Non-symplectic automorphisms of 3-power order on $K3$ surfaces

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(Communicated by Shigefumi MORI, M.J.A., Sept. 13, 2010)

Abstract: We classify non-symplectic automorphisms of 3-power order on algebraic $K3$ surfaces which act trivially on the Néron-Severi lattice, i.e., we describe their fixed locus. Moreover we give Weierstrass equations of $K3$ surfaces with a non-symplectic automorphism of 3-power order.

Key words: $K3$ surface; non-symplectic automorphism.

1. Introduction. Let X be a smooth compact complex surface. If its canonical line bundle K_X is trivial and $\dim H^1(X, \mathcal{O}_X) = 0$ then X is called a $K3$ *surface*. In the following, for an algebraic $K3$ surface X, we denote by S_X, T_X and ω_X the Néron-Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form on X, respectively.

An automorphism of X is *symplectic* if it acts trivially on $\mathbf{C}\omega_X$. In particular, this paper is devoted to study of *non*-symplectic automorphisms of 3-power order which act trivially on S_X.

We suppose that g is a non-symplectic automorphism of order I on X such that $g^*\omega_X = \zeta_I \omega_X$ where ζ_I is a primitive I-th root of unity. Then g^* has no non-zero fixed vectors in $T_X \otimes \mathbf{Q}$ and hence $\phi(I)$ divides $\operatorname{rank} T_X$, where ϕ is the Euler function. In particular $\phi(I) \le \operatorname{rank} T_X$ and hence $I \le 66$ ([Ni1], Theorem 3.1 and Corollary 3.2).

Non-symplectic automorphisms have been studied by several authors e.g. Nikulin [Ni1,Ni2], Vorontsov [Vo], Kondo [Ko], Xiao [Xi], Oguiso, Zhang [OZ1,OZ2], Artebani, Sarti [AS] and Taki [Ta]. Recently, we have the classification of non-symplectic automorphisms of prime order on $K3$ surfaces [AST]. In particular we characterize their fixed loci in terms of the invariants of p-elementary lattices. Then Schütt [Sc] classified $K3$ surfaces with non-symplectic automorphisms which the order is 2-power and equals the rank of the transcendental lattice.

We know the following

2010 Mathematics Subject Classification. Primary 14J28; Secondary 14J50.

doi: 10.3792/pjaa.86.125

Proposition 1.1 [Vo,Ko]. *Let k be a positive integer. Assume that there exists a non-symplectic automorphism φ of order p^k on X which acts trivially on S_X. Then S_X is a p-elementary lattice, that is, S_X^*/S_X is a p-elementary group where $S_X^* = \operatorname{Hom}(S_X, \mathbf{Z})$.*

In general, the inverse of Proposition 1.1 is not true. For example, $S_X = U(3) \oplus E_8(3)$ is a 3-elementary lattice. But X has no non-symplectic automorphisms of order 3 which act trivially on S_X. (See [AS,Ta].)

If I is 3-power then $I = 3, 9, 27$. Non-symplectic automorphisms of order 3 have been classified by Artebani, Sarti [AS] and Taki [Ta]. They proved the following

Theorem 1.2 [AS,Ta]. *Let r be the Picard number of X and let s be the minimal number of generators of S_X^*/S_X.*

X has a non-symplectic automorphism φ of order 3 which acts trivially on S_X if and only if $22 - r - 2s \ge 0$. Moreover the fixed locus $X^\varphi := \{x \in X | \varphi(x) = x\}$ has the form

$$X^\varphi = \begin{cases} \{P_1, P_2, P_3\} & \text{if } S_X = U(3) \oplus E_6^*(3) \\ \{P_1, \dots, P_M\} \amalg C^{(g)} \amalg E_1 \amalg \cdots \amalg E_K & \text{otherwise} \end{cases}$$

and $M = r/2 - 1$, $g = (22 - r - 2s)/4$, $K = (2 + r - 2s)/4$, where we denote by P_i an isolated point, $C^{(g)}$ a non-singular curve of genus g and by E_j a non-singular rational curve.

