On the mean value of general Cochrane sum

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Abstract: In this paper, the general Cochrane sum is defined. The main purpose is to study the mean square value problem of general Cochrane sum by using the estimates for trigonometric sums and the properties of general Kloosterman sum, and finally give a sharp asymptotic formula.

Key words: Cochrane sum; Legendre's symbol; Kloosterman sum; Gauss sum.

1. Introduction. For a positive integer q and an arbitrary integer h, the Cochrane sum is defined as

$$C(h,q)=\sum_{a=1}^{q}{}'\left(\left(\frac{\bar{a}}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where $\sum'$ denotes the summation over all a such that $(a,q)=1$,

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

and $\bar{a}$ is the integer satisfying $\bar{a}a\equiv 1 \pmod q$. Wenpeng Zhang and Yuan Yi [4] gave the following upper bound estimation:

$$|C(h,q)|\ll \sqrt{q}d(q)\ln^2 q,$$

and

$$\sum_{h=1}^{p-1}C^2(h,p)=\frac{5}{144}p^2+O\left(p\exp\left(\frac{4\ln p}{\ln\ln p}\right)\right),$$

where $d(q)$ is the divisor function and $\exp(y)=e^y$.

In this paper, we define the general Cochrane sum as follows:

Suppose that p is a prime satisfying $p\equiv 1 \pmod 4$, h is an integer such that $(h,p)=1$ and χ_1 is Legendre's symbol for p. The general Cochrane sum $C(h,\chi_1,p)$ is defined by

$$C(h,\chi_1,p)=\sum_{a\le p-1}\chi_1(a)\left(\left(\frac{\bar{a}}{p}\right)\right)\left(\left(\frac{ah}{p}\right)\right).$$

We will consider the mean square value of $C(h,\chi_1,p)$. In fact, we have the following

Theorem. *Let p be a prime satisfying $p\equiv 1 \pmod 4$. Then*

(1)
$$\sum_{h\le p-1}C^2(h,\chi_1,p)=\frac{1}{180}p^2\prod_{p_1\in\mathcal{A}}\left(\frac{p_1^2+1}{p_1^2-1}\right)^2+O(p^{1+\epsilon}),$$

where $\mathcal{A}$ is the set of quadratic residues of p, p_1 is prime which is not equal to p.

For a general even Dirichlet character χ modulo p, whether there exists an asymptotic formula for $\sum_{h\le p-1}C^2(h,\chi,p)$ is still an open problem.

The following symbols will be used in the proof of Theorem:

$d(n)$ is the divisor function; $e(x)=e^{2\pi ix}$; $r(n)=\sum_{d|n}\chi_1(d)$;

Let χ be the Dirichlet character modulo q. Then $G(m,\chi)=\sum_{\substack{n\le q\\(n,q)=1}}\chi(n)e(mn/q)$ denotes the Gauss sum and $\tau(\chi)=G(1,\chi)$; $S(m,n,\chi;q)=\sum_{\substack{a\le q\\(a,q)=1}}\chi(a)e\left(\frac{ma+n\bar{a}}{q}\right)$ is the general Kloosterman sum;

$[x]$ is the largest integer not exceeding x, $\{x\}=x-[x]$, $\langle x\rangle=\min(\{x\},1-\{x\})$;

ϵ always denotes a sufficiently small positive constant which may be different at different places.

2. Some lemmas. To complete the proof of the Theorem, we need several Lemmas.

Lemma 1. *Let U be a positive integer, α be a real number. Then we have*

(2)
$$\left|\sum_{\mu=1}^{U}e(\alpha\mu)\right|\le\min\left(U,\frac{1}{2\langle\alpha\rangle}\right).$$

Proof. (See Reference [3]). □

Lemma 2. *Assuming $N>p$, we have*

(3)
$$C(h,\chi_1,p)=-\frac{1}{2\pi^2}\sum_{n\le N^2}{}'\frac{r(n)}{n}\{S(nh,1,\chi_1;p)-S(-nh,1,\chi_1;p)\}+O\left(\frac{p^2}{N}\ln N\right),$$

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where $\sum'$ denotes the summation over all n such that $(n,p)=1$.

