A note on the denominators of Bernoulli numbers

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Abstract: We show that

$$\gcd(2!S(2n+1,2),\dots,(2n+1)!S(2n+1,2n+1)) = \text{denominator of } B_{2n},$$

where $S(n,k)$ is the Stirling number of the second kind and B_n is the Bernoulli number.

Key words: Bernoulli numbers; Stirling numbers.

1. Introduction. Let $S(n,k)$ be the Stirling number of the second kind, which counts the number of partitions of a set with n elements in k disjoint nonempty subsets. Put $\widetilde{S}(n,k) = k!S(n,k)$. Let B_m be the mth Bernoulli number.

Theorem 1. *The formula*

$$\gcd(\widetilde{S}(2n+1,2),\dots,\widetilde{S}(2n+1,2n+1)) = \text{denominator of } B_{2n},$$

holds.

It is interesting to note that there are already classical formulas expressing the Bernoulli number in terms of Stirling numbers such as

$$B_{2n} = \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}(k-1)!S(2n+1,k)}{k} = 1 - \sum_{k=2}^{2n+1} \frac{(-1)^k \widetilde{S}(2n+2,k)}{k^2}$$

(see, for example, Chapter 1 in [2]). We shall not use this formula in our argument.

Proof. We use the von Staudt–Clausen theorem ([1,3]) which states that

$$\text{denominator of } B_{2n} = \prod_{(p-1)\mid 2n} p.$$

We first show that the right–hand side divides each of the numbers $\widetilde{S}(2n+1,k)$. Let p be a prime such that $p-1 \mid 2n$. To proceed, we recall that

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n,$$

to derive that

$$\widetilde{S}(2n+1,k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^{2n+1}. \tag{1}$$

Let $j \in \{1,\dots,2n\}$. By Fermat's Little Theorem and since $p-1 \mid 2n$, we get

$$j^{2n+1} \equiv j \pmod p. \tag{2}$$

Hence, inserting the above congruence (2) for $j = 1,2,\dots,k$ into (1), we get

$$\widetilde{S}(2n+1,k) \equiv \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j \pmod p.$$

However, the last sum above

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j = 0 \tag{3}$$

for $k \ge 2$ as it can be seen by putting $x = 1$ into the identity

$$k(x-1)^{k-1} = \frac{d}{dx}(x-1)^k = \frac{d}{dx}\left(\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} x^j\right) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j x^{j-1}.$$

This shows that the right-hand side divides the left-hand side.

Now we show that the left-hand side divides the right–hand side. Note that in the left-hand side, the last term inside the gcd is $\widetilde{S}(2n+1,2n+1) = (2n+1)!$, which implies that every prime p dividing the left-hand side satisfies $p \le 2n+1$. Let p^t be the

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exact power of p appearing in the left-hand side. We show that $t=1$ and that $(p-1)\mid 2n$, statements which together imply the desired conclusion. We then have

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^{2n+1}\equiv 0 \pmod{p^t}, \tag{4}$$

for $k=2,\dots,2n+1$. Making $k=2$ in (4) above we get

$$-2+2^{2n+1}\equiv 0 \pmod{p^t}. \tag{5}$$

Making $k=3$ in (4) above we get

$$3-3\cdot 2^{2n+1}+3^{2n+1}\equiv 0 \pmod{p^t}, \tag{6}$$

and inserting also (5) into (6), we get the congruence $3^{2n+1}\equiv 3 \pmod{p^t}$. So, let us show by induction on $k=1,2,3,\dots,2n+1$ that $k^{2n+1}\equiv k \pmod{p^t}$. Assume that $k\ge 4$ and that the above congruences are satisfied for $1,2,\dots,k-1$. Formula (4) together with the induction hypothesis implies that

$$\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}j+k^{2n+1}\equiv 0 \pmod{p^t}$$

which together with the identity (3) gives that $k^{2n+1}\equiv k \pmod{p^t}$. Hence, it is indeed the case that

$$k^{2n+1}\equiv k \pmod{p^t},$$

for $k=1,\dots,2n+1$. Making $k=p$, we get that $p^{2n+1}\equiv p \pmod{p^t}$, showing that $t=1$. Finally, since $p\le 2n+1$, it follows that $1,2,\dots,2n+1$ cover all residue classes modulo p, therefore we have $a^{2n+1}\equiv a \pmod{p}$ for all integers a. In particular, $a^{2n}\equiv 1 \pmod{p}$ for all integers a coprime to p, implying that $(p-1)\mid 2n$ because the multiplicative group modulo p is cyclic of order $p-1$. This concludes the proof of the theorem. □

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