Non-existence of certain Diophantine quadruples in rings of integers of pure cubic fields

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Abstract: In this paper we derive some elements of the rings of integers in the cubic fields of the form $\mathbf{Q}(\sqrt[3]{d})$, where d is even, which cannot be written as a difference of two squares in the considered ring. We show that corresponding Diophantine quadruples do not exist for such elements, what supports the hypothesis mainly proved for the ring of integers and for certain quadratic fields.

Key words: Diophantine quadruples; cubic fields.

1. Introduction. A problem of proving the existence of Diophantine quadruples can be placed among the most interesting ancient problems in the number theory. It was first considered by the Greek mathematician Diophantus of Alexandria in the third century, who has discovered beautiful properties of the set $\{1, 33, 68, 105\}$. Originally, a problem of finding the Diophantine quadruple with a property $D(w)$, or the $D(w)$–quadruple for short, consisted of deriving a set $\{w_1, w_2, w_3, w_4\}$ of four nonzero integers with the property that $w_i \cdot w_j + w$ is a perfect square, for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, and Diophantus observed that the mentioned set is a $D(256)$–quadruple.

Many authors have studied this problem since then and it has been shown by Dujella in [3] that if $w \not\equiv 2 \pmod 4$ and $w \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$ then there exists a $D(w)$–quadruple. He has also derived many useful polynomial formulas for Diophantine quadruples, which can be found in [4]. The fact that there are no $D(4k+2)$–quadruples, where $k \in \mathbf{Z}$, is a consequence of work of some other authors ([1,9,10]).

Analogous problem can also be studied in other rings, especially in rings of integers of number fields. In particular, in a series of papers [6], [7] and [8], Franušić mainly solved this problem for non-imaginary quadratic number fields of the form $\mathbf{Q}(\sqrt{d})$. She showed that for an integer w in such quadratic field there exists a Diophantine quadruple with the property $D(w)$ if and only if w can be represented as a difference of squares of two integers, up to finitely many exceptions. One may ask whether this is still true if the ring of integers in $\mathbf{Q}(\sqrt{d})$ is replaced by the ring of integers in other number fields and this question is at present far from being solved. The purpose of this paper is to provide some results in that direction for cubic number fields.

The results in [6–8] are based on the description of differences of two squares in non–imaginary quadratic number fields, given in [5]. Description given there relies on the solvability of Pellian equations which arise as determinants of systems of certain linear equations. This also presents a main advantage of the quadratic number field case. Namely, analogous systems of linear equations that appear in the case of higher dimensional number fields lead to equations that are more complicated than Pellian's. We will restrict our attention to finding elements in the rings of integers of pure cubic number fields $\mathbf{Q}(\sqrt[3]{d})$, where d is even, that cannot be written as a difference of two squares. Depending on the structure of the ring of integers in $\mathbf{Q}(\sqrt[3]{d})$ there are two possibilities, which are considered separately. In each case we determine certain classes of elements that cannot be written as a difference of squares and show the non–existence of related Diophantine quadruples.

Non–existence of such Diophantine quadruples follows from the fact that systems of congruences which we obtain do not have a solution and is proved using case–by–case consideration.

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As a consequence, we prove that there exist no $D(4k+2)$-quadruples, $k \in \mathbf{Z}$, in the ring of integers in $\mathbf{Q}(\sqrt[3]{d})$ for even d, that is different from the case $\mathbf{Q}(i)$ or $\mathbf{Q}(\sqrt{2})$.

We now describe the content of the paper in more detail. In the following section we recall basic facts related to the structure of rings of integers in pure cubic fields, while in the third section we derive some elements of such rings that cannot be written as a difference of squares and consider related Diophantine quadruples.

2. Preliminaries. First we describe the rings of integers in the numbers fields which we consider. We will study the rings of integers in the cubic number fields of the form $\mathbf{Q}(\sqrt[3]{d})$, where $d \in \mathbf{Z}$. Such cubic number fields are usually called pure cubic number fields. Clearly, we may suppose that d is cube-free integer greater than 1. We now recall the description of the rings of integers in pure cubic number fields, which goes back as far as [2].