Oguiso and Zhang [OZ1] have proved that the $K3$ surface with non-symplectic automorphisms of order 27 is unique. Then we mainly study non-symplectic automorphisms of order 9.

And the main purpose of this paper is to prove the following theorem.

Theorem 1.3.

(1) *X has a non-symplectic automorphism φ of order 9 acting trivially on S_X if and only if* $S_X = U \oplus A_2$, $U \oplus E_8$, $U \oplus E_6 \oplus A_2$ *or* $U \oplus E_8 \oplus E_6$. *Moreover the fixed locus X^φ has the form*

$$X^\varphi = \begin{cases} \{P_1, P_2, \dots, P_6\} & \text{if } S_X = U \oplus A_2, \\ \{P_1, P_2, \dots, P_{10}\} \amalg E_1 & \\ \quad \text{if } S_X = U \oplus E_8 \text{ or } U \oplus E_6 \oplus A_2, & \\ \{P_1, P_2, \dots, P_{14}\} \amalg E_1 \amalg E_2 & \\ \quad \text{if } S_X = U \oplus E_8 \oplus E_6. & \end{cases}$$

(2) *X has a non-symplectic automorphism φ of order 27 acting trivially on S_X if and only if $S_X = U \oplus A_2$. Moreover the fixed locus X^φ has the form $X^\varphi = \{P_1, P_2, \dots, P_6\}$.*

Here we denote by P_i an isolated point and by E_j a non-singular rational curve.

Remark 1.4. We have already had the classification of non-symplectic automorphisms of 5-power order on $K3$ surfaces. If I is 5-power then $I = 5, 25$. Non-symplectic automorphisms of order 5 have been classified by Artebani, Sarti and Taki [AST]. Oguiso and Zhang [OZ1] have proved that the $K3$ surface with non-symplectic automorphisms of order 25 is unique.

In Section 2, we shall give the classification of an even hyperbolic 3-elementary lattices admitting a primitive embedding in $K3$ lattice. As a result, we get all lattices which are the Néron-Severi lattice of $K3$ surfaces with non-symplectic automorphisms of 3-power order which act trivially on S_X. In Section 3, we see that the number of isolated fixed points is determined by the Picard number of X. Here we use mainly the Lefschetz formula. In Section 4, we check that the existence and non-existence of $K3$ surfaces with a non-symplectic automorphism of 3-power order. And we give Weierstrass equations of $K3$ surfaces with a non-symplectic automorphism of 3-power order acting trivially on S_X. In Section 5, we see fixed locus of non-symplectic automorphisms.

2. The Néron-Severi and p-elementary lattices. A lattice L is a free abelian group of finite rank r equipped with a non-degenerate symmetric bilinear form, which will be denoted by $\langle\ ,\ \rangle$. The bilinear form $\langle\ ,\ \rangle$ determines a canonical embedding $L \subset L^* = \mathrm{Hom}(L, \mathbf{Z})$. We denote by A_L the factor group L^*/L which is a finite abelian group. $L(m)$ is the lattice whose bilinear form is the one on L multiplied by m.

We denote by U the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature $(1,1)$, and by A_m or E_n an even negative definite lattice associated with the Dynkin diagram of type A_m or E_n ($m \geq 1$, $n = 6,7,8$).

Let p be a prime number. A lattice L is called *p-elementary* if $A_L \simeq (\mathbf{Z}/p\mathbf{Z})^s$, where s is the minimal number of generator of A_L. For a p-elementary lattice we always have the inequality $s \leq r$, since $|L^*/L| = p^s$, $|L^*/pL^*| = p^r$ and $pL^* \subset L \subset L^*$.

Example 2.1. For all prime p, lattices E_8, $E_8(p)$, U and $U(p)$ are p-elementary. A_2 and E_6 are 3-elementary.