Proof. Note that

$$((x)) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}, \quad \sin x = \frac{1}{2i}\left(e^{xi}-e^{-xi}\right).$$

So we have

$$\begin{aligned}(4)\quad & C(h,\chi_1,p)\\ &= \sum_{a\le p-1}\chi_1(a)\left(\left(\frac{\bar a}{p}\right)\right)\left(\left(\frac{ah}{p}\right)\right)\\ &= \frac{1}{\pi^2}\sum_{a\le p-1}\chi_1(a)\sum_{m,n=1}^{\infty}{}'\frac{\sin\left(\frac{2\pi m\bar a}{p}\right)\sin\left(\frac{2\pi nha}{p}\right)}{mn}\\ &= -\frac{1}{4\pi^2}\sum_{a\le p-1}\chi_1(a)\left\{\sum_{m=1}^{\infty}{}'\frac{e\left(\frac{m\bar a}{p}\right)-e\left(\frac{-m\bar a}{p}\right)}{m}\right\}\\ &\quad\times\left\{\sum_{n=1}^{\infty}{}'\frac{e\left(\frac{nha}{p}\right)-e\left(\frac{-nha}{p}\right)}{n}\right\}.\end{aligned}$$

From Lemma 1, we can obtain

$$\left|\sum_{N<m\le t}{}' e\left(\frac{m\bar a}{p}\right)-\sum_{N<m\le t}{}' e\left(\frac{-m\bar a}{p}\right)\right| \ll p,$$

so by Abel's identity we can easily get

$$\left|\sum_{m>N}{}'\frac{1}{m}\left(e\left(\frac{m\bar a}{p}\right)-e\left(\frac{-m\bar a}{p}\right)\right)\right| \ll \frac{p}{N};$$

$$\left|\sum_{n>N}{}'\frac{1}{n}\left(e\left(\frac{nha}{p}\right)-e\left(\frac{-nha}{p}\right)\right)\right| \ll \frac{p}{N}.$$

Combing (4) and note that $N>p$, we have

$$\begin{aligned}& C(h,\chi_1,p)\\ &= -\frac{1}{4\pi^2}\sum_{m,n\le N}{}'\frac{1}{mn}\sum_{a=1}^{p-1}\chi_1(a)\\ &\quad\times\left\{e\left(\frac{m\bar a}{p}\right)-e\left(\frac{-m\bar a}{p}\right)\right\}\\ &\quad\times\left\{e\left(\frac{nha}{p}\right)-e\left(\frac{-nha}{p}\right)\right\}+O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{4\pi^2}\sum_{m,n\le N}{}'\frac{1}{mn}\sum_{a\le p-1}\chi_1(a)\left\{e\left(\frac{m\bar a+nha}{p}\right)\right.\\ &\quad - e\left(\frac{m\bar a-nha}{p}\right)-e\left(\frac{nha-m\bar a}{p}\right)\\ &\quad\left. + e\left(-\frac{m\bar a+nha}{p}\right)\right\}+O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{4\pi^2}\sum_{m,n\le N}{}'\frac{1}{mn}\{S(nh,m,\chi_1;p)\\ &\quad - S(-nh,m,\chi_1;p)-S(nh,-m,\chi_1;p)\\ &\quad + S(-nh,-m,\chi_1;p)\}+O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{2\pi^2}\sum_{m,n\le N}{}'\frac{1}{mn}\sum_{a\le p-1}\chi_1(a)\\ &\quad\times\left\{e\left(\frac{m\bar a+nha}{p}\right)-e\left(\frac{m\bar a-nha}{p}\right)\right\}\\ &\quad + O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{2\pi^2}\sum_{m,n\le N}{}'\frac{1}{mn}\sum_{a\le p-1}\chi_1(ma)\\ &\quad\times\left\{e\left(\frac{\{\bar a+mnha}{p}\right)-e\left(\frac{\bar a-mnha}{p}\right)\right\}\\ &\quad + O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{2\pi^2}\sum_{m,n\le N}{}'\frac{1}{mn}\chi_1(m)\{S(mnh,1,\chi_1;p)\\ &\quad - S(-mnh,1,\chi_1;p)\}+O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{2\pi^2}\sum_{n\le N^2}{}'\frac{1}{n}\sum_{m|n}\chi_1(m)\{S(nh,1,\chi_1;p)\\ &\quad - S(-nh,1,\chi_1;p)\}+O\left(\frac{p^2}{N}\ln N\right)\\ &= -\frac{1}{2\pi^2}\sum_{n\le N^2}{}'\frac{r(n)}{n}\{S(nh,1,\chi_1;p)\\ &\quad - S(-nh,1,\chi_1;p)\}+O\left(\frac{p^2}{N}\ln N\right).\end{aligned}$$