It is easy to see that there exist unique relatively prime positive integers a and b such that $d = ab^2$ and ab is square-free. Next, we define $\alpha = \sqrt[3]{d}$ and $\beta = \sqrt[3]{a^2b}$ (note that $\alpha^2 = b\beta$).

If $a^2 \not\equiv b^2 \pmod 9$, then the ring of integers in $\mathbf{Q}(\sqrt[3]{d})$ equals $\mathbf{Z}[1,\alpha,\beta]$. Otherwise, the ring of integers in $\mathbf{Q}(\sqrt[3]{d})$ is given by $\mathbf{Z}[\alpha,\beta,\gamma]$, where $\gamma = \frac{1+a\alpha+b\beta}{3}$. We take a moment to verify the form of perfect squares in mentioned rings of integers.

First we consider the case $a^2 \not\equiv b^2 \pmod 9$. Obviously, $\beta^2 = a\alpha$ and $\alpha\cdot\beta = a\cdot b$. For $x,y,z \in \mathbf{Z}$, it can be directly verified that the following holds

$$(x+y\alpha+z\beta)^2 = x^2+2yzab+(2xy+z^2a)\alpha+(2xz+y^2b)\beta.$$

Thus, each element of $\mathbf{Z}[1,\alpha,\beta]$ which can be written as a difference of two squares is of the form

$$\begin{aligned} &x_1^2-x_2^2+2ab(y_1z_1-y_2z_2)+\\ &(2(x_1y_1-x_2y_2)+a(z_1^2-z_2^2))\alpha+ \qquad (1)\\ &(2(x_1z_1-x_2z_2)+b(y_1^2-y_2^2))\beta \end{aligned}$$

for some $x_1,y_1,z_1,x_2,y_2,z_2 \in \mathbf{Z}$.

Now we consider more complicated case $a^2 \equiv b^2 \pmod 9$. Observe that in this case $1 = 3\gamma - a\alpha - b\beta$. It is not hard to see that $\alpha\cdot\beta = 3ab\gamma - a^2b\alpha - ab^2\beta$. The proof of the following relations is straightforward:

$$\alpha\gamma = ab^2\gamma - \frac{a^2b^2-1}{3}\alpha - ab\frac{b^2-1}{3}\beta$$

$$\beta\gamma = a^2b\gamma - ab\frac{a^2-1}{3}\alpha - \frac{a^2b^2-1}{3}\beta$$

$$\gamma^2 = \frac{1+2a^2b^2}{3}\gamma + \frac{a(b^2+1-2a^2b^2)}{9}\alpha + \frac{b(a^2+1-2a^2b^2)}{9}\beta.$$

Note that since a and b are relatively prime and $a^2 \equiv b^2 \pmod 9$ it follows that 3 does not divide d. Thus, $a^2 \equiv 1 \pmod 3$ and $b^2 \equiv 1 \pmod 3$, and we obtain that 3 divides all of $a^2b^2-1, a^2-1, b^2-1, 1+2a^2b^2$. Furthermore, $(a^2,b^2) \bmod 9 \in \{(1,1),(4,4),(7,7)\}$. Now a direct verification shows that 9 divides both $a^2+1-2a^2b^2$ and $b^2+1-2a^2b^2$.

Using previous formulas we obtain that for $x,y,z \in \mathbf{Z}$ the following holds:

$$\begin{aligned}(x\alpha+y\beta+z\gamma)^2 &= \Big(ay^2-2a^2bxy-2xz\frac{a^2b^2-1}{3}\\ &\quad -2yzab\frac{a^2-1}{3}+z^2\frac{a(b^2+1-2a^2b^2)}{9}\Big)\alpha\\ &\quad +\Big(bx^2-2ab^2xy-2xzab\frac{b^2-1}{3}\\ &\quad -2yz\frac{a^2b^2-1}{3}+z^2\frac{b(a^2+1-2a^2b^2)}{9}\Big)\beta\\ &\quad +\Big(6abxy+2ab^2xz+2a^2byz+\frac{1+2a^2b^2}{3}z^2\Big)\gamma.\end{aligned}$$

