Semicomplete vector fields on non-Kähler surfaces

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Abstract: We investigate semicomplete meromorphic vector fields on complex surfaces, those where the solutions of the associated ordinary differential equations have no multivaluedness. We prove that if a non-Kähler compact complex surface has such a vector field, then, up to a bimeromorphic transformation, either the vector field is holomorphic, has a first integral or preserves a fibration. This extends previous results of Rebelo and the author to the non-Kähler setting.

Key words: Semicompleteness; Enoki surface; foliation.

1. Introduction. In the beginning of the 20th century, Painlevé drew attention to the general problem of determining the algebraic ordinary differential equations whose general solution is uniform, starting from those of smaller orders [14]. Differential equations given by rational vector fields on algebraic manifolds of dimension n are natural examples of such equations (they give autonomous algebraic differential equations of order n). Already for these, and already in small dimensions, Painlevé's program is far from being achieved.

More generally, one may consider this problem for meromorphic vector fields on general compact complex manifolds, and not only algebraic ones. For meromorphic vector fields on compact complex Kähler surfaces, the situation is well understood. These are the subject of Rebelo and the author's Theorem B in [6]:

Theorem 1.1. *Let X be a semicomplete meromorphic vector field on the compact complex Kähler surface S. Then, up to a bimeromorphic transformation, X is holomorphic, X has a first integral or S has a rational or elliptic fibration preserved by X (with each component of the locus of poles of X contained in a fiber).*

(In it, Rebelo's notion of *semicompleteness* [15] is used to formalize the notion of "vector field whose general solution is uniform"). There remained the problem of understanding those semicomplete meromorphic vector fields on complex compact surfaces which are not Kähler. The aim of this article is to extend Theorem 1.1 to the non-Kähler case. Our result is the following one:

Theorem 1.2. *Let S be a compact complex non-Kähler surface, X a semicomplete meromorphic vector field on S. Up to a bimeromorphic transformation, either X is holomorphic, X has a first integral or there is an elliptic fibration preserved by X (with each component of the locus of poles of X contained in a fiber).*

We will show that, under the hypothesis of the theorem, if the algebraic dimension of S is one, its algebraic reduction gives either a first integral of X or a fibration preserved by X; if it is zero, we will prove that, up to a bimeromorphic transformation, X is holomorphic (and is thus to be found within the classification of holomorphic vector fields on compact complex surfaces [4, Thm. 0.3]).

Together, Theorems 1.1 and 1.2 describe semicomplete meromorphic vector fields on all compact complex surfaces.

2. Proof of the theorem. We begin the proof of Theorem 1.2. Let S be a non-Kähler compact complex surface and X a semicomplete meromorphic vector field on S. Since semicompleteness is a birational invariant [6, Cor. 12], we may suppose that S is minimal and that X is strictly meromorphic (for otherwise it would automatically satisfy Theorem 1.2). Since S is not Kähler, its algebraic dimension, that we will denote by $a(S)$, is strictly smaller than two [1, Ch. IV, Section 5].

If $a(S) = 1$, there is an *algebraic reduction* of S, a fibration $\Pi : S \to B$ such that B is an algebraic

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curve and such that any meromorphic function on S is the pullback by Π of an algebraic function on B. Its generic fiber is an elliptic curve and every curve in S is in a fiber (Theorems 4.2 and 4.3 in [9, Section 4]). The vector field X naturally induces a meromorphic vector field Y on B such that $\Pi_* X = Y$ (Π is a first integral if $Y \equiv 0$). The curve of poles of X is contained in the fibers of Π and thus, since X is holomorphic in a neighborhood of a generic fiber, the fibration is naturally preserved by X. Further, since X is semicomplete, so is Y and is thus holomorphic [6, Lemma 2]. This proves Theorem 1.2 in this case. Example 3.1 will illustrate this situation.