Even indefinite $p(>2)$-elementary lattices were classified as follows:

Theorem 2.2 [RS]**.** *An even indefinite p-elementary lattice of rank n for $p \neq 2$ and $n \geq 2$ is uniquely determined by its discriminant (i.e., the number s).*

For $p \neq 2$ a hyperbolic lattice corresponding to a given value of $s \leq n$ exist if and only if the following conditions are satisfied: $n \equiv 0 \pmod 2$ *and*

$$\begin{cases} \text{for } s \equiv 0 \pmod 2, & n \equiv 2 \pmod 4, \\ \text{for } s \equiv 1 \pmod 2, & p \equiv (-1)^{n/2-1} \pmod 4. \end{cases}$$

And moreover $n > s > 0$, if $n \not\equiv 2 \pmod 8$.

Let ϕ be the Euler function. Then $\phi(9) = 6$. Since $\phi(9)$ divides $\mathrm{rank}\, T_X$, $\mathrm{rank}\, T_X = 18,\ 12,\ 6$. (see Section 1 and [Ni1].) Hence if X has a non-symplectic automorphisms of order 9 then $\mathrm{rank}\, S_X = 4,\ 10,\ 16$. In the same way, if X has a non-symplectic automorphisms of order 27 then $\mathrm{rank}\, S_X = 4$.

By Theorem 1.2, X has a non-symplectic automorphism φ of order 3 which acts trivially on S_X if and only if $22 - \mathrm{rank}\, S_X - 2s \geq 0$. Hence if X has a non-symplectic automorphism of order 3^k which act trivially on S_X then $22 - \mathrm{rank}\, S_X - 2s \geq 0$.

Table I is a list of 3-elementary lattices which satisfy $22 - \mathrm{rank}\, S_X - 2s \geq 0$ and $\mathrm{rank}\, S_X = 4,\ 10,\ 16$. Hence if X has a non-symplectic automorphisms of order 9 (resp. 27) which act trivially on S_X then S_X is one of the lattices in Table I (resp. $U \oplus A_2$ or $U(3) \oplus A_2$).

Remark 2.3. Let $\{e, f\}$ be a basis of U (resp. $U(3)$) with $\langle e, e\rangle = \langle f, f\rangle = 0$ and $\langle e, f\rangle = 1$ (resp. $\langle e, f\rangle = 3$). If necessary replacing e by $\varphi(e)$,

Table I. 3-elementary lattices

Rank S_X	s	S_X	T_X
4	1	$U \oplus A_2$	$U^{\oplus 2} \oplus E_6 \oplus E_8$
4	3	$U(3) \oplus A_2$	$U \oplus U(3) \oplus E_6 \oplus E_8$
10	0	$U \oplus E_8$	$U^{\oplus 2} \oplus E_8$
10	2	$U \oplus E_6 \oplus A_2$	$U \oplus U(3) \oplus E_8$
10	4	$U \oplus A_2^{\oplus 4}$	$U \oplus U(3) \oplus E_6 \oplus A_2$
10	6	$U(3) \oplus A_2^{\oplus 4}$	$A_2(-1) \oplus A_2^{\oplus 5}$
16	1	$U \oplus E_8 \oplus E_6$	$U^{\oplus 2} \oplus A_2$
16	3	$U \oplus E_8 \oplus A_2^{\oplus 3}$	$A_2(-1) \oplus A_2^{\oplus 2}$

where φ is a composition of reflections induced from non-singular rational curves on X, we may assume that e is represented by the class of an elliptic curve F and the linear system $|F|$ defines an elliptic fibration $\pi : X \to \mathbf{P}^1$. Note that π has a section $f - e$ in case U. In case $U(3)$, there are no (-2)-vectors r with $\langle r, e\rangle = 1$, and hence π has no sections.

It follows from Remark 2.3 and Table I that X has an elliptic fibration $\pi : X \to \mathbf{P}^1$. In the following, we fix such an elliptic fibration.

The following lemma follows from [PS, §3 Corollary 3] and the classification of singular fibers of elliptic fibrations [Kd].