This proves Lemma 2. □

Lemma 3. *For any Dirichlet character χ modulo p, we have*

$$|S(m,n,\chi;p)| \ll (m,n,p)^{\frac{1}{2}}p^{\frac{1}{2}+\epsilon},$$

where (m,n,p) denotes the greatest common divisor of m, n and p.

Proof. (See References [1] and [2]). □

3. Proof of the theorem. In this section we will complete the proof of the Theorem. Making use of Lemma 2 and Lemma 3, we have

(5)

$$\begin{aligned}
&\sum_{h=1}^{p-1} C^2(h,\chi_1;p)\\
&=\frac{1}{4\pi^4}\sum_{h=1}^{p-1}{\sum_{m\le N^2}}'\frac{r(m)}{m}\\
&\quad\times\{S(mh,1,\chi_1;p)-S(-mh,1,\chi_1;p)\}\\
&\quad\times{\sum_{n\le N^2}}'\frac{r(n)}{n}\{S(nh,1,\chi_1;p)-S(-nh,1,\chi_1;p)\}\\
&\quad+O\left(\frac{p^{\frac{7}{2}+\epsilon}}{N}\ln^3 N\right)+O\left(\frac{p^5}{N^2}\ln^2 N\right)\\
&=\frac{1}{4\pi^4}\left\{{\sum_{m,n\le N^2}}'\frac{r(m)r(n)}{mn}\sum_{h=1}^{p-1}S(mh,1,\chi_1;p)\right.\\
&\quad\times S(nh,1,\chi_1;p)-{\sum_{m,n\le N^2}}'\frac{r(m)r(n)}{mn}\\
&\quad\times\sum_{h=1}^{p-1}S(mh,1,\chi_1;p)S(-nh,1,\chi_1;p)\\
&\quad-{\sum_{m,n\le N^2}}'\frac{r(m)r(n)}{mn}\sum_{h=1}^{p-1}S(-mh,1,\chi_1;p)\\
&\quad\times S(nh,1,\chi_1;p)+{\sum_{m,n\le N^2}}'\frac{r(m)r(n)}{mn}\\
&\quad\left.\times\sum_{h=1}^{p-1}S(-mh,1,\chi_1;p)S(-nh,1,\chi_1;p)\right\}\\
&\quad+O\left(\frac{p^{\frac{7}{2}+\epsilon}}{N}\ln^3 N\right)+O\left(\frac{p^5}{N^2}\ln^2 N\right)\\
&=\frac{1}{4\pi^4}(M_1-M_2-M_3+M_4)\\
&\quad+O\left(\frac{p^{\frac{7}{2}+\epsilon}}{N}\ln^3 N\right)+O\left(\frac{p^5}{N^2}\ln^2 N\right),
\end{aligned}$$

where we have used the estimation $\sum\limits_{n\le x}\frac{d(n)}{n}\ll\ln^2 x$.

We need to estimate M_i $(i=1,2,3,4)$ respectively.