We will henceforth suppose that $a(S) = 0$. Our aim is to prove that the vector field is, up to a bimeromorphic transformation, holomorphic. Let $b_i(S)$ denote the i-th Betti number of S. A theorem of Kodaira [10, Thm. 11] states that if S is a non-Kähler compact complex surface of vanishing algebraic dimension then $b_1(S) = 1$ (S belongs to the class VII_0). Two cases appear:

First case, $b_2(S) = 0$. Since X is strictly meromorphic, the curve of poles of X is a nonempty curve on S. A theorem of Kodaira affirms that if S is a minimal surface with no nonconstant meromorphic functions with $b_1(S) = 1$, $b_2(S) = 0$, and containing at least one curve, S is a Hopf surface [11, Thm. 34]. We will prove that *a meromorphic vector field on a Hopf surface of vanishing algebraic dimension is holomorphic.* The Hopf surface has a finite nonramified cover $\widehat{S}$ that is the quotient of $\mathbf{C}^2 \setminus \{0\}$ under the action of $(z, w) \mapsto (\alpha z + \lambda w^n, \beta w)$, with either $\lambda = 0$ or $\alpha = \beta^n$ (further, since we are supposing that the algebraic dimension is zero, $\alpha^j \neq \beta^k$ if $\lambda = 0$). Let $\widehat{X}$ be the lift of X to $\widehat{S}$. The surface $\widehat{S}$ admits two holomorphic vector fields Y_1 and Y_2, linearly independent almost everywhere: if $\lambda = 0$, they are given by $z\partial/\partial z$ and $w\partial/\partial w$; if $\lambda \neq 0$, by $nz\partial/\partial z + w\partial/\partial w$ and $w^n\partial/\partial z$. There exist meromorphic functions f_1 and f_2 on $\widehat{S}$ such that $\widehat{X} = f_1 Y_1 + f_2 Y_2$. Since f_1 and f_2 are necessarily constants, $\widehat{X}$ is holomorphic and thus X is holomorphic as well.

Second case, $b_2(S) > 0$. The semicompleteness hypothesis on X will allow us to give a more precise description of its curve of poles. The following is a key element in the proof of Theorem 1.1 and involves no global hypothesis on S:

Theorem 2.1 ([6], Thm. A)**.** *Let X be a semicomplete meromorphic vector field on the surface S, Z a compact connected component of the locus of poles of X. Up to a bimeromorphic transformation, Z is either empty, a rational curve of vanishing self-intersection, or supports a divisor D of elliptic fiber type, a divisor such that $D \cdot D = 0$ and $\chi(D) = 0$.*

Enoki constructed some minimal compact complex surfaces $S_{n,\alpha,t}$ ($n > 0$, $0 < |\alpha| < 1$, $t \in \mathbf{C}^n$), generally called *Enoki surfaces.* They are non-Kähler surfaces with $b_1(S_{n,\alpha,t}) = 1$ and $b_2(S_{n,\alpha,t}) = n$. Each one of them has a cycle formed by n rational curves C_i, $i \in \mathbf{Z}/n\mathbf{Z}$, with $C_i \cdot C_{i+1} = 1$, $C_i \cdot C_i = -2$ and $C_i \cdot C_j = 0$ if $|i - j| > 1$. The surface $S_{n,\alpha,t}$ has the divisor $D_{n,\alpha,t} = \sum_i C_i$. It is of vanishing self-intersection and has the combinatorics of Kodaira's elliptic fiber I_n.

The particular cases $S_{n,\alpha,0}$ are unramified covers of some surfaces previously constructed by Inoue [7], and are called *parabolic Inoue surfaces.* All of these parabolic Inoue surfaces have holomorphic vector fields and one elliptic curve [2, Thm. 1.31]. Reciprocally, if an Enoki surface has a holomorphic vector field or a curve other than those in the support of $D_{n,\alpha,t}$, it is a parabolic Inoue surface [13, Thm. 7.1].

Enoki provided the following characterization of these surfaces [5]:

Theorem 2.2 (Enoki)**.** *Let S be a minimal compact complex surface with $b_1(S) = 1$ and $b_2(S) = n$. If S has a divisor $D \neq 0$ with $D \cdot D = 0$, then S is biholomorphic to $S_{n,\alpha,t}$ for some n, α, t and $D = mD_{n,\alpha,t}$ for some $m \neq 0$.*

Theorems 2.1 and 2.2 imply that *if S is a compact complex surface with $b_1(S) = 1$ and $b_2(S) \neq 0$ endowed with a semicomplete meromorphic vector field X, its minimal model is an Enoki surface, and the curve of poles of X is the support of the divisor $D_{n,\alpha,t}$.* Theorem 1.2 is a consequence of the following

Proposition 2.1. *Let S be an Enoki surface, X a meromorphic vector field on S. Then X is holomorphic* (*in particular, S is a parabolic Inoue surface*)*.*

Before proceeding to the proof of this proposition, let us recall some facts about Enoki surfaces and their foliations. Enoki surfaces belong to the class VII_0; they contain *global spherical shells* and may thus be obtained by Kato's construction [8]. Let us go through this construction following [2].