Similarly as in the first case, each element of $\mathbf{Z}[\alpha,\beta,\gamma]$ that can be written as a difference of two squares is of the form

$$\begin{aligned}&\Big(a\Big(y_1^2-y_2^2-2ab(x_1y_1-x_2y_2)\\ &-2b\frac{a^2-1}{3}(y_1z_1-y_2z_2)+\frac{b^2+1-2a^2b^2}{9}(z_1^2-z_2^2)\Big)\\ &\quad -2\frac{a^2b^2-1}{3}(x_1z_1-x_2z_2)\Big)\alpha\\ (2)\quad &+\Big(b\Big(x_1^2-x_2^2-2ab(x_1y_1-x_2y_2)\\ &-2a\frac{b^2-1}{3}(x_1z_1-x_2z_2)+\frac{a^2+1-2a^2b^2}{9}(z_1^2-z_2^2)\Big)\\ &\quad -2\frac{a^2b^2-1}{3}(y_1z_1-y_2z_2)\Big)\beta\\ &\quad +\Big(2ab(3(x_1y_1-x_2y_2)+b(x_1z_1-x_2z_2)\\ &\quad +a(y_1z_1-y_2z_2))+\frac{1+2a^2b^2}{3}(z_1^2-z_2^2)\Big)\gamma,\end{aligned}$$

where $x_1,y_1,z_1,x_2,y_2,z_2 \in \mathbf{Z}$.

3. Non-existence of some Diophantine quadruples. In this section we will derive the non-existence of some Diophantine quadruples under the assumption that d is even. As before, we write $d = ab^2$ where ab is square-free and $(a, b) = 1$. Note that the assumption that d is even implies that exactly one of the positive integers a and b is even.

Again, we will discuss case $a^2 \not\equiv b^2 \pmod 9$ first. Since $2ab \equiv 0 \pmod 4$, it follows from (1) that each element $x + y\alpha + z\beta \in \mathbf{Z}[1, \alpha, \beta]$ which can be written as a difference of squares satisfies $x \equiv x_1^2 - x_2^2 \pmod 4$, for some $x_1, x_2 \in \mathbf{Z}$. It is well known that $x_1^2 - x_2^2 \not\equiv 2 \pmod 4$, so an element of the form $4x + 2 + y\alpha + z\beta$, $x, y, z \in \mathbf{Z}$ cannot be written as a difference of two squares in the ring $\mathbf{Z}[1, \alpha, \beta]$.

The first result on the non-existence of some Diophantine quadruples in the pure cubic fields is given by the following lemma:

Lemma 3.1. *Let $w = x + y\alpha + z\beta \in \mathbf{Z}[1, \alpha, \beta]$ denote an element of the ring of integers in a pure cubic field $\mathbf{Q}(\sqrt[3]{d})$.*
(i) If a is even and y is odd, then there is no $D(w)$–quadruple in $\mathbf{Z}[1, \alpha, \beta]$.
(ii) If b is even and z is odd, then there is no $D(w)$–quadruple in $\mathbf{Z}[1, \alpha, \beta]$.

Proof. We comment only the part (i) because (ii) can be proved in a completely analogous way.

Suppose, on the contrary, that the set $\{x_i + y_i\alpha + z_i\beta : i = 1, 2, 3, 4\}$ is a $D(w)$–quadruple in $\mathbf{Z}[1, \alpha, \beta]$.

Using the formulas obtained in the previous section, we deduce that for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, there exist integers c_{ij}, d_{ij}, e_{ij} such that

$$x_i y_j + x_j y_i + a z_i z_j + y = 2c_{ij}d_{ij} + a e_{ij}^2.$$

Since a is even and y is odd, it follows that $x_i y_j + x_j y_i$ is an odd integer for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Now one can get a contradiction in the same way as in the proof of Proposition 1 in [11]. □

Observe that if a is even then (1) implies that $x + (2y + 1)\alpha + z\beta$, where $x, y, z \in \mathbf{Z}$, is not a difference of two squares in $\mathbf{Z}[1, \alpha, \beta]$. Also, if b is even then $x + y\alpha + (2z + 1)\beta$, where $x, y, z \in \mathbf{Z}$, is not a difference of two squares in $\mathbf{Z}[1, \alpha, \beta]$.