(6)

$$\begin{aligned}
M_1&=\sum_{h=1}^{p-1}{\sum_{m,n\le N^2}}'\frac{r(m)r(n)}{mn}\\
&\quad\times S(mh,1,\chi_1;p)S(nh,1,\chi_1;p)\\
&=\sum_{h\le p-1}{\sum_{m\le N^2}}'\frac{r(m)}{m}\sum_{a\le p-1}\chi_1(a)e\left(\frac{mha+\bar a}{p}\right)\\
&\quad\times{\sum_{n\le N^2}}'\frac{r(n)}{n}\sum_{b\le p-1}\chi_1(b)e\left(\frac{nhb+\bar b}{p}\right)\\
&={\sum_{m\le N^2}}'\frac{r(m)}{m}{\sum_{n\le N^2}}'\frac{r(n)}{n}\sum_{a\le p-1}\chi_1(a)e\left(\frac{\bar a}{p}\right)\\
&\quad\times\sum_{b\le p-1}\chi_1(b)e\left(\frac{\bar b}{p}\right)\sum_{h\le p-1}e\left(\frac{ma+nb}{p}h\right)\\
&=p{\sum_{m\le N^2}}'{\sum_{n\le N^2}}'\sum_{a\le p-1}\sum_{\substack{b\le p-1\\ ma\equiv -nb\ (\mathrm{mod}\, p)}}\frac{r(m)r(n)}{mn}\\
&\quad\times\chi_1(ab)e\left(\frac{\bar a+\bar b}{p}\right)-\left({\sum_{m\le N^2}}'\frac{r(m)}{m}\right)^2\tau^2(\overline{\chi_1})\\
&=p{\sum_{m\le N^2}}'{\sum_{n\le N^2}}'\sum_{a\le p-1}\sum_{\substack{b\le p-1\\ a\equiv -b\ (\mathrm{mod}\, p)}}\frac{r(m)r(n)}{mn}\\
&\quad\times\chi_1(\overline{mn}ab)e\left(\frac{m\bar a+n\bar b}{p}\right)+O(p\ln^4 N)\\
&=\chi_1(-1)p{\sum_{m\le N^2}}'{\sum_{n\le N^2}}'\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}\\
&\quad\times\sum_{a\le p-1}\chi_1(a^2)e\left(\frac{m\bar a-n\bar a}{p}\right)+O(p\ln^4 N)\\
&=p{\sum_{m\le N^2}}'{\sum_{n\le N^2}}'\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}\\
&\quad\times\sum_{a\le p-1}\overline{\chi_1}(a^2)e\left(\frac{m-n}{p}a\right)+O(p\ln^4 N)\\
&=p{\sum_{m\le N^2}}'{\sum_{n\le N^2}}'\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m-n,\overline{\chi_1}^2)\\
&\quad+O(p\ln^4 N)\\
&=p{\sum_{m\le N^2}}'{\sum_{n\le N^2}}'\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m-n,\chi_0)\\
&\quad+O(p\ln^4 N).
\end{aligned}$$

When $p\nmid(m-n)$, $G(m-n,\chi_0)=\tau(\chi_0)=\mu(p)=-1$, so we have

$$p\left|{\sum_{\substack{m\le N^2\\ p\nmid(m-n)}}}'{\sum_{n\le N^2}}'\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m-n,\chi_0)\right| \ll p\left(\sum_{m\le N^2}\frac{d(m)}{m}\right)^2\ll p\ln^4 N. \tag{7}$$

When $p|(m-n)$, $G(m-n,\chi_0)=p-1$, we get

$$(8)\quad p\sum_{\substack{m\le N^2\\ p|(m-n)}}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m-n,\chi_0)$$
$$= p(p-1)\sum_{\substack{m\le N^2\\ m\equiv n \ (\mathrm{mod}\, p)}}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}$$
$$= p(p-1)\sum_{n\le N^2}{}' \left(\frac{r(n)}{n}\right)^2$$
$$+ p(p-1)\sum_{\substack{m\le N^2\\ m\equiv n \ (\mathrm{mod}\ p)\\ m\neq n}}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}$$
$$= p(p-1)\sum_{n\le N^2}{}' \left(\frac{r(n)}{n}\right)^2 + O\left(p^2 \sum_{k\le \frac{N^2}{p}} \sum_{n\le N^2} \frac{d(kp+n)d(n)}{(kp+n)n}\right)$$
$$= p(p-1)\sum_{n=1}^{\infty}{}' \frac{r^2(n)}{n^2} + O(p^2N^{-2+\epsilon}) + O(pN^{\epsilon})$$
$$= p(p-1)\sum_{n=1}^{\infty}{}' \frac{r^2(n)}{n^2} + O(pN^{\epsilon}),$$

where we have used the estimation $\sum_{n\le x} d^2(n) \ll x\ln^3 x$, $d(n) \ll n^{\epsilon}$ and the Abel's identity.