Lemma 2.4. *Assume that $S_X = U(m) \oplus K_1 \oplus \cdots \oplus K_r$, where $m = 1$ or 3, and K_i is a lattice isomorphic to A_2, E_6 or E_8. Then π has a reducible singular fiber with corresponding Dynkin diagram K_i.*

3. The number of isolated fixed points. In this Section, we shall see that the number of isolated fixed points of non-symplectic automorphism of order 9.

Lemma 3.1. *Let X be an algebraic $K3$ surface and φ a non-symplectic automorphism of order 9 on X. Then we have:*

(1) *$\varphi^* \mid T_X \otimes \mathbf{C}$ can be diagonalized as:*

$$\begin{pmatrix} \zeta I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta^2 I_q & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^5 I_q & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^7 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta^8 I_q \end{pmatrix},$$

where I_q is the identity matrix of size q, ζ is a primitive 9-th root of unity.

(2) *Let P be an isolated fixed point of φ on X. Then φ^* can be written as*

$$\begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix} \quad (i + j \equiv 1 \mod 9)$$

under some appropriate local coordinates around P.

(3) *Let C be an irreducible curve in X^φ and Q a point on C. Then φ^* can be written as*

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

under some appropriate local coordinates around Q. In particular, fixed curves are non-singular.

Proof. (1) This follows form [Ni1, Theorem 3.1].

(2), (3) Since φ^* acts on $H^0(X, \Omega_X^2)$ as a multiplication by ζ, it acts on the tangent space of a fixed point as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix}$$

where $i + j \equiv 1 \pmod 9$. □

Thus the fixed locus of φ consists of disjoint union of non-singular curves and isolated points. Hence we can express the irreducible decomposition of X^φ as

$$X^\varphi = \{P_1, \ldots, P_M\} \amalg C_1 \amalg \cdots \amalg C_N,$$

where P_j is an isolated point and C_k is a non-singular curve.

Lemma 3.2. *Let r be the Picard number of X. Then $\chi(X^\varphi) = r + 2$.*

Proof. We apply the topological Lefschetz formula:

$$\chi(X^\varphi) = \sum_{i=0}^{4} (-1)^i \operatorname{tr}(\varphi^* | H^i(X, \mathbf{R})).$$

Since φ^* acts trivially on S_X, $\operatorname{tr}(\varphi^*|S_X) = r$. By Lemma 3.1 (1), $\operatorname{tr}(\varphi^*|T_X) = q(\zeta + \zeta^2 + \zeta^4 + \zeta^5 + \zeta^7 + \zeta^8) = -q(1 + \zeta^3 + \zeta^6) = 0$. Hence we can calculate the right-hand side of the Lefschetz formula as follows: $\sum_{i=0}^{4} (-1)^i \operatorname{tr}(\varphi^*|H^i(X, \mathbf{R})) = 1 - 0 + \operatorname{tr}(\varphi^*|S_X) + \operatorname{tr}(\varphi^*|T_X) - 0 + 1 = r + 2$. □

By Table I and Lemma 2.4, the elliptic fibration $\pi : X \to \mathbf{P}^1$ has a reducible singular fiber. In the following, we check a detail of Theorem 1.2.

Lemma 3.3. *We put $\sigma = \varphi^3$. All isolated fixed points of σ lie on reducible singular fibers. In particular, these are intersection points of compo-*

nents of reducible singular fibers or a point of the component of a singular fiber of type II^* *which is multiplicity* 3 *and meet the component with multiplicity* 6.

Proof. Since σ also acts trivially on S_X, σ preserves reducible singular fibers. Hence intersection points of components of reducible singular fibers are fixed by σ. We will check the claim for each S_X individually.