Let $B_0 = \{v \in \mathbf{C}^2; |v| \le \epsilon\}$ be the closed ball and consider the sequence of blowups

$$B_n \xrightarrow{\Pi_n} \cdots \xrightarrow{\Pi_1} B_0,$$

where Π_i is the blowup of p_{i-1}, $p_0 = 0$, $C_i = \Pi_i^{-1}(p_{i-1})$, $p_i \in C_i$. Let $\Pi : B_n \to B_0$, $\Pi = \Pi_n \circ \cdots \circ \Pi_1$. Let $\sigma : B_0 \to B_n$ such that, $\sigma(B_0)$ is in the interior of B_n. Let $p_n = \sigma(0)$, and suppose that $p_n \in C_n$. The *Kato surface* S is the compact complex surface resulting from the identification of the two boundary components of $B_n \setminus \sigma(\mathrm{int}(B_0))$ —considered as a real four-dimensional manifold-with-boundary— by the map $\sigma \circ \Pi$.

The above data can be recovered from the germ of $F = \Pi \circ \sigma$ at its fixed point 0. There is a natural correspondence between the objects in $(\mathbf{C}^2, 0)$ that are invariant by F and the objects in S.

The Kato surface of the above construction is an Enoki surface if, furthermore [2, Thm. 3.33],

(a) $p_i \notin \cup_{j<i} C_j$ and
(b) p_1 is not in the strict transform of $\sigma^{-1}(C_n)$ under Π_1,

or, equivalently, if the trace of $DF|_0$ is nonzero [2, Thm. 3.30].

We will henceforth assume that S is an Enoki surface. In suitable coordinates (x, y), the germ of F may be written as

$$F(x, y) = (xy^n + P(y), ty), \tag{1}$$

for some $t \in \mathbf{C}$, $0 < |t| < 1$, and some polynomial P of degree $n - 1$ [3, Thm. 1.19]. Here, $y = 0$ is the curve that maps via σ into C_n. The parabolic Inoue case corresponds to $P \equiv 0$ in (1); in this case, both the holomorphic vector field $x\partial/\partial x$ and the germ of curve $x = 0$ are preserved by F and induce, respectively, a holomorphic vector field Y and an elliptic curve in S. In all cases, since the meromorphic one-form dy/y on $(\mathbf{C}^2, 0)$ is invariant by F, it induces a global meromorphic one-form on S. On its turn, the kernel of this form induces a holomorphic foliation with singularities $\mathcal{F}$ on S (compare with [13, Section 3]).

According to [12, Section 2.3], an Enoki surface has no foliation other than this foliation $\mathcal{F}$. Let us give a short proof of this fact. Let $\mathcal{G}$ be a foliation on S. By Levi's extension principle, there exists a meromorphic one-form $\omega = \alpha dx + \beta dy$ on $(\mathbf{C}^2, 0)$ generating it. In order for $\mathcal{G}$ to be preserved by F, $F^*\omega = h\omega$ for some meromorphic function h. If $\beta \equiv 0$, $\omega = \alpha dx$ and $F^*(\alpha dx) = (\alpha \circ F) dF_1 = (\alpha \circ F)[(\partial F_1/\partial x)dx + (\partial F_1/\partial y)dy]$. From formula (1), $\partial F_1/\partial y$ does not vanish identically, and $\mathcal{G}$ cannot be preserved. If β is not identically zero, up to multiplying by a meromorphic function, we may suppose that $\omega = \alpha dx + dy/y$. Since dy/y is invariant by F, we must have that $h \equiv 1$ and that the one-form αdx is preserved. Repeating the previous arguments shows that this is impossible unless $\alpha \equiv 0$. This proves the uniqueness of the foliation.

