A simple proof of convolution identities of Bernoulli numbers

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Abstract: T. Agoh and K. Dilcher proved convolution identities of Bernoulli numbers in 2007. Their proof was complicated calculations in more than 10 pages, which were based on the relation between the Stirling numbers of second kind and the Bernoulli numbers. In this short paper, we give a simple proof of it. Essentially, the proof is based on just one formula on a new kind of generating function.

Key words: Bernoulli numbers; convolution identity; generating function.

Set

$$f(t) := \frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$$

(*i.e.*, we use the convention $B_1 = -\frac{1}{2}$). In this paper (which is based on a letter to Noriyuki Otsubo in March/2014), we give a simple proof of the following identity due to Agoh and Dilcher:

Theorem 0.1. *Let l, m, n be non-negative integers. Put*

$$\delta_{n,m>0} := \begin{cases} 1 & \text{if } n, m > 0, \\ 0 & \text{if } (n = 0, m \neq 0) \text{ or } (n \neq 0, m = 0), \\ -1 & \text{if } n = m = 0. \end{cases}$$

Then, we have

$$\begin{aligned}(1)\quad \sum_{0\leq j\leq l} \binom{l}{j} B_{n+j} B_{m+l-j} &= l \sum_{0\leq k\leq n} (-1)^{n-k} \binom{n}{k} \frac{B_{n+m+1-k} B_{l+k-1}}{n+m+1-k} \\ &+ l \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \frac{B_{n+m+1-k} B_{l+k-1}}{n+m+1-k} \\ &+ m \sum_{0\leq k\leq n} (-1)^{n-k} \binom{n}{k} \frac{B_{n+m-k} B_{l+k}}{n+m-k} \\ &+ n \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \frac{B_{n+m-k} B_{l+k}}{n+m-k} \\ &- l \binom{n+m}{n}^{-1} \frac{B_{l+n+m}}{n+m+1} \\ &- \delta_{n,m>0} \binom{n+m}{n}^{-1} B_{l+n+m}.\end{aligned}$$

Let X denote a new indeterminate, and put $D := \frac{d}{dt}$. We consider a new kind of generating function, *i.e.*, a generating function (with respect to the derivatives) of the generating function of Bernoulli numbers:

$$F(X) = F[f(t)](X) := \sum_{n\geq 0} D^n(f(t)) \frac{X^n}{n!} = f(t+X),$$

where the last equality comes from Taylor's expansion.

This new kind of generating function seems useful. For example, if we substitute $t + X$ for t in classical Euler's quadratic relation $f(t)^2 = -tDf(t) - (t-1)f(t)$, then we have $f(t+X)^2 = -(t+X)Df(t+X) - (t+X-1)f(t+X)$, *i.e.*, $F(X)^2 = -(t+X)DF(X) - (t+X-1)F(X)$. When we compare the coefficients of X^1 (resp. X^2 etc.), we obtain $2fD(f) = -tD^2(f) - D(f) - (t-1)D(f) - f$ (resp. $(D(f))^2 + fD^2(f) = -\frac{t}{2} D^3(f) - D^2(f) - \frac{t-1}{2} D^2(f) - D(f)$ etc.), which gives us quadratic relations among Bernoulli numbers other than Euler's one, etc.

In this short paper, we give a simple proof of the convolution identity of Agoh-Dilcher [AD] in 2007 by using this generating function (their original proof was complicated calculations in more than 10 pages, which were based on the relation between the Stirling numbers of second kind and the Bernoulli numbers).

Proof. The formula (1) is derived just from the following single key identity:

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$$\frac{1}{e^{t+X}-1}\cdot\frac{1}{e^{t+Y}-1} = \frac{1}{e^{t+X}-1}\frac{1}{e^{-X+Y}-1} + \frac{1}{e^{t+Y}-1}\frac{1}{e^{-Y+X}-1}.$$

