On the sum of powers of two consecutive Fibonacci numbers

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Abstract: Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. In this note, we prove that if s is an integer number such that $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large integer n, then $s = 1$ or 2.

Key words: Fibonacci numbers; linear forms in logarithms.

1. Introduction. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

The Fibonacci numbers are well-known for possessing wonderful and amazing properties (consult [5, pp. 53–56] and [2] together with their very extensive annotated bibliography for additional references and history). In 1963, the Fibonacci Association was created to provide enthusiasts an opportunity to share ideas about these intriguing numbers and their applications. Also, in the issues of *The Fibonacci Quarterly* we can find many new facts, applications, and relationships about Fibonacci numbers. Some well-known properties of this sequence can be proved by using elementary techniques, but several stronger results have been proved by using refined tools in number theory, as for instance, the problem of showing that the only perfect powers in that sequence are 0, 1, 8 and 144, see [1] and its generalization, see [4].

Among the several pretty algebraic identities involving Fibonacci numbers, we are interested in the following one

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \text{ for all } n \geq 0.$$

In particular, this naive identity (which can be proved easily by induction) tell us that the sum of the square of two consecutive Fibonacci numbers is still a Fibonacci number. So, natural questions arise: Does the same property hold for $F_n^3 + F_{n+1}^3$? And for $F_n^4 + F_{n+1}^4$? And so on?

The aim of this paper is to determine when the sum $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large integer n. Our main result is the following

Theorem 1. *Let s be a positive integer. If $F_n^s + F_{n+1}^s$ is a Fibonacci number for all sufficiently large n, then $s = 1$ or 2.*

Let us describe in a few words our strategy to prove Theorem 1. First, we write $(\sqrt{5})^s(F_n^s + F_{n+1}^s)$ as $P(\alpha) + Q(\alpha)$, where $P, Q \in \mathbf{Z}[x]$ are polynomials with degree ns and $(n+1)s$, respectively and $\alpha = (1+\sqrt{5})/2 = 1.61803\ldots$. The main trick is to divide the previous sum by α^{ns} and to see that the result will tend to $(\sqrt{5})^{-s}(1+\alpha^s)$ when n tends to infinity. Therefore, if $F_n^s + F_{n+1}^s$ is a Fibonacci number for n sufficiently large, say F_{ℓ_n}, then we can find a Diophantine equation satisfied by s and an integer number t (which will depend on ℓ_n and s). Finally, we use linear forms in logarithms of two algebraic numbers to prove that the only solutions for a such equation are $s = 1$ or 2.

2. The proof of the Theorem. According to the Binet's formula, for $n \geq 1$

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}.$$

Thus the Binomial Theorem yields

$$(\sqrt{5})^s \cdot \left(\frac{F_n^s + F_{n+1}^s}{\alpha^{ns}}\right) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{k(n+1)} \alpha^{-2kn} + \sum_{k=0}^{s} \binom{s}{k} (-1)^{k(n+2)} \alpha^{s-2k(n+1)}.$$

Since

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$$\lim_{n\to\infty}\sum_{k=0}^{s}\binom{s}{k}(-1)^{k(n+1)}\alpha^{-2kn}=1$$

and

$$\lim_{n\to\infty}\sum_{k=0}^{s}\binom{s}{k}(-1)^{k(n+2)}\alpha^{s-2k(n+1)}=\alpha^{s},$$

we have

$$\lim_{n\to\infty}\frac{F_n^s+F_{n+1}^s}{\alpha^{ns}}=\frac{1+\alpha^s}{(\sqrt{5})^s}.$$

On the other hand, if we suppose the existence of $N_0>0$ and a subsequence $(\ell_n)_{n\geq 0}\subseteq\mathbf{N}$ such that $F_n^s+F_{n+1}^s=F_{\ell_n}$, for all $n\geq N_0$, then

$$\begin{aligned}(2.1)\qquad \frac{1+\alpha^s}{(\sqrt{5})^s}&=\lim_{n\to\infty}\frac{F_{\ell_n}}{\alpha^{ns}}\\&=\lim_{n\to\infty}\frac{\alpha^{\ell_n-ns}-(-1)^{\ell_n}\alpha^{-\ell_n-ns}}{\sqrt{5}}\\&=\lim_{n\to\infty}\frac{\alpha^{\ell_n-ns}}{\sqrt{5}}.\end{aligned}$$

