Life span of solutions for a quasilinear parabolic equation with initial data having positive limit inferior at infinity

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Abstract: We present a new upper bound of the life span of positive solutions of a quasilinear parabolic equation for the initial data having positive limit inferior at space infinity. The upper bound is expressed by the data in limit inferior, not in every direction, but around a specific direction. It is also shown that the minimal time blow-up occurs when the initial data attain its maximum at space infinity.

Key words: Life span; quasilinear parabolic equation; Cauchy problem; blow-up.

1. Introduction. We consider the Cauchy problem for a quasilinear parabolic equation

$$(1)\qquad \begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases}$$

where $1 < m < p$, $n \geq 1$ and an initial datum $u_0(x)$ is a bounded continuous function on $\mathbf{R}^n$.

It is well known that a unique bounded non-negative weak solution of (1) exists locally in time [1, 2, 8, 13]. Here, we state the definition of a weak solution of (1).

Definition 1. By a weak solution of equation (1) in $\mathbf{R}^n \times (0,T)$, we mean a function $u(x,t)$ in $\mathbf{R}^n \times [0,T)$ such that

(i) $u(x,t) \geq 0$ in $\mathbf{R}^n \times [0,T)$ and in $BC(\mathbf{R}^n \times [0,\tau])$ (bounded continuous) for each $0 < \tau < T$.

(ii) For any bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary $\partial\Omega$, $0 < \tau < T$ and non-negative function $\varphi \in C^{2,1}(\bar{\Omega} \times [0,T))$ which vanishes on the boundary $\partial\Omega$,

$$(2)\qquad \int_\Omega u(x,\tau)\varphi(x,\tau)dx - \int_\Omega u(x,0)\varphi(x,0)dx = \int_0^\tau \int_\Omega \{u\varphi_t + u^m \Delta\varphi + u^p\varphi\} dxdt - \int_0^\tau \int_{\partial\Omega} u^m \partial_\nu \varphi dSdt,$$

where ν denotes the outer unit normal to the boundary.

A supersolution [or subsolution] is similarly defined with the equality in (2) replaced by $\geq$ [or $\leq$].

We define the life span T^* as

$$(3)\qquad T^* = \sup\{T > 0;\ (1) \text{ possesses a unique weak solution in } \mathbf{R}^n \times (0,T)\}.$$

If $T^* = \infty$, the solution is global. On the other hand, if $T^* < \infty$, the solution is not global in time in the sense that it blows up at $t = T^*$ such as

$$(4)\qquad \limsup_{t \to T^*} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

The blow-up and the global existence of solutions are studied by Galaktionov–Kurdyumov–Mikhailov–Samarskii [4], Galaktionov [3], Kawanago [7], Mochizuki–Suzuki [11], Mochizuki [9] and Mukai–Mochizuki–Huang [12]. And the following results are known to hold.

(i) Let $p \in (m, m + 2/n]$. Then, $T^* < \infty$ for every nontrivial solution of (1).

(ii) Let $p \in (m + 2/n, \infty)$. Then, $T^* = \infty$ for some initial data $u_0(x) \not\equiv 0$.

Especially for the non-decaying initial data, it was shown that the solution of (1) blows up in finite time for any $p > m$.

In this paper, we present new upper bounds on the life span of positive solutions of (1) for non-decaying initial data.

Recently, several studies have been made on the life span of solutions for (1). (See [10, 12, 16], and references therein.)

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Mukai-Mochizuki-Huang [12] proved the following results when an initial datum takes the form $u_0(x) = \lambda\phi(x)$, where $\lambda > 0$ and $\phi(x)$ is a bounded continuous function on $\mathbf{R}^n$.

(i) If $\|\phi\|_{L^\infty(\mathbf{R}^n)} = \phi(0) > 0$, then

$$\lim_{\lambda\to\infty} \lambda^{p-1}T^* = \frac{1}{p-1}\,\phi(0)^{-(p-1)}.$$

(ii) If $\|\phi\|_{L^\infty(\mathbf{R}^n)} = \lim_{|x|\to\infty} \phi(x) = \phi_\infty > 0$, then

$$\lim_{\lambda\to 0} \lambda^{p-1}T^* = \frac{1}{p-1}\,\phi_\infty^{-(p-1)}.$$

The purpose of this paper is to give a sharp upper of the life span of solution for (1) with the initial data having positive limit inferior at space infinity.

