On Diophantine quintuple conjecture

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(Communicated by Shigefumi MORI, M.J.A., May 12, 2014)

Abstract: In this note, we prove that if $\{a,b,c,d,e\}$ with $a<b<c<d<e$ is a Diophantine quintuple, then $d<10^{74}$.

Key words: Diophantine m-tuples; Pell equations; upper bound.

A set of m distinct positive integers $\{a_1,\ldots,a_m\}$ is called a Diophantine m-tuple if a_ia_j+1 is a perfect square. Diophantus studied sets of positive rational numbers with the same property, particularly he found the set of four positive rational numbers $\{\frac{1}{16},\frac{33}{16},\frac{17}{4},\frac{105}{16}\}$. But the first Diophantine quadruple was found by Fermat. In fact, Fermat proved that the set $\{1,3,8,120\}$ is a Diophantine quadruple, called *Fermat's set*. Moreover, Baker and Davenport [1] proved that the set $\{1,3,8,120\}$ cannot be extended to a Diophantine quintuple.

Several results of the generalization of the result of Baker and Davenport are obtained. In 1997, Dujella [2] proved that the Diophantine triples of the form $\{k-1,k+1,4k\}$, for $k\ge 2$, cannot be extended to a Diophantine quintuple. The Baker-Davenport's result corresponds to $k=2$. In 1998, Dujella and Pethö [4] proved that the Diophantine pair $\{1,3\}$ cannot be extended to a Diophantine quintuple. In 2008, Fujita [7] obtained a more general result by proving that the Diophantine pairs $\{k-1,k+1\}$, for $k\ge 2$, cannot be extended to a Diophantine quintuple. A folklore conjecture is

Conjecture. There does not exist a Diophantine quintuple.

In 2004, Dujella [5] proved that there are only finitely many Diophantine quintuples. Assuming that $\{a,b,c,d,e\}$ is a Diophantine quintuple with $a<b<c<d<e$, the following upper bounds of the element d are known:

i) $d<10^{2171}$ by Dujella [5].
ii) $d<10^{830}$ by Fujita [8].
iii) $d<10^{100}$ by Filipin and Fujita [9].
iv) $d<3.5\cdot 10^{94}$ by Elsholtz, Filipin and Fujita [6].

Moreover, by using upper bound of d, corresponding upper bound of number of Diophantine quintuples are obtained, 10^{1930}, 10^{276}, 10^{96} and $6.8\cdot 10^{32}$, respectively.

In this paper, we prove the following result.

Theorem 1. *If $\{a,b,c,d,e\}$ is a Diophantine quintuple with $a<b<c<d<e$, then $d<10^{74}$.*

From now on, we will assume that $\{a,b,c,d,e\}$ is a Diophantine quintuple with $a<b<c<d<e$. Let us consider a Diophantine triple $\{A,B,C\}$. We define the positive integers R,S,T by

$$AB+1=R^2,\quad AC+1=S^2,\quad BC+1=T^2.$$

In order to extend the Diophantine triple $\{A,B,C\}$ to a Diophantine quadruple $\{A,B,C,D\}$, we have to solve the system

$$AD+1=x^2,\quad BD+1=y^2,\quad CD+1=z^2,$$

in integers x,y,z. Eliminating D, we obtain the following system of Pellian equations.

$$Az^2-Cx^2=A-C, \tag{1}$$

$$Bz^2-Cy^2=B-C. \tag{2}$$

All solutions of (1) and (2) are respectively given by $z=v_m$ and $z=w_n$ for some integer $m,n\ge 0$, where

$$v_0=z_0,\quad v_1=Sz_0+Cx_0,\quad v_{m+2}=2Sv_{m+1}-v_m,$$
$$w_0=z_1,\quad w_1=Tz_1+Cy_1,\quad w_{n+2}=2Tw_{n+1}-w_n,$$

with some integers z_0,z_1,x_0,y_1.

By Lemma 3 of [5], we have the following relations between m and n.

Lemma 1. *If $v_{2m}=w_{2n}$, then $n\le m\le 2n$.*

We will give a new lower bound of m in this paper.

Lemma 2. *If $B\ge 8$ and $v_{2m}=w_{2n}$ has solutions for $m\ge 3, n\ge 2$, then $m>0.48B^{-1/2}C^{1/2}$.*

2010 Mathematics Subject Classification. Primary 11D09.
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doi: 10.3792/pjaa.90.84

Proof. By Lemma 4 in [3] and $z_0 = z_1 = \lambda \in \{1, -1\}$, we have

$$Am^2 + \lambda Sm \equiv Bn^2 + \lambda Tn \pmod{4C}.$$

Suppose that $m \le 0.48B^{-1/2}C^{1/2}$. From the relation $n \le m$, we get

$$\max\{Am^2, Bn^2\} \le Bm^2 \le 0.25B \cdot B^{-1}C < 0.25C$$

and

$$\begin{aligned}\max\{Sm, Tn\} &\le Tm < 0.48(BC+1)^{1/2}B^{-1/2}C^{1/2}\\ &< 0.5(BC)^{1/2}B^{-1/2}C^{1/2} = 0.5C.\end{aligned}$$

