Transcendence of special values of log double sine function

By Hidekazu TANAKA

Department of Mathematics, Tokyo Institute of Technology,
2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan

(Communicated by Kenji FUKAYA, M.J.A., Oct. 14, 2014)

Abstract: In 2009 [4,5], S. Gun, M. R. Murty, P. Rath studied transcendental values of the logarithm of the gamma function. They showed that for any rational number x with $0 < x < \frac{1}{2}$, the number $\log\Gamma(x) + \log\Gamma(1-x)$ is transcendental with at most one possible exception. In this paper, we study transcendental values of log double sine function using their method.

Key words: Double sine function; transcendency; Catalan constant.

1. Introduction. Recall that multiple sine functions $S_r(x)$ are constructed as

$$S_r(x) = \prod_{n_1,\dots,n_r\geq 0}(n_1+\cdots+n_r+x) \times \left(\prod_{n_1,\dots,n_r\geq 1}(n_1+\cdots+n_r-x)\right)^{(-1)^{r-1}},$$

where $\prod$ denotes the regularized product of Deninger [3]:

$$\prod_{\lambda\in\Lambda}\lambda = \exp\left(-\frac{d}{ds}\sum_{\lambda\in\Lambda}\lambda^{-s}\bigg|_{s=0}\right).$$

We refer to [2,6–8] for the details of multiple sine functions. We are interested with special values of log multiple sine functions to investigate unknown special values of zeta functions.

In [4,5](2009), S. Gun, M. R. Murty and P. Rath investigated the transcendency of

$$\log\Gamma(x) + \log\Gamma(1-x)$$

for any rational number $x \in (0, \frac{1}{2})$. By Lerch's formula [9] we have

$$\begin{aligned}\log\Gamma(x) + \log\Gamma(1-x) &= \log\pi - \log\sin(\pi x)\\ &= \log(2\pi) - \log S_1(x).\end{aligned}$$

Here we study an analogue of this result for double sine function $S_2(x)$. Using the definition of $S_2(x)$ we have

$$S_2(x) = S_2(2-x)^{-1}.$$

Hence, we may restrict ourselves to the domain $0 < x < 1$ for $\log S_2(x)$. This domain corresponds to the domain $0 < x < \frac{1}{2}$ for $\log\Gamma(x) + \log\Gamma(1-x) = \log\pi - \log\sin(\pi x)$.

2000 Mathematics Subject Classification. Primary 11M36.

Theorem 1.1. *Let* $\alpha \in (0,1) - \{\frac{1}{6}, \frac{5}{6}\}$ *be a rational number.*
(1) *For any positive integer* k*, the number*

$$\log|S_2(\alpha+k)|$$

is transcendental with at most one possible exceptional k.
(2) *For any non-positive integer* k*, the number*

$$\log|S_2(\alpha+k)|$$

is transcendental with at most one possible exceptional k.

Remark 1.2. Let G be the Catalan constant, that is, $G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.9159\cdots > 0$. Then Kurokawa and Koyama [7, Example 2.9 (a)] obtained

$$S_2\left(\frac{1}{4}\right) = 2^{\frac{3}{8}}\exp\left(-\frac{G}{2\pi}\right). \tag{1.1}$$

Corollary 1.3. (1) *Let* k *be any positive integer. Then the number*

$$\log\left|S_2\left(\frac{1}{4}+k\right)\right| = \frac{3-4k}{8}\log 2 - \frac{G}{2\pi}$$

is transcendental with at most one possible exceptional k.
(2) *Let* k *be any non-positive integer. Then the number*

$$\log\left|S_2\left(\frac{1}{4}+k\right)\right| = \frac{3+4k}{8}\log 2 - \frac{G}{2\pi},$$

is transcendental with at most one possible exceptional k.

doi: 10.3792/pjaa.90.133

2. Proofs.

We prepare the following lemma:

Lemma 2.1 ([7, (a) of Theorem 2.1]). *Let $x \in \mathbf{R}$. We have*

$$S_r(x+1) = \frac{S_r(x)}{S_{r-1}(x)}, \tag{2.1}$$

where we put

$$S_0(x) = -1.$$

Lemma 2.2. *Let k_i be distinct positive integers for $i = 1, 2$ and $\alpha \in (0,1) - \{\frac{1}{6}, \frac{5}{6}\}$ be a rational number. Put $g(k_i) := \log|S_2(\alpha + k_i)|$. Then we have*

$$g(k_1) \neq g(k_2).$$

Proof of Lemma 2.2. Assume $k_1 > k_2 > 0$ and $g(k_1) = g(k_2)$. Since by (2.1)

$$\begin{aligned} g(k_i) := \log|S_2(\alpha + k_i)| &= \log|S_2(\alpha)| \\ &- \sum_{l=0}^{k_i-1} \log|S_1(\alpha + l)| \end{aligned}$$

for $i = 1, 2$, we have

$$\sum_{l=k_2}^{k_1-1} \log|S_1(\alpha + l)| = 0$$

from $g(k_1) = g(k_2)$. Since $\alpha \in (0,1) - \{\frac{1}{6}, \frac{5}{6}\}$ is a rational number and

$$\begin{aligned} \prod_{l=k_2}^{k_1-1} |S_1(\alpha + l)| &= \prod_{l=k_2}^{k_1-1} |2\sin(\pi(\alpha + l))| \\ &= (2\sin(\alpha\pi))^{k_1-k_2} \neq 1, \end{aligned}$$

we have a contradiction. So $g(k_1) \neq g(k_2)$. □

Proof of Theorem 1.1. Now we prove (1) of Theorem 1.1. Suppose otherwise, namely, there exist positive integers k_1, k_2 $(k_1 > k_2)$ such that both values $g(k_1)$ and $g(k_2)$ are algebraic. Then using Lemma 2.2 we see that the algebraic number $g(k_1) - g(k_2)$ is expressed as

$$0 \neq g(k_1) - g(k_2) = \sum_{l=k_2}^{k_1-1} \log|S_1(\alpha + l)|. \tag{2.2}$$

We recall a famous result of Baker.

Lemma 2.3 ([1, Theorem 2.2]). *Any non-vanishing linear combination of logarithms of algebraic numbers with algebraic coefficients is transcendental.*

From (2.2) and Lemma 2.3 $g(k_1) - g(k_2)$ is transcendental. This gives a contradiction. Next we prove (2) of Theorem 1.1. By the definition we have

$$S_2(2-x) = S_2(x)^{-1},$$

so that for $\alpha \in (0,1) - \{\frac{1}{6}, \frac{5}{6}\}$ and for positive integer k

$$S_2(\alpha + k)^{-1} = S_2(2 - (\alpha + k)) = S_2(\alpha' + k'),$$

where $\alpha' = 1 - \alpha$ and $k' = 1 - k$. Hence, (2) of Theorem 1.1 is given by (1) of Theorem 1.1. □

Proof of Corollary 1.3. Corollary 1.3 is given by Theorem 1.1, (1.1) and (2.1). □

References

[1] A. Baker, *Transcendental number theory*, Cambridge Univ. Press, London, 1975.

[2] E. W. Barnes, On the theory of the multiple gamma function, Trans. Cambridge Philos. Soc. **19** (1904), 374–425.

[3] C. Deninger, Local L-factors of motives and regularized determinants, Invent. Math. **107** (1992), no. 1, 135–150.

[4] S. Gun, M. R. Murty and P. Rath, Transcendence of the log gamma function and some discrete periods, J. Number Theory **129** (2009), no. 9, 2154–2165.

[5] S. Gun, M. R. Murty and P. Rath, Corrigendum to “Transcendence of the log gamma function and some discrete periods” [J. Number Theory **129** (9) (2009) 2154–2165], J. Number Theory **130** (2010), 1251.

[6] S. Koyama and N. Kurokawa, Values of the double sine function, J. Number Theory **123** (2007), no. 1, 204–223.

[7] N. Kurokawa and S. Koyama, Multiple sine functions, Forum Math. **15** (2003), no. 6, 839–876.

[8] N. Kurokawa and M. Wakayama, Extremal values of double and triple trigonometric functions, Kyushu J. Math. **58** (2004), no. 1, 141–166.

[9] M. Lerch, Další studie v oboru Malmsténovských řad, Rozpravy České Akad. **3** (1894), no. 28, 1–61.