The outline of the remainder of this paper is as follows. In Section 2, we prepare several notations and state the main results: Theorems 1 and 2. In Sections 3 and 4, we prove Theorems 1 and 2 by improving the method in Yamauchi [19] and Ozawa–Yamauchi [14], respectively.

2. Main results. In order to state the main results, we prepare several notations. For $\xi \in S^{n-1}$, and $\delta \in (0, \sqrt{2})$, we set the conic neighborhood $\Gamma_\xi(\delta)$:

$$\Gamma_\xi(\delta) = \left\{\eta \in \mathbf{R}^n\backslash\{0\}; \left|\xi - \frac{\eta}{|\eta|}\right| < \delta\right\}, \tag{5}$$

and set $S_\xi(\delta) = \Gamma_\xi(\delta) \cap S^{n-1}$. Define

$$u_{0,\infty}(\theta) = \liminf_{r\to+\infty} u_0(r\theta)$$

for $\theta \in S^{n-1}$. We note that $u_{0,\infty} \in L^\infty(S^{n-1})$.

Now, we state a main result.

Theorem 1. *Let $n \geq 2$. Assume that there exist $\xi \in S^{n-1}$ and $\delta > 0$ such that*

$$\operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} u_{0,\infty}(\theta) > 0.$$

Then the weak solution for (1) *blows up in finite time, and the blow-up time is estimated as*

$$T^* \leq \frac{1}{p-1}\left(\operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} u_{0,\infty}(\theta)\right)^{1-p}. \tag{6}$$

Once we prove Theorem 1, we can show the following corollaries immediately.

Corollary 1. *Suppose that $\|u_{0,\infty}\|_{L^\infty(S^{n-1})} > 0$. Assume that for arbitrary small $\varepsilon > 0$, there exist $\xi \in S^{n-1}$ and $\delta > 0$ such that*

$$\operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} u_{0,\infty}(\theta) \geq \|u_{0,\infty}\|_{L^\infty(S^{n-1})} - \varepsilon. \tag{7}$$

Then the weak solution for (1) *blows up in finite time, and the blow-up time is estimate as*

$$T^* \leq \frac{1}{p-1}\,\|u_{0,\infty}\|_{L^\infty(S^{n-1})}^{1-p}. \tag{8}$$

Proof of Corollary 1. For arbitrary small $\varepsilon > 0$, we obtain

$$T^* \leq \frac{1}{p-1}\left(\|u_{0,\infty}\|_{L^\infty(S^{n-1})} - \varepsilon\right)^{1-p} \tag{9}$$

from Theorem 1. Taking $\varepsilon \to 0$, we obtain the desired result. □

In particular, the following result holds if $u_{0,\infty}$ is continuous on whole S^{n-1}.

Corollary 2. *Suppose that $\|u_{0,\infty}\|_{L^\infty(S^{n-1})} > 0$. Assume that $u_{0,\infty} \in C(S^{n-1})$. Then the weak solution for* (1) *blows up in finite time, and the blow-up time is estimated as*

$$T^* \leq \frac{1}{p-1}\,\|u_{0,\infty}\|_{L^\infty(S^{n-1})}^{1-p}. \tag{10}$$

Proof of Corollary 2. For $u_{0,\infty} \in C(S^{n-1})$, inequality (7) in Corollary 1 holds. □

Remark 1. From the comparison principle, we easily obtain the lower bound of the life span:

$$T^* \geq \frac{1}{p-1}\,\|u_0\|_{L^\infty(\mathbf{R}^n)}^{1-p}. \tag{11}$$

In addition to the same hypothesis as in Corollary 1, assume that $0 \leq u_0 \leq \|u_{0,\infty}\|_{L^\infty(S^{n-1})}$. Then we have

$$T^* = \frac{1}{p-1}\,\|u_{0,\infty}\|_{L^\infty(S^{n-1})}^{1-p}; \tag{12}$$

that is, the so-called minimal time blow-up occurs. Related researchers are provided in [5, 6, 10, 15–18].

Theorem 2. *Let $n = 1$. Assume that*

$$\max\left\{\liminf_{x\to+\infty} u_0(x), \liminf_{x\to-\infty} u_0(x)\right\} > 0.$$

Then the weak solution of (1) *blows up in finite time T^*, and the blow-up time is estimated as*

$$T^* \leq \frac{1}{p-1} \times \left(\max\left\{\liminf_{x\to+\infty} u_0(x), \liminf_{x\to-\infty} u_0(x)\right\}\right)^{1-p}. \tag{13}$$

3. Proof of Theorem 1. For $\xi \in S^{n-1}$ and $\delta > 0$ as in the theorem, we first determine the sequences $\{a_j\} \subset \mathbf{R}^n$ and $\{R_j\} \subset (0, \sqrt{2})$. Let

$\{a_j\} \subset \mathbf{R}^n$ be a sequence satisfying that $|a_j| \to \infty$ as $j \to \infty$, and that $a_j/|a_j| = \xi$ for any $j \in \mathbf{N}$. Put $R_j = (\delta\sqrt{4-\delta^2}/2)|a_j|$.

