Inequalities of Ono numbers and class numbers associated to imaginary quadratic fields

By Kenichi SHIMIZU

Kenmei Girls' Junior and Senior High School, 68, Honmachi, Himeji, Hyogo 670-0012, Japan

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Abstract: We denote by h_D the class number and by p_D the Ono number of the imaginary quadratic fields $\mathbf{Q}(\sqrt{-D})$. Sairaiji-Shimizu [2] showed that there are infinitely many imaginary quadratic fields such that the inequality $h_D > c^{p_D}$ holds for any real number. On the other hand we have the possibility that $h_D \leqq c^{p_D}$ holds for infinitely many imaginary quadratic fields for the same real number c. In this paper, given a real number c, we consider whether $h_D \leqq c^{p_D}$ holds for infinitely many imaginary quadratic fields or not.

Key words: Ono number; class number.

1. Introduction. Given a square-free integer $d > 0$, we define D by

$$D := \begin{cases} 4d & \text{if } d \equiv 1,2 \pmod 4 \\ d & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and call $-D$ the discriminant of the imaginary quadratic field $K_D = \mathbf{Q}(\sqrt{-D})$. We denote by h_D the class number of K_D. Let $\nu(n)$ be the number of (not necessarily different) prime factors of an integer n, then we define the Ono number p_D as follows:

$$p_D := \begin{cases} \max\{\nu(f_D(x)) \mid x \text{ are integers} \\ \text{in the interval } 0 \leqq x \leqq D/4 - 1\} \\ \qquad\qquad\qquad\qquad \text{if } d \neq 1,3 \\ 1 \qquad\qquad\qquad\quad\;\; \text{if } d = 1,3, \end{cases}$$

where we define $f_D(x)$ by

$$f_D(x) := \begin{cases} x^2 + d & \text{if } d \equiv 1,2 \pmod 4 \\ x^2 + x + (1+d)/4 & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

A motivation of this study was raised by the inequality

$$h_D \leqq 2^{p_D},$$

which conjectured by T. Ono [1]. Sairaiji-Shimizu [2] showed that the inequality $h_D \leqq 2^{p_D}$ does not hold for all D, by giving infinite many imaginary quadratic fields such that $h_D > c^{p_D}$ holds for any real number. Further in [3] we also showed that $h_D \leqq 2^{p_D}$ holds for all D if $D \equiv 7 \pmod 8$.

We consider the supremum c_0 of real numbers c such that the inequality $h_D \leqq c^{p_D}$ holds for only finitely many D. At first we show:

Proposition 2.1. *There is a constant c_0 which satisfies the following conditions* (1) *and* (2).
(1) *If $c < c_0$, then there are finitely many D such that $h_D \leqq c^{p_D}$.*
(2) *If $c > c_0$, then there are infinitely many D such that $h_D \leqq c^{p_D}$.*

We want to calculate the constant c_0, but we can not do now. In this paper, we show the following theorems.

Theorem 2.4. *The inequality $c_0 \leqq \sqrt{2}$ holds.*

Theorem 3.3. *The inequality $\sqrt[4]{2} \leqq c_0$ holds.*

In Section 2 we discuss an upper bound for c_0 and we give the Proof of Theorem 2.4. In Section 3 we discuss a lower bound for c_0 and we give the Proof of Theorem 3.3.

2. An upper bound for c_0.

At first we consider the existence of the following real number c_0.

Proposition 2.1. *There is a constant c_0 which satisfies the following conditions* (1) *and* (2).
(1) *If $c < c_0$, then there are finitely many D such that $h_D \leqq c^{p_D}$.*
(2) *If $c > c_0$, then there are infinitely many D such that $h_D \leqq c^{p_D}$.*

Proof. Put $S := \{c \mid h_D \leqq c^{p_D}$ holds for finitely many $D\}$. Since there are only finitely many D such that $h_D = 1$, we have $1 \in S$. Since in [3] we have the fact that $h_D \leqq 2^{p_D}$ holds for infinitely many D, we see that $S \subset [1, 2)$. Thus there exists the supremum c_0 of S, and we have the assertion. □

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For giving an upper bound for c_0, we show Propositions 2.2 and 2.3.

Proposition 2.2. *For real numbers ℓ, m, $a > 1$ and $k > 0$, if there are infinitely many D such that $p_D > k\log_a(\ell D + m)$, then the inequality $c_0 \leqq \sqrt[2k]{a}$ holds.*

Proof. Siegel [4] showed that the inequality $h_D < (3/\pi)\sqrt{D}\log D$ holds for all D. By this inequality and the assumption of this proposition, there are infinitely many D such that

$$\frac{p_D}{\log h_D} > \frac{k\log_a(\ell D + m)}{\log((3/\pi)\sqrt{D}\log D)},$$

that is,

$$\frac{p_D \log a}{\log h_D} > \frac{k\log(\ell D + m)}{\log((3/\pi)\sqrt{D}\log D)}.$$

Putting

$$\phi(D) = \frac{k\log(\ell D + m)}{\log((3/\pi)\sqrt{D}\log D)},$$

we have

$$\begin{aligned}\phi(D) &= \frac{k\log(\ell D + m)}{\log(3/\pi) + (1/2)\log D + \log\log D}\\ &= \frac{k\log(\ell D + m)/\log D}{\log(3/\pi)/\log D + 1/2 + \log\log D/\log D}.\end{aligned}$$

Since

$$\lim_{D\to\infty} \log(\ell D + m)/\log D = 1,$$

$$\lim_{D\to\infty} \log(3/\pi)/\log D = 0,$$

and

$$\lim_{D\to\infty} \log\log D/\log D = 0,$$

we have

$$\lim_{D\to\infty} \phi(D) = 2k.$$

For any η such that $0 < \eta < 2k$, there are infinitely many D such that

$$\frac{p_D\log a}{\log h_D} \geqq 2k - \eta.$$

This inequality implies

$$\frac{p_D\log a}{2k - \eta} \geqq \log h_D,$$

and consequently

$$h_D \leqq a^{\frac{p_D}{2k-\eta}},$$

that is, there are infinitely many D such that

$$h_D \leqq a^{\frac{p_D}{2k-\eta}}.$$

Hence, for $\varepsilon = \varepsilon(\eta) > 0$ there are infinitely many D such that $h_D \leqq a^{(\frac{1}{2k}+\varepsilon)p_D}$.

