}\\
&\quad \geq 1,
\end{aligned}
$$

which implies the conclusion. □

Lemma 3.2. *If*

$$\forall m.(b_i^{-1}a_{i+1}^m \geq 1),$$

then

$$\forall m.(b_{i-1}^{-1}a_i^m \geq 1).$$

Proof. Because

$$a_{i+1} = b_i^{-1}a_ib_ia_i$$

and

$$a_i^{k_i} = b_ia_ib_ib_{i-1}^{-1},$$

we have

$$
\begin{aligned}
&b_{i-1}^{-1}a_i^m\\
&\quad = b_i^{-1}a_i^{-1}b_i^{-1}a_i^{m+k_i}\\
&\quad = b_i^{-1}a_{i+1}^{-1}b_i^{-1}a_i^{m+k_i+1}.
\end{aligned}
$$

By Lemma 3.1, we have

$$\exists m_1.(a_{i+1}^{-m_1}b_i^{-1}a_i^{m+k_i+1} \geq 1).$$

By assumption, we have $b_i^{-1}a_{i+1}^{m_1-1} \geq 1$. Therefore, we conclude that

$$
\begin{aligned}
&b_{i-1}^{-1}a_i^m\\
&\quad = (b_i^{-1}a_{i+1}^{m_1-1})(a_{i+1}^{-m_1}b_i^{-1}a_i^{m+k_i+1})\\
&\quad \geq 1.
\end{aligned}
$$

□

Let us apply the fixed point method on a_1. This is a common technique in the proofs of many non-left-orderability results. If the fundamental group $\pi_1(\Sigma_2(L(k_1,k_2,\ldots,k_n)))$ has a left order $\leq$, then [4] there exists a homomorphism ρ from this group to $\mathrm{Homeo}_+(\mathbf{R})$ with no global fixed points, and $g \leq h$ if and only if $\rho(g)(0) \leq \rho(h)(0)$. Then for the element a_1, there are two situations:

(1) If $\rho(a_1)$ has a fixed point s, then there is a left total preorder $\leq_{a_1}$, defined as $g \leq h$ if and only if $\rho(g)(s) \leq \rho(h)(s)$, with $a_1 \geq_{a_1} 1$ and $a_1^{-1} \geq_{a_1} 1$.

(2) Otherwise, any conjugate of a_1 has the same sign as a_1.

We assume the first situation, then $\leq_{a_1}$ is a total preorder which is invariant under left multiplication. Without loss of generality, we assume that $b_n \leq_{a_1} 1$, then

$$\forall m.(b_n^{-1}a_1^m \geq_{a_1} 1).$$

By inductive applications of Lemma 3.2, we have

$$\forall m.(b_i^{-1}a_{i+1}^m \geq_{a_1} 1)$$

for any $i = 1,\ldots,n$. By Lemma 3.1, we have the relation

$$\forall m_2\exists m_1.(a_{i+1}^{-m_1}b_i^{-1}a_i^{m_2} \geq_{a_1} 1)$$

for any $i = 1,\ldots,n$.

Because

$$a_{i+1} = b_i^{-1}a_ib_ia_i$$

and

$$a_i^{k_i} = b_ia_ib_ib_{i-1}^{-1},$$

we have

$$
\begin{aligned}
&b_i^{-1}a_i^{k_i}\\
&\quad = a_ib_ib_{i-1}^{-1}\\
&\quad = b_ia_{i+1}a_i^{-1}b_{i-1}^{-1}\\
&\quad = (b_ia_{i+1})(b_{i-1}a_i)^{-1}.
\end{aligned}
$$

Hence we have

$$\forall m_2\exists m_1.((b_{i-1}a_i)^{-1}a_i^{m_2} \geq_{a_1} (b_ia_{i+1})^{-1}a_{i+1}^{m_1}).$$

By induction, we have

$$\forall m_2\exists m_1.((b_1a_2)^{-1}a_2^{m_2} \geq_{a_1} (b_na_1)^{-1}a_1^{m_1}),$$

and especially we get

$$\exists m_1.((b_1a_2)^{-1} \geq_{a_1} (b_na_1)^{-1}a_1^{m_1}).$$

Then we have

$$\exists m_1.(1 \geq_{a_1} b_1^{-1}a_1^{m_1+k_1}),$$

which implies $b_1 \geq_{a_1} 1$. Because

$$\forall m.(b_i^{-1}a_{i+1}^m \geq_{a_1} 1)$$

implies $b_1^{-1} \geq_{a_1} 1$, every fixed point of a_1 is also a fixed point of b_1.

By

$$a_{i+1} = b_i^{-1}a_ib_ia_i,$$

every fixed point of a_1 is also a fixed point of a_2. By symmetry and induction, any fixed point of a_1 is a global fixed point.

Now we assume any conjugate of a_1 is positive. By

$$a_{i+1} = b_i^{-1} a_i b_i a_i,$$

if any conjugate of a_i is positive, then any conjugate of a_{i+1} is positive. By induction, any conjugate of a_i is positive. Since

$$a_{i+1} a_i^{-1} = b_i^{-1} a_i b_i \geq 1,$$

and the fact that the product of all $a_{i+1}a_i^{-1}$ is identity, we have $a_i = a_{i+1}$ for all $i = 1, \ldots, n$. Furthermore, we have $a_i = 1$. Then the fundamental group is finite, so it is not left-orderable.

Therefore, we proved the non-left-orderability.

Acknowledgements. The authors thank Wei Chen, Nathan Dunfield, Heng Liao, Luigi Lunardon and Hannah Turner for helpful discussions.

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