The following theorem provides a non–existence of $D(w)$–quadruples for a class of elements w that are not representable as a difference of squares in $\mathbf{Z}[1, \alpha, \beta]$.

Theorem 3.2. *Let $w = 4x + 2 + y\alpha + z\beta$, $x, y, z \in \mathbf{Z}$, denote an element of the ring of integers $\mathbf{Z}[1, \alpha, \beta]$ in a pure cubic field $\mathbf{Q}(\sqrt[3]{d})$, where d is even. Then there exist no $D(w)$–quadruples in $\mathbf{Z}[1, \alpha, \beta]$.*

Proof. Suppose that the set $\{x_i + y_i\alpha + z_i\beta : i = 1, 2, 3, 4\}$ has the property $D(4x + 2 + y\alpha + z\beta)$. Then for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, there exist $c_{ij}, d_{ij}, e_{ij} \in \mathbf{Z}$ such that the following equalities hold:

$$x_i x_j + ab(y_i z_j + y_j z_i) + 4x + 2 = c_{ij}^2 + 2ab d_{ij} e_{ij}$$
$$x_i y_j + x_j y_i + a z_i z_j + y = 2c_{ij} d_{ij} + a e_{ij}^2$$
$$x_i z_j + x_j z_i + b y_i y_j + z = 2c_{ij} e_{ij} + b d_{ij}^2.$$

Exactly one of the numbers a and b is even, and let us first assume that a is even. The previous lemma implies that y is even and consequently $x_i y_j + x_j y_i$ is even for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

On the other hand, since ab is even and square-free, from the first equality we deduce that for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, one of the following holds:

$$\text{(3)} \qquad \begin{aligned} x_i x_j + 2(y_i z_j + y_j z_i) &\equiv 2 \pmod 4, \\ x_i x_j + 2(y_i z_j + y_j z_i) &\equiv 3 \pmod 4. \end{aligned}$$

We will show that there do not exist integers x_i, y_i, z_i, $i = 1, 2, 3, 4$, such that the above conditions are fulfilled.

There are several cases to discuss:

- Suppose that x_i is even for $i = 1, 2, 3, 4$.

It follows that $2(y_i z_j + y_j z_i) \equiv 2 \pmod 4$ for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, and consequently $y_i z_j + y_j z_i$ is odd for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. It can be seen that this is impossible in the same way as in the proof of the previous lemma.

- Suppose that there exist $i \in \{1, 2, 3, 4\}$ such that x_i is odd and $j, k \in \{1, 2, 3, 4\}$, $j \neq k$, such that x_j and x_k are even.

We may assume that x_1 is odd and x_2, x_3 are even. Then, $x_1 y_i + x_i y_1 \equiv 0 \pmod 2$, $i \in \{2, 3\}$, implies that both y_2 and y_3 are even and hence, $x_2 x_3 + 2(y_2 z_3 + y_3 z_2) \equiv 0 \pmod 4$, a contradiction.

- Suppose that there exists at most one $i \in \{1, 2, 3, 4\}$ such that x_i is even.

We may assume that x_1, x_2, x_3 are odd. Since $x_i y_j + x_j y_i \equiv 0 \pmod 2$ for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, we obtain either $y_i \equiv 0 \pmod 2$ for $i \in \{1, 2, 3\}$ or $y_i \equiv 1 \pmod 2$ for $i \in \{1, 2, 3\}$.

Let us first assume $y_i \equiv 0 \pmod 2$ for $i \in \{1, 2, 3\}$. It follows directly from (3) that $x_i x_j \equiv$

$3 \pmod 4$ holds for $i,j \in \{1,2,3\}$, $i \neq j$. On the other hand, there exist $i,j \in \{1,2,3\}$, $i \neq j$ such that $x_i \equiv x_j \pmod 4$, leading to $x_ix_j \equiv 1 \pmod 4$, which is impossible.

Now we assume $y_i \equiv 1 \pmod 2$ for $i \in \{1,2,3\}$. If $z_1 \equiv z_2 \equiv z_3 \pmod 2$, then we again obtain from (3) that $x_ix_j \equiv 3 \pmod 4$ holds for $i,j \in \{1,2,3\}$, $i \neq j$, a contradiction.