For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\Delta$ on $B_{R_j}(0) = \{x \in \mathbf{R}^n; |x| < R_j\}$ with zero Dirichlet boundary condition under the normalization $\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1$. Moreover, let μ_{R_j} be the corresponding first eigenvalue of the eigenfunction. For the solutions for (1), we define

$$w_j(t) = \int_{B_{R_j}(0)} u(x+a_j, t)\rho_{R_j}(x)dx. \tag{14}$$

Here, we shall focus on the upper bound of the life span of w_j.

Translating both sides of the equation (1) by a_j, by the definition of a weak solution, we have

$$\begin{aligned} w_j(\tau) - w_j(0) \geq \int_0^\tau \int_{B_{R_j}(0)} \{&-\mu_{R_j} u^m(x+a_j,t) \\ &+ u^p(x+a_j,t)\}\rho_{R_j}(x)dxdt. \end{aligned} \tag{15}$$

Let $T^*_{w_j}$ be the life span of w_j. Then we have the following proposition.

Proposition 1. *If*

$$w_j(0) > \mu_{R_j}^{1/(p-m)}, \tag{16}$$

then $u(x+a_j,t)$ is never global in t, and we have

$$T^*_{w_j} \leq \int_{w_j(0)}^\infty \frac{1}{-\mu_{R_j}\xi^m + \xi^p} d\xi.$$

Proof. See [11, Proposition 2.3]. □

Here, we introduce the properties of the initial value $\{w_j(0)\}$.

Proposition 2. *We have*

$$\liminf_{j\to+\infty} w_j(0) \geq \operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} u_{0,\infty}(\theta). \tag{17}$$

Proof. Changing the variable and using the relation $\rho_{\pi/2}(x) = (2R_j/\pi)^n \rho_{R_j}(2R_j x/\pi)$, we have

$$\begin{aligned} w_j(0) &= \int_{B_{R_j}(0)} u_0(x+a_j)\rho_{R_j}(x)dx \\ &= \left(\frac{2R_j}{\pi}\right)^n \int_{B_{\pi/2}(0)} u_0\left(\frac{2R_j}{\pi}x + a_j\right) \\ &\quad \times \rho_{R_j}\left(\frac{2R_j}{\pi}x\right)dx \\ &= \int_{B_{\pi/2}(0)} u_0\left(\frac{2R_j}{\pi}x + a_j\right)\rho_{\pi/2}(x)dx. \end{aligned} \tag{18}$$

□

Here, we prepare the following lemma to prove Proposition 2.

Lemma 1. *For $x \in B_{\pi/2}(0)$, the following properties hold.*

(i) $\dfrac{(2R_j/\pi)x + a_j}{|(2R_j/\pi)x + a_j|} = \dfrac{(2R_k/\pi)x + a_k}{|(2R_k/\pi)x + a_k|}$ *for any $j, k \in \mathbf{N}$.*

(ii) $(2R_j/\pi)x + a_j \in B_{R_j}(a_j) \subset \Gamma_\xi(\delta)$.

(iii) $|(2R_j/\pi)x + a_j| \to \infty$ *as $j \to \infty$.*

Proof. See [19, Lemma 1]. □

Proof of Proposition 2. For fixed $x \in B_{\pi/2}(0)$, put $\theta = \dfrac{(2R_j/\pi)x + a_j}{|(2R_j/\pi)x + a_j|}$. We note that θ is independent of $j \in \mathbf{N}$ from Lemma 1 (i). Moreover, $\theta \in S_\xi(\delta)$ from Lemma 1 (ii). Then, by Lemma 1 (iii), we have

$$\begin{aligned} &\liminf_{j\to\infty} u_0\left(\frac{2R_j}{\pi}x + a_j\right) \\ &\quad = \liminf_{j\to\infty} u_0\left(\left|\frac{2R_j}{\pi}x + a_j\right|\theta\right) \\ &\quad \geq \liminf_{r\to\infty} u_0(r\theta) = u_{0,\infty}(\theta). \end{aligned} \tag{19}$$

By Fatou's lemma, we obtain

$$\begin{aligned} &\liminf_{j\to\infty} w_j(0) \\ &\quad \geq \int_{B_{\pi/2}(0)} \liminf_{j\to\infty} u_0\left(\frac{2R_j}{\pi}x + a_j\right)\rho_{\pi/2}(x)dx \\ &\quad \geq \operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} u_{0,\infty}(\theta). \end{aligned} \tag{20}$$

Hence, we obtain (17). □

Now let us prove Theorem 1.