We obtain that

$$Am^2 - Bn^2 = \lambda(Tn - Sm).$$

This implies

$$\begin{aligned}\lambda(Tn+Sm)(Am^2-Bn^2) &= T^2n^2 - S^2m^2\\ &= (BC+1)n^2 - (AC+1)m^2\\ &= C(Bn^2 - Am^2) + n^2 - m^2.\end{aligned}$$

It follows that

$$m^2 - n^2 = (C + \lambda(Tn+Sm))(Bn^2 - Am^2).$$

If $Bn^2 - Am^2 = 0$, then $m = n$, it is impossible. Hence,

$$m^2 - n^2 = |m^2 - n^2| \ge |C + \lambda(Tn + Sm)|.$$

The case $\lambda = 1$ provides $m^2 > C$, it is a contradiction to $m < 0.48B^{-1/2}C^{1/2}$. From $Tn + Sm < 2Tn < C$, we need to consider

$$\begin{aligned}m^2 - n^2 = |m^2 - n^2| &\ge |C - (Tn+Sm)|\\ &= C - (Tn + Sm).\end{aligned}$$

Therefore, we get the inequality

$$\begin{aligned}C &\le Tn + Sm + m^2 - n^2 \le 2Tm + 0.75m^2\\ &< 0.96(BC+1)^{1/2}B^{-1/2}C^{1/2} + 0.173B^{-1}C < C\end{aligned}$$

when $B \ge 8$. We have a contradiction. This completes the proof. □

Proof of Theorem 1. Assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$. In [4], Dujella and Pethö have shown that the pair $\{1, 3\}$ cannot extend to a Diophantine quintuple. This helps us to assume that $b \ge 8$.

We choose

$$A = a,\ B = b,\ C = d,\ D = e$$

in the Diophantine quintuple $\{a, b, c, d, e\}$. This implies the system of Pellian equations (1) and (2) has a positive integer solution (x, y, z) with $|z| > 1$. Equivalently, there are positive integers j and k satisfying $v_j = w_k$. By Lemma 5 and Lemma 6 of [8], we have $j \equiv k \equiv 0 \pmod 2$, $k \ge 4$, $z_0 = z_1 = \pm 1$. We set $j = 2m$ and $k = 2n$. Using Lemma 2, we have $m > 0.48B^{-1/2}C^{1/2}$.

It is known that $d \ge d^+ > 4abc > 4b^2$, where $d^+ = a + b + c + 2abc + 2rst$. It results $B = b < d^{1/2}/2 = C^{1/2}/2$. Hence, we have

$$m \ge 0.678C^{1/4}. \tag{3}$$

On the other hand, by used Theorem 2.1 in [10] of Matveev, we have the relative upper bound (cf. Proposition 4.2 of [8])

$$\frac{2m}{\log(351 \cdot 2m)} < 2.786 \cdot 10^{12} \cdot \log^2 C. \tag{4}$$

Combining (3) and (4), we obtain

$$C^{1/4} < 2.06 \cdot 10^{12} \cdot \log^2 C \cdot \log(476C^{1/4}).$$

Therefore, we have $d = C < 10^{74}$. This completes the proof of Theorem 1. □

Acknowledgments. The authors express their gratitude to Prof. Alain Togbé for his valuable comments. They thank the anonymous referee for constructive suggestions to improve an earlier draft of this paper, in particular, for revising an error on (4). The authors were supported by Natural Science Foundation of China (Grant No. 11301363), and Sichuan provincial scientific research and innovation team in Universities (Grant No. 14TD0040), and the Natural Science Foundation of Education Department of Sichuan Province (Grant No. 13ZA0037 and No. 13ZB0036).

References

[1] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[2] A. Dujella, The problem of the extension of a parametric family of Diophantine triples, Publ. Math. Debrecen **51** (1997), no. 3–4, 311–322.

[3] A. Dujella, An absolute bound for the size of Diophantine m-tuples, J. Number Theory **89** (2001), no. 1, 126–150.

[4] A. Dujella and A. Pethö, A generalization of a theorem of Baker and Davenport, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 195, 291–306.

[5] A. Dujella, There are only finitely many Diophantine quintuples, J. Reine Angew. Math. **566** (2004), 183–214.

[6] C. Elsholtz, A. Filipin and Y. Fujita, On Diophantine quintuples and $D(-1)$-quadruples, Monatsh. Math. (to appear).

[7] Y. Fujita, The extensibility of Diophantine pairs $\{k-1, k+1\}$, J. Number Theory **128** (2008), no. 2, 322–353.
[8] Y. Fujita, The number of Diophantine quintuples, Glas. Mat. Ser. III **45(65)** (2010), no. 1, 15–29.
[9] A. Filipin and Y. Fujita, The number of Diophantine quintuples II, Publ. Math. Debrecen **82** (2013), no. 2, 293–308.
[10] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II, Izv. Math. **64** (2000), no. 6, 1217–1269.