Combining (6), (7) and (8), we obtain

$$(9)\qquad M_1 = p(p-1)\sum_{n=1}^{\infty}{}' \frac{r^2(n)}{n^2} + O(pN^{\epsilon}).$$

By the same method, we can get

$$(10)\qquad M_4 = p(p-1)\sum_{n=1}^{\infty}{}' \frac{r^2(n)}{n^2} + O(pN^{\epsilon}).$$

For M_2, we have

(11)
$$M_2 = \sum_{h=1}^{p-1} \sum_{m,n\le N^2}{}' \frac{r(m)r(n)}{mn} \times S(mh,1,\chi_1;p)S(-nh,1,\chi_1;p)$$
$$= \sum_{h\le p-1}\sum_{m\le N^2}{}' \frac{r(m)}{m}\sum_{a\le p-1}\chi_1(a)e\left(\frac{mha+\bar a}{p}\right) \times \sum_{n\le N^2}{}' \frac{r(n)}{n}\sum_{b\le p-1}\chi_1(b)e\left(\frac{-nhb+\bar b}{p}\right)$$
$$= \sum_{m\le N^2}{}' \frac{r(m)}{m}\sum_{n\le N^2}{}' \frac{r(n)}{n}\sum_{a\le p-1}\chi_1(a)e\left(\frac{\bar a}{p}\right) \times \sum_{b\le p-1}\chi_1(b)e\left(\frac{\bar b}{p}\right)\sum_{h\le p-1} e\left(\frac{ma-nb}{p}h\right)$$
$$= p\sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \sum_{\substack{a\le p-1\\ ma\equiv nb \ (\mathrm{mod}\, p)}} \sum_{b\le p-1} \frac{r(m)r(n)}{mn} \times \chi_1(ab)e\left(\frac{\bar a+\bar b}{p}\right) - \left(\sum_{m\le N^2}{}' \frac{r(m)}{m}\right)^2 \tau^2(\overline{\chi_1})$$
$$= p\sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \sum_{\substack{a\le p-1\\ a\equiv b \ (\mathrm{mod}\, p)}} \sum_{b\le p-1} \frac{r(m)r(n)}{mn} \times \chi_1(\overline{mn}ab)e\left(\frac{m\bar a+n\bar b}{p}\right) + O(p\ln^4 N)$$
$$= p\sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn} \times \sum_{a\le p-1}\chi_1(a^2)e\left(\frac{m\bar a+n\bar a}{p}\right) + O(p\ln^4 N)$$
$$= p\sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn} \times \sum_{a\le p-1}\overline{\chi_1}(a^2)e\left(\frac{m+n}{p}a\right) + O(p\ln^4 N)$$
$$= p\sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m+n,\overline{\chi_1}^2) + O(p\ln^4 N)$$
$$= p\sum_{m\le N^2}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m+n,\chi_0) + O(p\ln^4 N).$$

When $p\nmid(m+n)$, $G(m+n,\chi_0) = \tau(\chi_0) = \mu(p) = -1$, so we get

$$(12)\quad p\left|\sum_{\substack{m\le N^2\\ p\nmid(m+n)}}{}' \sum_{n\le N^2}{}' \overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m+n,\chi_0)\right| \ll p\left(\sum_{m\le N^2}\frac{d(m)}{m}\right)^2 \ll p\ln^4 N.$$