It remains to consider the case $z_i \equiv z_j \not\equiv z_k \pmod 2$, where $\{i,j,k\} = \{1,2,3\}$. Using (3), we get $x_ix_j \equiv 3 \pmod 4$ and $x_ix_k \equiv x_jx_k \equiv 1 \pmod 4$. The last congruences yield $x_i \equiv x_j \pmod 4$ and, in consequence, $x_ix_j \equiv 1 \pmod 4$, which is impossible.

Let us now assume that b is even. By the previous lemma, z is even and it directly follows that $x_iz_j + x_jz_i$ is even for $i,j \in \{1,2,3,4\}$, $i \neq j$. Also, in the same way as before we obtain that for all $i,j \in \{1,2,3,4\}$, $i \neq j$, one of (3) holds. Using the same procedure, it can be concluded that there is no solution. This completes the proof. □

In the rest of this section we study the remaining case $a^2 \equiv b^2 \pmod 9$. If a is even, it is a direct consequence of the formula (2) that an element of the form $(2x+1)\alpha + y\beta + z\gamma \in \mathbf{Z}[\alpha,\beta,\gamma]$, where $x \in \mathbf{Z}$, is not representable as a difference of two squares in the ring $\mathbf{Z}[\alpha,\beta,\gamma]$. On the other hand, if b is even then the same can be concluded for an element of the form $x\alpha + (2y+1)\beta + z\gamma \in \mathbf{Z}[\alpha,\beta,\gamma]$, where $y \in \mathbf{Z}$.

The following lemma is an analogue of Lemma 3.1 for the pure cubic fields of a different type.

Lemma 3.3. *Let $w = x\alpha + y\beta + z\gamma \in \mathbf{Z}[\alpha,\beta,\gamma]$ denote an element of the ring of integers in a pure cubic field $\mathbf{Q}(\sqrt[3]{d})$, with even $d = ab^2$, where ab is square-free and $a^2 \equiv b^2 \pmod 9$.*
(i) If a is even and x is odd, then there is no $D(w)$–quadruple in $\mathbf{Z}[\alpha,\beta,\gamma]$.
(ii) If b is even and y is odd, then there is no $D(w)$–quadruple in $\mathbf{Z}[\alpha,\beta,\gamma]$.

Proof. We will again comment only the part (i). Suppose the assertion of the lemma is false. Then there exists a set $\{x_i\alpha + y_i\beta + z_i\gamma : i = 1,2,3,4\}$ which is a $D(w)$–quadruple in $\mathbf{Z}[\alpha,\beta,\gamma]$. Using formulas obtained in the previous section, it may be concluded that there exist integers c_{ij}, d_{ij}, e_{ij} such that equality

$$\begin{aligned}
&ay_iy_j - a^2bx_iy_j - a^2bx_jy_i - \frac{a^2b^2-1}{3}x_iz_j \\
&\quad - \frac{a^2b^2-1}{3}x_jz_i - ab\frac{a^2-1}{3}y_iz_j \\
&\quad - ab\frac{a^2-1}{3}y_jz_i + \frac{a(b^2+1-2a^2b^2)}{9}z_iz_j + x = \\
&ad_{ij}^2 - 2a^2bc_{ij}d_{ij} - 2\frac{a^2b^2-1}{3}c_{ij}e_{ij} - 2ab\frac{a^2-1}{3}d_{ij}e_{ij} \\
&\quad + \frac{a(b^2+1-2a^2b^2)}{9}e_{ij}^2
\end{aligned}$$

holds for $i,j \in \{1,2,3,4\}$, $i \neq j$.

Since a is even and x is odd, we conclude that $x_iz_j + x_jz_i$ has to be an odd integer for $i,j \in \{1,2,3,4\}$, $i \neq j$, and it can be seen in the same way as in the proof of Lemma 3.1 that this is impossible. □

It is not hard to see, using the Chinese remainder theorem, that $1 + 2a^2b^2 \equiv 9 \pmod{24}$ holds and consequently $\frac{1+2a^2b^2}{3} \equiv 3 \pmod 4$. Since ab is even, formula (2) shows that element $x\alpha + y\beta + (4z+2)\gamma \in \mathbf{Z}[\alpha,\beta,\gamma]$, for $x,y,z \in \mathbf{Z}$, cannot be represented as a difference of two squares in $\mathbf{Z}[\alpha,\beta,\gamma]$ (note that $\frac{1+2a^2b^2}{3}(z_1^2 - z_2^2) \equiv 0,\ 1$ or $3 \pmod 4$).