Proof of Theorem 1. By Propositions 1 and 2, we see that

$$\begin{aligned} \limsup_{j\to\infty} T^*_{w_j} &\leq \limsup_{j\to\infty} \int_{w_j(0)}^\infty \frac{1}{-\mu_{R_j}\xi^m + \xi^p} d\xi \\ &\leq \frac{1}{p-1}\left(\operatorname*{ess.inf}_{\theta\in S_\xi(\delta)} u_{0,\infty}(\theta)\right)^{1-p}. \end{aligned} \tag{21}$$

On the other hand, we have

$$\limsup_{j\to\infty} T^*_{w_j} \geq \limsup_{j\to\infty} T^* = T^*. \tag{22}$$

Indeed, for fixed $j \in \mathbf{N}$ and $t \in (0, T^*)$, if u remains bounded then w_j is finite. This completes the proof. □

4. Proof of Theorem 2. Let $a_j = j$ or $-j$. Put $R_j = j/2$. For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\frac{\partial^2}{\partial x^2}$ on $(-R_j, R_j)$ with zero Dirichlet boundary condition under the normalization $\int_{-R_j}^{R_j} \rho_{R_j}(x)dx = 1$. Moreover, let μ_{R_j} be the corresponding first eigenvalue. For the solutions for (1), we define

$$w_j(t) = \int_{-R_j}^{R_j} u(x + a_j, t)\rho_{R_j}(x)dx. \tag{23}$$

Here, we shall focus on the upper bound of the life span of w_j.

Translating both sides of the equation (1) by a_j, by the definition of a weak solution, we have

$$w_j(\tau) - w_j(0) \geq \int_0^\tau \int_{-R_j}^{R_j} \{-\mu_{R_j} u^m(x + a_j, t) + u^p(x + a_j, t)\}\rho_{R_j}(x)dxdt. \tag{24}$$

The rest of the proof is the same as in that of Theorem 1. We show the corresponding proposition used in the rest of the proof.

Proposition 3. *We have*

$$\liminf_{j\to+\infty} w_j(0) \geq \max\left\{\liminf_{x\to+\infty} u_0(x), \liminf_{x\to-\infty} u_0(x)\right\}. \tag{25}$$

Proof. Since $x + a_j \to +\infty$ or $-\infty$ $(j \to \infty)$ for $x \in (-R_j, R_j)$, by Fatou's lemma we obtain

$$\begin{aligned} \liminf_{j\to\infty} w_j(0) &\geq \int_{-\pi/2}^{\pi/2} \liminf_{j\to\infty} u_0\left(\frac{2R_j}{\pi}x + a_j\right)\rho_{\pi/2}(x)dx \\ &\geq \liminf_{x\to+\infty \text{ or } -\infty} u_0(x) \int_{-\pi/2}^{\pi/2} \rho_{\pi/2}(x)dx \\ &= \liminf_{x\to+\infty \text{ or } -\infty} u_0(x). \end{aligned} \tag{26}$$

□

Finally, let us prove Theorem 2.

Proof of Theorem 2. Let $T^*_{w_j}$ be the life span of w_j. By Propositions 1 and 3, we see that

$$\begin{aligned} &\limsup_{j\to\infty} T^*_{w_j} \\ &\leq \limsup_{j\to\infty} \int_{w_j(0)}^{\infty} \frac{1}{-\mu_{R_j}\xi^m + \xi^p} d\xi \\ &\leq \frac{1}{p-1}\left(\max\left\{\liminf_{x\to+\infty} u_0(x), \liminf_{x\to-\infty} u_0(x)\right\}\right)^{1-p}. \end{aligned} \tag{27}$$

On the other hand, we have

$$\limsup_{j\to\infty} T^*_{w_j} \geq \limsup_{j\to\infty} T^* = T^*. \tag{28}$$

Indeed, for fixed $j \in \mathbf{N}$ and $t \in (0, T^*)$, if u remains bounded the w_j is finite. This completes the proof. □

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