When $p|(m+n)$, $G(m+n,\chi_0) = p-1$, we have

$$(13)\quad p\left|\sideset{}{'}\sum_{\substack{m\le N^2\\ p|(m+n)}}\sideset{}{'}\sum_{n\le N^2}\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}G(m+n,\chi_0)\right|$$
$$= p(p-1)\left|\sideset{}{'}\sum_{\substack{m\le N^2\\ m\equiv -n \,(\mathrm{mod}\, p)}}\sideset{}{'}\sum_{n\le N^2}\overline{\chi_1}(mn)\frac{r(m)r(n)}{mn}\right|$$
$$\ll p(p-1)\sum_{m=1}^{p-1}\frac{d(m)d(p-m)}{m(p-m)}$$
$$+\,p(p-1)\sum_{1\le m\le N^2}\ \sum_{(1+\frac{m}{p})\le k\le \frac{N^2}{p}}\frac{d(m)d(kp-m)}{m(kp-m)}$$
$$\ll pN^{\epsilon},$$

where we have used the estimate

$$\sum_{m=1}^{p-1}\frac{d(m)d(p-m)}{m(p-m)}\ll p^{-1+\epsilon}.$$

From (11), (12) and (13), we can get

$$(14)\qquad M_2\ll pN^{\epsilon}.$$

Similarly, we have

$$(15)\qquad M_3\ll pN^{\epsilon}.$$

Let $N=p^3$, from (5), (9), (10), (14) and (15), we can obtain

(16)
$$\sum_{h\le p-1}C^2(h,\chi_1,p)=\frac{1}{2\pi^4}p(p-1)\sideset{}{'}\sum_{n=1}^{\infty}\frac{r^2(n)}{n^2}+O(p^{1+\epsilon}).$$

Noting that $r(n)$ is a multiplicative function, we can express the convergent series $\sideset{}{'}\sum_{n=1}^{\infty}\frac{r^2(n)}{n^2}$ as:

$$(17)\quad \sideset{}{'}\sum_{n=1}^{\infty}\frac{r^2(n)}{n^2}$$
$$=\prod_{p_2\notin\mathcal{A}}\left(1+\frac{1}{p_2^4}+\cdots+\frac{1}{p_2^{4n}}+\cdots\right)$$
$$\times\prod_{p_1\in\mathcal{A}}\left(1+\frac{2^2}{p_1^2}+\cdots+\frac{(n+1)^2}{p_1^{2n}}+\cdots\right).$$

Combining (16) and (17), we finally get

$$\sum_{h\le p-1}C^2(h,\chi_1,p)$$
$$=\frac{1}{2\pi^4}p(p-1)\prod_{p_2\notin\mathcal{A}}\left(1+\frac{1}{p_2^4}+\cdots+\frac{1}{p_2^{4n}}+\cdots\right)$$
$$\times\prod_{p_1\in\mathcal{A}}\left(1+\frac{2^2}{p_1^2}+\cdots+\frac{(n+1)^2}{p_1^{2n}}+\cdots\right)$$
$$+O(p^{1+\epsilon})$$
$$=\frac{1}{2\pi^4}p(p-1)\prod_{p_2\notin\mathcal{A}}\frac{p_2^4}{p_2^4-1}\prod_{p_1\in\mathcal{A}}\frac{p_1^2(p_1^2+1)}{(p_1^2-1)^2}$$
$$\times\left(1+\frac{1}{p_1^2}+\cdots+\frac{1}{p_1^{2n}}+\cdots\right)+O(p^{1+\epsilon})$$
$$=\frac{\zeta(2)}{2\pi^4}\frac{(p-1)^2(p+1)}{p}\prod_{p_2\notin\mathcal{A}}\frac{p_2^2}{p_2^2+1}\prod_{p_1\in\mathcal{A}}\frac{p_1^2(p_1^2+1)}{(p_1^2-1)^2}$$
$$+O(p^{1+\epsilon})$$
$$=\frac{\zeta(2)}{2\pi^4}\frac{(p-1)^2(p+1)}{p}\frac{p^2+1}{p^2}\frac{\zeta(4)}{\zeta(2)}\prod_{p_1\in\mathcal{A}}\left(\frac{p_1^2+1}{p_1^2-1}\right)^2$$
$$+O(p^{1+\epsilon})$$
$$=\frac{1}{180}p^2\prod_{p_1\in\mathcal{A}}\left(\frac{p_1^2+1}{p_1^2-1}\right)^2+O(p^{1+\epsilon}).$$

This completes the proof of the Theorem. □

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