Let $c(\varepsilon) = a^{\frac{1}{2k}+\varepsilon}$, then it holds that $\sqrt[2k]{a} < c(\varepsilon)$ and $h_D \leqq c(\varepsilon)^{p_D}$ for infinitely many D.

Thus given a real number $c > \sqrt[2k]{a}$, then there is a positive number ε such that $\sqrt[2k]{a} < c(\varepsilon) \leqq c$, and it holds $h_D \leqq c(\varepsilon)^{p_D} \leqq c^{p_D}$ for infinitely many D. Hence we get

$$c_0 \leqq \sqrt[2k]{a}.$$

□

Proposition 2.3. *There are infinitely many D such that $p_D > \log_2(D/4 - 1)$.*

Proof. By Sairaiji-Shimizu [3], we have the inequality $p_D > \log_{q_D}(D/4 - 1)$ for $D > 4$. If $d \equiv 7 \pmod 8$, then $q_D = 2$. Hence there are infinitely many D such that $p_D > \log_2(D/4 - 1)$. □

By Propositions 2.2 and 2.3, we immediately obtain the following theorem.

Theorem 2.4. *The inequality $c_0 \leqq \sqrt{2}$ holds.*

3. A lower bound for c_0. For giving a lower bound for c_0, we show Propositions 3.1 and 3.2.

Proposition 3.1. *For real numbers ℓ, m, $a > 1$ and $k > 0$, if there exists a constant D_1 such that $p_D < k\log_a(\ell D + m)$ for all $D > D_1$, then $\sqrt[2k]{a} \leqq c_0$.*

Proof. Siegel [4] showed the following formula related to class numbers, that is,

$$\lim_{n\to\infty} \frac{\log h_D}{\log\sqrt{D}} = 1.$$

For any $\varepsilon > 0$, there exists a constant D_2 depending on ε such that the inequality

$$1 - \varepsilon < \frac{\log h_D}{\log\sqrt{D}}$$

holds for all $D > D_2$. From this, we have

$$\frac{1-\varepsilon}{2}\log D < \log h_D.$$

By this inequality and the assumption of this proposition, for all $D > \max\{D_1, D_2\}$ we have

$$\frac{p_D}{\log h_D} < \frac{k\log_a(\ell D + m)}{\frac{1-\varepsilon}{2}\log D}$$

$$= \frac{2k}{(1-\varepsilon)\log a} \cdot \frac{\log(\ell D + m)}{\log D}$$
$$= \frac{1}{\log a^{\frac{1-\varepsilon}{2k}}} \cdot \frac{\log(\ell D + m)}{\log D}.$$

Since

$$\lim_{D\to\infty} \log(\ell D + m)/\log D = 1,$$

we obtain

$$\lim_{D\to\infty} \frac{1}{\log a^{\frac{1-\varepsilon}{2k}}} \cdot \frac{\log(\ell D + m)}{\log D} = \frac{1}{\log a^{\frac{1-\varepsilon}{2k}}}.$$

Hence for any $\eta > 0$ there is a constant D_3 depending on η, we get

$$\frac{1}{\log a^{\frac{1-\varepsilon}{2k}-\eta}} > \frac{p_D}{\log h_D}$$

for all $D > \max\{D_1, D_2, D_3\}$, that is,

$$p_D \log a^{\frac{1-\varepsilon}{2k}-\eta} < \log h_D.$$

Therefore we get

$$a^{(\frac{1-\varepsilon}{2k}-\eta)p_D} < h_D,$$

and consequently

$$a^{(\frac{1}{2k}-\frac{\varepsilon}{2k}-\eta)p_D} < h_D.$$

Let $c(\varepsilon,\eta) = a^{\frac{1}{2k}-\frac{\varepsilon}{2k}-\eta}$, then we have $c(\varepsilon,\eta) < \sqrt[2k]{a}$ and $c(\varepsilon,\eta)^{p_D} < h_D$ for all $D > \max\{D_1, D_2, D_3\}$. Hence there are only finitely many D such that $h_D \leqq c(\varepsilon,\eta)^{p_D}$.

Thus given a real number $c < \sqrt[2k]{a}$, then there are positive numbers ε and η such that $c \leqq c(\varepsilon,\eta) < \sqrt[2k]{a}$, and it implies that $h_D \leqq c^{p_D}$ holds for finitely many D. Therefore we get $\sqrt[2k]{a} \leqq c_0$. □

Proposition 3.2 (Sairaiji-Shimizu [3])**.** *The inequality $p_D < 2\log_2 D$ holds for all D.*

By Propositions 3.1 and 3.2, we immediately obtain the following theorem.

Theorem 3.3. *The inequality $\sqrt[4]{2} \leqq c_0$ holds.*

From Theorems 2.4 and 3.3 we have showed that the inequality $\sqrt[4]{2} \leqq c_0 \leqq \sqrt{2}$ holds. We want to obtain sharper lower bounds and upper bounds for c_0, and determine the value c_0 itself. Furthermore we wonder whether c_0 is an algebraic number or a transcendental number, and whether the inequality $h_D \leqq c_0^{p_D}$ holds for infinitely many D or not.

References

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