Since ℓ_n-ns is an integer and $|\alpha|>1$, we have that ℓ_n-ns must be constant with respect to n, say t, for n sufficiently large. Thus (2.1) yields

$$(2.2)\qquad (\sqrt{5})^{s-1}\alpha^t=1+\alpha^s.$$

Of course, this equality is valid for $(t,s)\in\{(1,2),(2,1)\}$. Our goal is to prove that (2.2) is not true if $s>2$. In this case, $2\alpha^s>1+\alpha^s=(\sqrt{5})^{s-1}\alpha^t>2\alpha^t$ and so $s>t$. Note that (2.2) can be rewritten into the form $(\sqrt{5})^{s-1}\alpha^{t-s}-1=\alpha^{-s}$. Put

$$(2.3)\qquad \Lambda=(s-1)\log\sqrt{5}-(s-t)\log\alpha.$$

Then $e^{\Lambda}-1=\alpha^{-s}>0$, which implies $\Lambda>0$. Therefore, $\Lambda<e^{\Lambda}-1=\alpha^{-s}$ and thus

$$(2.4)\qquad \log\Lambda<-s\log\alpha.$$

Now, we will determine a lower bound for the linear form in logarithms (2.3) *à la* Baker. So we choose to use a result due to Laurent [3, Corollary 2] with $m=22$ and $C_2=19.2$. First let us introduce some notations. Let α_1,α_2 be real algebraic numbers, with $|\alpha_i|\geq 1$, b_1,b_2 be positive integer numbers and

$$\Lambda=b_2\log\alpha_2-b_1\log\alpha_1.$$

As usual, the *logarithmic height* of an n-degree algebraic number α is defined as

$$h(\alpha)=\frac{1}{n}\left(\log|a|+\sum_{j=1}^{n}\log\max\{1,|\alpha^{(j)}|\}\right),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbf{Z}$) and $(\alpha^{(j)})_{1\leq j\leq n}$ are the conjugates of α. Let A_i be real numbers such that

$$\log A_i\geq\max\{h(\alpha_i),|\log\alpha_i|/D,1/D\},\ i\in\{1,2\},$$

where D is the degree of the number field $\mathbf{Q}(\alpha_1,\alpha_2)$ over $\mathbf{Q}$. Define

$$b'=\frac{b_1}{D\log A_2}+\frac{b_2}{D\log A_1}.$$

Laurent's result asserts that if $\Lambda\neq 0$, then

$$\log|\Lambda|\geq -19.2\cdot D^4(\max\{\log b'+0.38,m/D,1\})^2\times\log A_1\log A_2.$$

We take

$$D=2,\ b_1=s-t,\ b_2=s-1,\ \alpha_1=\alpha,\ \alpha_2=\sqrt{5}.$$

We can choose $\log A_1=0.25$ and $\log A_2=1$. So we get

$$b'=\frac{s-t}{2}+\frac{s-1}{0.5}\leq\frac{5(s-1)}{2}.$$

As $\Lambda\neq 0$, by Corollary 2 of [3] we get

$$(2.5)\ \log|\Lambda|\geq -76.8\cdot(\max\{\log(5(s-1)/2)+0.38,11\})^2.$$

Now, we combine the estimates (2.4) and (2.5) for yielding

$$(2.6)\ \ s\log\alpha<76.8\cdot\left(\max\{\log(5(s-1)/2)+0.38,11\}\right)^2.$$

- If $s\leq 16379$, then $\log(5(s-1)/2)+0.38\leq 11$. Therefore, inequality (2.6) gives $s\leq 19311$.
- If $s>16379$, then inequality (2.6) becomes

$$s\log\alpha<76.8\cdot(\log(5(s-1)/2)+0.38)^2.$$

We deduce that $s\leq 20022$.

For the remaining possibilities, we define the function $\mathcal{T}:\mathbf{N}\to\mathbf{R}$ by

$$\mathcal{T}(s):=\frac{\log\left(\frac{1+\alpha^s}{(\sqrt{5})^{s-1}}\right)}{\log\alpha}.$$

Thus in view of the relation in (2.2), if s satisfies the hypothesis of the theorem, then $\mathcal{T}(s)$ must be an integer. To finish, we use Mathematica to compute all values of this function, for $3\leq s\leq 20022$. We see that $\mathcal{T}(s)$ is never an integer in this range. This completes the proof of Theorem 1.

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