Assume $S_X = U \oplus A_2$. By [Ta, Lemma 3.5] π has a singular fiber of type IV. By Theorem 1.2, $X^\sigma = C^{(4)} \amalg \mathbf{P}^1 \amalg \{P_1\}$. Now X^σ contains $C^{(4)}$. This implies that the automorphism σ acts trivially on the base of π and the section (cf. Remark 2.3) is fixed by σ. Since an automorphism of order 3 on a smooth fiber has 3 fixed points, $C^{(4)}.F = 2$ where F is a fiber of π. Thus $C^{(4)}$ does not pass the intersection point. Hence a singular fiber of type IV has exactly one isolated fixed point P_1 at the intersection point of the three components of the singular fiber. This settles Lemma 3.3 in the case $S_X = U \oplus A_2$.

Assume $S_X = U \oplus E_8$. By Theorem 1.2, $X^\sigma = C^{(3)} \amalg \coprod_{i=1}^{3} \mathbf{P}^1_i \amalg \coprod_{j=1}^{4}\{P_j\}$. Note π has a singular fibers of type II^*. The component D_6 with multiplicity 6 is pointwisely fixed by σ. Since X^σ contains $C^{(3)}$, σ acts trivially on the base of π, the section (cf. Remark 2.3) is fixed by σ, and $C^{(3)}$ is a double section, that is, $C^{(3)}.F = 2$ where F is a fiber of π.

If F is a singular fiber of type II^* then $C^{(3)}$ meets the component with multiplicity 2 which meets the component with multiplicity 4. Indeed, if $C^{(3)}$ meets another component D of F with multiplicity ≤ 2 then it is easy to see that D has three or more fixed points. Hence $C^{(3)}$ meets another pointwisely fixed curve D, a contradiction.

Therefore σ fixes the 5 intersection points $Q_1, \cdots, Q_5$ of $F \setminus D_6$ and a point Q_6 of the component with multiplicity 3 which meets D_6. Since X^σ contains exactly 4 isolated points $P_1, \ldots, P_4$, F contains one pointwisely fixed component containing Q_i and Q_j $(\exists i, j \leq 5)$ and $\{P_1, \ldots, P_4\} = \{Q_k | k \neq i, j\}$. This settles Lemma 3.3 in the case $S_X = U \oplus E_8$.

In other cases we can check the claim by similar arguments. □

Corollary 3.4. *Let P be an isolated fixed point of φ^3. Then $\varphi(P) = P$.*

Proof. By Lemma 3.3 P is a special point on reducible singular fibers. Since φ preserves such a singular fiber, these points are fixed by φ. □

Proposition 3.5. *Let r be the Picard number of X. Then the number of isolated points M is* $(2r + 10)/3$.

Proof. First we calculate the holomorphic Lefschetz number $L(\varphi)$ in two ways as in [AS1, page 542] and [AS2, page 567]. That is

$$L(\varphi) = \sum_{i=0}^{2} \mathrm{tr}(\varphi^* | H^i(X, \mathcal{O}_X)),$$

$$L(\varphi) = \sum_{j=1, u+v=10, u \leq v}^{m_{u,v}} a(P_j^{u,v}) + \sum_{l=1}^{N} b(C_l),$$

where $P_j^{u,v}$ is an isolated point of type $\begin{pmatrix} \zeta^u & 0 \\ 0 & \zeta^v \end{pmatrix}$.

Here

$$\begin{aligned} a(P_j^{u,v}) &:= \frac{1}{\det(1 - \varphi^* | T_{P_j^{u,v}})} \\ &= \frac{1}{\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \zeta^u & 0 \\ 0 & \zeta^v \end{pmatrix}\right)} \\ &= \frac{1}{(1 - \zeta^u)(1 - \zeta^v)}, \\ b(C_l) &:= \frac{1 - g(C_l)}{1 - \zeta} - \frac{\zeta C_l^2}{(1 - \zeta)^2} \\ &= \frac{(1 + \zeta)(1 - g(C_l))}{(1 - \zeta)^2}, \end{aligned}$$

where T_{P_j} is the tangent space of X at P_j, $g(C_l)$ is the genus of C_l.