By this identity, we have

$$\begin{aligned}\frac{t+X}{e^{t+X}-1}\frac{t+Y}{e^{t+Y}-1} &= \frac{t+X}{e^{t+X}-1}\frac{t}{e^{-X+Y}-1} + \frac{t+Y}{e^{t+Y}-1}\frac{t}{e^{-Y+X}-1}\\ &\quad+ \frac{t+X}{e^{t+X}-1}\frac{Y}{e^{-X+Y}-1} + \frac{t+Y}{e^{t+Y}-1}\frac{X}{e^{-Y+X}-1},\end{aligned}$$

i.e.,

$$\begin{aligned}(2)\quad & F(X).F(Y)\\ &= F(X).\frac{tf(-X+Y)}{-X+Y} + F(Y).\frac{tf(-Y+X)}{-Y+X}\\ &\quad + F(X).\frac{Yf(-X+Y)}{-X+Y} + F(Y).\frac{Xf(-Y+X)}{-Y+X}\\ &= F(X).\frac{t(f(-X+Y)-1)}{-X+Y}\\ &\quad + F(Y).\frac{t(f(-Y+X)-1)}{-Y+X}\\ &\quad + F(X).\frac{Y(f(-X+Y)-1)}{-X+Y}\\ &\quad + F(Y).\frac{X(f(-Y+X)-1)}{-Y+X}\\ &\quad - \frac{t(F(X)-F(Y))}{X-Y} - \frac{YF(X)-XF(Y)}{X-Y}.\end{aligned}$$

We compare the coefficients of $\frac{X^n}{n!}\frac{Y^m}{m!}$ in the identity (2). By noting

$$\begin{aligned}\frac{f(-X+Y)-1}{-X+Y} &= \sum_{i\geq 0}\frac{B_{i+1}}{i+1}\frac{(-X+Y)^i}{i!}\\ &= \sum_{i\geq 0}\frac{B_{i+1}}{i+1}\frac{1}{i!}\sum_{0\leq j\leq i}(-1)^j\binom{i}{j}X^jY^{i-j}\\ &= \sum_{n\geq 0}\sum_{m\geq 0}(-1)^n\frac{B_{n+m+1}}{n+m+1}\frac{X^n}{n!}\frac{Y^m}{m!},\end{aligned}$$

$$\begin{aligned}&\frac{F(X)-F(Y)}{X-Y}\\ &= \sum_{k\geq 1}\frac{D^k(f)}{k!}(X^{k-1}+X^{k-2}Y+\cdots+Y^{k-1})\\ &= \sum_{n\geq 0}\sum_{m\geq 0}\binom{n+m}{n}^{-1}\frac{D^{n+m+1}(f)}{n+m+1}\frac{X^n}{n!}\frac{Y^m}{m!}, \text{ and}\end{aligned}$$

$$\begin{aligned}&\frac{YF(X)-XF(Y)}{X-Y}\\ &= -f + \sum_{k\geq 2}\frac{D^k(f)}{k!}(X^{k-1}Y+X^{k-2}Y^2+\cdots+XY^{k-1})\\ &= -f + \sum_{n\geq 1}\sum_{m\geq 1}\binom{n+m}{n}^{-1}D^{n+m}(f)\frac{X^n}{n!}\frac{Y^m}{m!},\end{aligned}$$

we have

$$\begin{aligned}&D^n(f)D^m(f)\\ &= \sum_{0\leq k\leq n}\binom{n}{k}(-1)^k\frac{B_{k+m+1}}{k+m+1}tD^{n-k}(f)\\ &\quad + \sum_{0\leq k\leq m}\binom{m}{k}(-1)^k\frac{B_{n+k+1}}{n+k+1}tD^{m-k}(f)\\ &\quad + \sum_{0\leq k\leq n}\binom{n}{k}(-1)^k\frac{mB_{k+m}}{k+m}D^{n-k}(f)\\ &\quad + \sum_{0\leq k\leq m}\binom{m}{k}(-1)^k\frac{nB_{n+k}}{n+k}D^{m-k}(f)\\ &\quad - \binom{n+m}{n}^{-1}\frac{tD^{n+m+1}(f)}{n+m+1}\\ &\quad - \delta_{n,m>0}\binom{n+m}{n}^{-1}D^{n+m}(f).\end{aligned}$$

Again, we compare the coefficients of $\frac{t^l}{l!}$ in this identity. Then, we obtain the formula (1). □

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