The non–existence of $D(w)$–quadruples for such w is established by the following theorem:

Theorem 3.4. *Let $w = x\alpha + y\beta + (4z+2)\gamma$, $x,y,z \in \mathbf{Z}$, denote an element of the ring of integers $\mathbf{Z}[\alpha,\beta,\gamma]$ in a pure cubic field $\mathbf{Q}(\sqrt[3]{d})$, where d is even. Then there exist no $D(w)$–quadruples in $\mathbf{Z}[\alpha,\beta,\gamma]$.*

Proof. Let us first consider the case when a is even. Suppose that the set $\{x_i\alpha + y_i\beta + z_i\gamma : i = 1,2,3,4\}$ has the property $D(x\alpha + y\beta + (4z+2)\gamma)$.

Lemma 3.3 shows that x is also even, and that $x_iz_j + x_jz_i$ is even for all $i,j \in \{1,2,3,4\}$, $i \neq j$.

Further, for all $i,j \in \{1,2,3,4\}$, $i \neq j$, there exist $c_{ij}, d_{ij}, e_{ij} \in \mathbf{Z}$ such that the following equality holds:

$$\begin{aligned}
&3ab(x_iy_j + x_jy_i) + ab^2(x_iz_j + x_jz_i) + a^2b(y_iz_j + y_jz_i) \\
&\quad + \frac{1+2a^2b^2}{3}z_iz_j + 4z + 2 \\
&= 6abc_{ij}d_{ij} + 2ab^2c_{ij}e_{ij} + 2a^2bd_{ij}e_{ij} + \frac{1+2a^2b^2}{3}e_{ij}^2.
\end{aligned}$$

Since both a and $x_iz_j + x_jz_i$ are even, we obtain the following congruence:

$$2(x_iy_j + x_jy_i) + 3z_iz_j + 2 \equiv 3e_{ij}^2 \pmod 4.$$

Multiplying both sides of the previous congruence by 3, we have

$$z_iz_j + 2(x_iy_j + x_jy_i) + 2 \equiv e_{ij}^2 \pmod 4,$$

which shows that one of the following holds:

$$z_iz_j + 2(x_iy_j + x_jy_i) \equiv 2 \pmod 4,$$
$$z_iz_j + 2(x_iy_j + x_jy_i) \equiv 3 \pmod 4$$

for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

These congruences are obtained just by replacing x_i, y_i, z_i in (3) with z_i, x_i, y_i, respectively. Thus, the proof of Theorem 3.2 immediately leads us to a contradiction.

The assumption that b is even leads to $y_iz_j + y_jz_i \equiv 0 \pmod 2$, for $i \neq j$, and to the fact that one of the following

$$z_iz_j + 2(x_iy_j + x_jy_i) \equiv 2 \pmod 4,$$
$$z_iz_j + 2(x_iy_j + x_jy_i) \equiv 3 \pmod 4$$

holds for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Clearly, the rest of the proof follows in the same way as in the previous case. □

We also note the following corollary, which can be regarded as a generalization of the result due to Brown *et al.*

Corollary 3.5. *If n and d are even integers with $n \equiv 2 \pmod 4$ and d cube-free, then there exist no $D(n)$–quadruples in the ring of integers in the field $\mathbf{Q}(\sqrt[3]{d})$.*

Proof. We again write $d = ab^2$ where ab is square-free and $(a, b) = 1$. Further, let $n = 4k + 2$, for some $k \in \mathbf{Z}$. If $a^2 \not\equiv b^2 \pmod 9$, claim of the corollary follows directly from Theorem 3.2.

Otherwise, we have $4k + 2 = x\alpha + y\beta + z\gamma$ for some $x, y, z \in \mathbf{Z}$. It is not hard to see that $z = 3(4k + 2) = 4(3k + 1) + 2$ and Theorem 3.4 completes the proof. □

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