Using the Serre duality $H^2(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X(K_X))^\vee$, we calculate from the first formula that $L(\varphi) = 1 + \zeta^8$. From the second formula, we obtain

$$\begin{aligned} L(\varphi) = &\sum_{u+v=10, u \leq v} \frac{m_{u,v}}{(1 - \zeta^u)(1 - \zeta^v)} \\ &+ \sum_{l=1}^{N} \frac{(1 + \zeta)(1 - g(C_l))}{(1 - \zeta)^2}. \end{aligned}$$

Combing these two formulae, we have

$$(\sharp) \quad \begin{cases} 1 &= m_{2,8} - m_{3,7} + m_{4,6} - 2m_{5,5}, \\ 1 &= m_{3,7} - 2\sum_{l=1}^{N}(1 - g(C_l)), \\ 1 &= m_{2,8} + m_{5,5} - 3\sum_{l=1}^{N}(1 - g(C_l)), \\ 2 &= 2m_{2,8} - m_{3,7} + m_{4,6} - m_{5,5} \\ &\quad -3\sum_{l=1}^{N}(1 - g(C_l)). \end{cases}$$

We remark that $\varphi^3(P^{u,v})$ is a fixed point of a non-symplectic automorphism of order 3. Since $\begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix}^3 = \begin{pmatrix} \zeta^{3i} & 0 \\ 0 & \zeta^{3j} \end{pmatrix}$, $\varphi^3(P^{2,8})$ and $\varphi^3(P^{5,5})$ are isolated fixed points of φ^3. In the same way, $\varphi^3(P^{3,7})$ and $\varphi^3(P^{4,6})$ are points on a irreducible fixed curve of φ^3. By Corollary 3.4, isolated fixed points of φ^3 are $P^{2,8}$ or $P^{5,5}$. By Theorem 1.2, we have

$$m_{2,8} + m_{5,5} = r/2 - 1. \tag{1}$$

Next we apply the topological Lefschetz formula: $\chi(X^\varphi) = \sum_{i=0}^4 (-1)^i \operatorname{tr}(\varphi^*|H^i(X, \mathbf{R}))$. The left-hand side is

$$\chi(X^\varphi) = M + \sum_{l=1}^{N} (2 - 2g(C_l)). \tag{2}$$

By (♯), (1), (2) and Lemma 3.2, we have $M = (2r + 10)/3$. □

4. Existence. We show the existence of $K3$ surfaces with a non-symplectic automorphism of 3-power order acting trivially on S_X. To do this, we shall give examples of such $K3$ surfaces. In this Section, we denote by ζ_ν a primitive ν-th root of 1.

Example 4.1 [Ko, (7.7)]. ($S_X = U \oplus A_2$) $X_1 : y^2 = x^3 + t\prod_{k=1}^9 (t - \zeta_{27}^{3k})$, $\varphi_1(x, y, t) = (\zeta_{27}^2 x, \zeta_{27}^3 y, \zeta_{27}^6 t)$.

Since φ_1 is a non-symplectic automorphism of order 27, φ_1^3 is of order 9. Moreover X_1 has a singular fiber of type IV and 10 singular fibers of type II.

Example 4.2 [Ko, (3.2)]. ($S_X = U \oplus E_8$) $X_2 : y^2 = x^3 - t^5 \prod_{k=1}^6 (t - \zeta_6^k)$, $\varphi_2(x, y, t) = (\zeta_9^2 x, \zeta_9^3 y, \zeta_9^6 t)$.

X_2 has a singular fiber of type II* and 7 singular fibers of type II.

Example 4.3. ($S_X = U \oplus E_6 \oplus A_2$) $X_3 : y^2 = x^3 - t^4 \prod_{k=1}^6 (t - \zeta_6^k)$, $\varphi_3(x, y, t) = (\zeta_9 x, \zeta_9^6 y, \zeta_9^3 t)$.

X_3 has a singular fiber of type IV*, a singular fiber of type IV and 6 singular fibers of type II.