The construction of the Enoki surface and of its foliation can be done simultaneously. Consider a nonsingular foliation $\mathcal{F}_0$ on $(\mathbf{C}^2, 0)$. Under the blowup of 0 by Π_1, this regular point of the foliation produces an exceptional divisor C_1 that is invariant by the induced foliation $\mathcal{F}_1$. There is only one singular point of the transformed foliation $\mathcal{F}_1$ along C_1. Let p_2 be a regular point of $\mathcal{F}_1$ in C_1 and continue the construction of the Enoki surface until we have a foliation $\mathcal{F}_n$ on B_n. Let σ map $\mathcal{F}_0$ to $\mathcal{F}_n$ (mapping $\{y = 0\}$ to C_n) and suppose that F is contracting. Through Kato's construction, this produces a general Enoki surface with a foliation (notice that condition (b) in the construction of the Enoki surface is automatically satisfied). The only singularities of the foliation are at the intersection of the divisors, where the foliation has, locally, a first integral (the holonomy is trivial).

Let us now come to the proof of Proposition 2.1. Let S be an Enoki surface, D its divisor and let X be a meromorphic vector field on S. If S is a parabolic Inoue surface, it has a holomorphic and nonzero vector field Y. Since X and Y are collinear and since S has no meromorphic functions, X and Y differ by a multiplicative constant, and X is holomorphic. Let us thus assume that S is not a parabolic Inoue surface and, in particular, that it has no curves other than those of the cycle in the support of D. Let X be a meromorphic vector field on S. By Levi's extension principle, it is associated to a meromorphic vector field in B_0. Since the foliation it induces is the unique foliation on the Enoki surface, the vector field is of the form $X_0 = g(x, y)\partial/\partial x$ for some meromorphic function g. In order for this vector field to induce a global one, we should have

$$\Pi_*^{-1} X_0 = \sigma_* X_0. \tag{2}$$

In particular, the curves of zeros and poles of g other than $y = 0$ are preserved by F and induce curves in S different from $C_1, \ldots, C_n$. Since we supposed that there are no further curves in S, X_0 must actually be of the form $h(x,y)y^q\partial/\partial x$, with h holomorphic and nonzero, $q \in \mathbf{Z}$. In the chart of the blowup $(x,y) = (sy, y)$, the vector field reads $h(sy,y)y^{q-1}\partial/\partial s$. (As a function of s and y, $h(sy,y)$ is holomorphic and nonzero at the origin.) After n blowups, the order of the transformed vector field along C_n is $q - n$. However, in order to satisfy (2), this number must equal q. This contradiction proves Proposition 2.1 and finishes the proof of Theorem 1.2. □

3. An example and a remark.

Example 3.1. Consider the meromorphic vector field $\widehat{X} = yx^{-1}(y\partial/\partial y - x\partial/\partial x)$ on $\mathbf{C}^2$. Outside $\{xy = 0\}$, it has the solutions $t \mapsto (-2ct^{1/2}, ct^{-1/2})$. Let $\widehat{\Pi}\colon \mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1$ be given by $(x,y) \mapsto x/y$. The image of $\widehat{X}$ under Π is the vector field $Y = 2\partial/\partial\xi$. Let S be the secondary Hopf surface obtained as the quotient of $\mathbf{C}^2 \setminus \{0\}$ under the action of the group generated by the maps $(x,y) \mapsto (2x, 2y)$ and $(x,y) \xrightarrow{\sigma} (-x,-y)$. These maps preserve $\widehat{X}$ and $\widehat{\Pi}$ and there is thus a well-defined vector field X on S and a map (elliptic fibration) $\Pi : S \to \mathbf{P}^1$, $\Pi_*(X) = Y$. The elliptic curves in S coming from $\{x = 0\}$ and $\{y = 0\}$ are, respectively, the curves of zeros and poles of X. Outside these curves, where X is holomorphic, the solutions are single-valued (σ cancels the multivaluedness of the solutions of $\widehat{X}$).

Remark 3.1. Enoki surfaces do not have strictly meromorphic vector fields, although they are not far from doing so. Consider the vector field $xy^q\partial/\partial x$, $q \in \mathbf{Z}$. In the chart $(x,y) = (sy, y)$ of the blowup of 0, the vector field reads again $sy^q\partial/\partial s$, so that, after n blowups, the vector field is equivalent to the original one. Equivalently, the vector field is preserved under the local ramified maps $(x,y) \mapsto (xy^n, y)$. However, the derivative at the origin of these maps has trace equal to 1, and they are not contracting.

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