Example 4.4 [Ko, (7.8)]. ($S_X = U \oplus E_8 \oplus E_6$) $X_4 : y^2 = x^3 - t^5 \prod_{k=1}^3 (t - \zeta_9^{3k})$, $\varphi_4(x, y, t) = (\zeta_9^2 x, \zeta_9^3 y, \zeta_9^3 t)$.

X_4 has a singular fiber of type II*, a singular fiber of type IV* and 3 singular fibers of type II.

It is easy to give Néron-Severi lattice S_X of these examples by checking singular fibers (see also Lemma 2.4.). And each irreducible singular fiber has no symmetry, φ_i acts on S_X trivially.

In the following, we treat cases where X has no non-symplectic automorphisms of 3-power order. The following Proposition has been proved by Oguiso and Zhang.

Proposition 4.5 [OZ1, §2]. *Let φ be a non-symplectic automorphism of 3-power order. Let ϕ be the Euler function. Then there exists, modulo isomorphisms, a unique $K3$ surface X such that $\phi(\operatorname{ord}\varphi) = \operatorname{rank} T_X$.*

Therefore we have the uniqueness of $K3$ surfaces with a non-symplectic automorphism of order 27. In particular, if $S_X = U(3) \oplus A_2$ then X has no non-symplectic automorphisms of order 27 which act trivially on S_X. Similarly, there exists the uniqueness of $K3$ surface with a non-symplectic automorphism of order 9 and $\operatorname{rank} S_X = 16$. Hence by Example 4.4, if $S_X = U \oplus E_8 \oplus A_2^{\oplus 3}$ then X has no non-symplectic automorphisms of order 9 which act trivially on S_X.

In the following, we treat non-symplectic automorphisms order 9 with $\operatorname{rank} S_X = 4$, 10.

Proposition 4.6. *If $S_X = U \oplus A_2^{\oplus 4}$ or $U(3) \oplus A_2^{\oplus 4}$, then X has no non-symplectic automorphisms of order 9 which act trivially on S_X.*

Proof. We assume that $S_X = U \oplus A_2^{\oplus 4}$ and X has a non-symplectic automorphism φ of order 9 which acts trivially on S_X. Then φ induces an automorphism $\bar{\varphi}$ on $\mathbf{P}^1$. We see the order of $\bar{\varphi}$. A priori $\operatorname{ord}\bar{\varphi} = 1$, 3 or 9. If $\operatorname{ord}\bar{\varphi} = 1$ then a smooth fiber E is $\bar{\varphi}$-stable and $\bar{\varphi}^*_{|E}\omega_E = \zeta_9 \omega_E$. But there exists no such elliptic curve. If $\operatorname{ord}\bar{\varphi} = 9$ then since X has 4 reducible singular fibers of type IV or of type I$_3$, $\bar{\varphi}$ does not permute these fibers. Thus $\operatorname{ord}\bar{\varphi} = 3$.

We remark that $\bar{\varphi}$ has exactly 2 isolated fixed points Q_1 and Q_2. Hence $\bar{\varphi}$ permutes 3 reducible singular fibers, and fixes a reducible singular fiber over Q_1 and irreducible singular fiber over Q_2. Since reducible singular fibers which X has are of type IV or of type I$_3$, φ has at most 4 fixed points on a fiber over Q_1 and at most 2 fixed points on a fiber over Q_2. Therefore φ has at most 6 fixed point on X. But this is a contradiction by Proposition 3.5.

Similarly we can see the same assertion in the case of $S_X = U(3) \oplus A_2^{\oplus 4}$. □

By Theorem 1.2, if $S_X = U(3) \oplus A_2$ then X has a non-symplectic automorphism of order 3 which acts trivially on S_X. The following lemma follows from [AS, Proposition 4.9].

Lemma 4.7 [AS]. *Let X be a $K3$ surface with $S_X = U(3) \oplus A_2$ then X is isomorphic to a smooth quartic in $\mathbf{P}^3$ with equations of the form $X : F_4(x_0, x_1, x_2) + F_1(x_0, x_1, x_2)x_3^3 = 0$,*

$g(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, \zeta_3 x_3)$ *where* F_i *is a homogeneous polynomials of degree* i.

Proposition 4.8. *If* $S_X = U(3) \oplus A_2$ *then* X *has no non-symplectic automorphisms of order* 9 *which act trivially on* S_X.

Proof. Let φ be a non-symplectic automorphism of order 9 which acts trivially on S_X. By Lemma 4.7, $\varphi^3 = g$. Hence $\varphi(x_0, x_1, x_2, x_3) = \varphi(f(x_0, x_1, x_2), \zeta_9 x_3)$ where f is a non-trivial automorphism of order 3 on $\mathbf{P}^2$. Thus we can put $f(x_0, x_1, x_2) = (x_0, x_1, \zeta_9^3 x_2)$, $(x_0, \zeta_9^3 x_1, \zeta_9^3 x_2)$ or $(x_0, \zeta_9^3 x_1, \zeta_9^6 x_2)$.

Since φ preserves X, if $f(x_0, x_1, x_2) = (x_0, x_1, \zeta_9^3 x_2)$ and $F_1(x_0, x_1, x_2) = G_1(x_0, x_1)$ then $f(F_4(x_0, \ x_1, x_2)) = x_2 G_3(x_0, x_1)$ where G_i is a homogeneous polynomials of degree i. Therefor $X^\varphi = \{(0,0,0,1)\} \amalg \{(0,0,1,0)\} \amalg \{(G_3(x_0, x_1) = 0) \cap (x_2 = x_3 = 0)\}$, i.e. X^φ has 5 isolated fixed points. But these are contradictions by Proposition 3.5. Similarly if $F_1(x_0, x_1, x_2) = x_2$ then X^φ does not have exactly 6 isolated points. In the same way, a similar assertion holds in the other cases. □

5. Fixed locus of non-symplectic automorphisms. By Proposition 4.5, we have the uniqueness of $K3$ surfaces with a non-symplectic automorphism of order 27. And it is easy to see the fixed locus is exactly 6 isolated points. In this section, we see fixed locus of non-symplectic automorphisms of order 9.

Proposition 5.1. *Let* $S_X = U \oplus A_2$, $U \oplus E_8$, $U \oplus E_6 \oplus A_2$ *or* $U \oplus E_8 \oplus E_6$. *Then* X *has a non-symplectic automorphism* φ *of order* 9 *acting trivially on* S_X. *Moreover* X^φ *has the form*

$$X^\varphi = \begin{cases} \{P_1, P_2, \dots, P_6\} & \text{if } S_X = U \oplus A_2, \\ \{P_1, P_2, \dots, P_{10}\} \amalg E_1 & \\ \quad \text{if } S_X = U \oplus E_8 \text{ or } U \oplus E_6 \oplus A_2, & \\ \{P_1, P_2, \dots, P_{14}\} \amalg E_1 \amalg E_2 & \\ \quad \text{if } S_X = U \oplus E_8 \oplus E_6. & \end{cases}$$

Proof. We will check the claims for each S_X individually.

Assume $U \oplus E_6 \oplus A_2$. Is is easy to see φ does not act trivially on the base of π (see also proof of Proposition 4.6.). Thus X^φ does not contain a nonsingular curve with genus greater than 2. Note π has a singular fiber of type IV*. The component with multiplicity 3 of the singular fiber is pointwisely fixed by φ. By Proposition 3.2 and Proposition 3.5, we have $X^\varphi = \{P_1, P_2, \dots, P_{10}\} \amalg E_1$.

Similarly in other cases we can calculate fixed locus by the same argument of the example. These results satisfy the assertion. □

Therefore, we have proved Theorem 1.3.

Acknowledgments. I would like to express my gratitude to Prof. Shigeyuki Kondo for giving me much advice. I also thank Dr. Hisanori Ohashi for pointing out some mistakes and valuable advice. And I would like to thank to the referee for pointing out some mistakes and useful comments.

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