Zeta and L-functions and Bernoulli polynomials of root systems

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Abstract: This article is essentially an announcement of the papers [7–10] of the authors, though some of the examples are not included in those papers. We consider what is called zeta and L-functions of root systems which can be regarded as a multi-variable version of Witten multiple zeta and L-functions. Furthermore, corresponding to these functions, Bernoulli polynomials of root systems are defined. First we state several analytic properties, such as analytic continuation and location of singularities of these functions. Secondly we generalize the Bernoulli polynomials and give some expressions of values of zeta and L-functions of root systems in terms of these polynomials. Finally we give some functional relations among them by our previous method. These relations include the known formulas for their special values formulated by Zagier based on Witten's work.

Key words: Multiple zeta-function; Witten zeta-function; root systems; simple Lie algeras; analytic continuation; functional relation.

1. Zeta and L-functions of root systems. Let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ be the set of all positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with rank r. The Witten zeta-function associated with $\mathfrak{g}$ is defined by

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.1}$$

where the summation runs over all finite dimensional irreducible representations φ of $\mathfrak{g}$. It is known that

$$\zeta_W(2k;\mathfrak{g}) = C_W(2k,\mathfrak{g})\pi^{2kn}$$

for any $k \in \mathbf{N}$, where n is the number of all positive roots and $C_W(2k,\mathfrak{g}) \in \mathbf{Q}$. This is called Witten's volume formula (Witten [20], Zagier [21]).

In this paper, we introduce its multi-variable version and character analogues defined as follows:

Let V be an r-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. We denote the norm of $v \in V$ by $\|v\| = \langle v, v\rangle^{1/2}$. The dual space V^* is identified with V via the inner product of V. Let Δ be a finite reduced root system in V and $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ its fundamental system. Let Δ_+ and Δ_- be the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \coprod \Delta_-$. Let $Q^\vee$ be the coroot lattice, P the weight lattice, P_+ the set of integral dominant weights and P_{++} the set of integral strongly dominant weights respectively defined by

$$Q^\vee = \bigoplus_{i=1}^r \mathbf{Z}\,\alpha_i^\vee, \quad P = \bigoplus_{i=1}^r \mathbf{Z}\,\lambda_i,$$

$$P_+ = \bigoplus_{i=1}^r \mathbf{N}_0\,\lambda_i, \quad P_{++} = \bigoplus_{i=1}^r \mathbf{N}\,\lambda_i,$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ are a basis dual to $\Psi^\vee$ satisfying $\langle \alpha_i^\vee, \lambda_j\rangle = \delta_{ij}$. Let

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha = \sum_{j=1}^r \lambda_j$$

be the lowest strongly dominant weight. Then $P_{++} = P_+ + \rho$.

We define the reflection σ_α with respect to a root $\alpha \in \Delta$ as

$$\sigma_\alpha : V \to V, \quad \sigma_\alpha : v \mapsto v - \langle \alpha^\vee, v\rangle\alpha$$

and for a subset $\Delta^* \subset \Delta$, let $W(\Delta^*)$ be the group generated by reflections σ_α for $\alpha \in \Delta^*$. Let $W = W(\Delta)$ be the Weyl group. Then $\sigma_j = \sigma_{\alpha_j}$ $(1 \le j \le r)$ generates W. Namely we have $W = W(\Psi)$. Any two fundamental systems Ψ, Ψ' are conjugate under W.

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Let $\mathrm{Aut}(\Delta)$ be the subgroup of all the automorphisms $GL(V)$ which stabilizes Δ (see [3, §12.2]). Then the Weyl group W is a normal subgroup of $\mathrm{Aut}(\Delta)$ and there exists a subgroup $\Omega \subset \mathrm{Aut}(\Delta)$ such that $\mathrm{Aut}(\Delta) = \Omega \ltimes W$. The group $\mathrm{Aut}(\Delta)$ is called the extended Weyl group. For $w \in \mathrm{Aut}(\Delta)$, we set $\Delta_w = \Delta_+ \cap w^{-1}\Delta_-$ and the length function $\ell(w) = |\Delta_w|$ (see [4, §1.6]). The subgroup Ω is characterized as $w \in \Omega$ if and only if $\ell(w) = 0$. Note that $w\Delta_w = \Delta_- \cap w\Delta_+ = -\Delta_{w^{-1}}$ and $\ell(w) = \ell(w^{-1})$.

Let $n = |\Delta_+|$ and r be the rank of Δ. Let $\overline{\Delta}$ be the quotient of Δ obtained by identifying α and $-\alpha$. For $\mathbf{s} = (s_\alpha)_{\alpha\in\overline{\Delta}} \in \mathbf{C}^n$ we define an action of $\mathrm{Aut}(\Delta)$ by $(w\mathbf{s})_\alpha = s_{w^{-1}\alpha}$. For $\mathbf{y} \in V$, $\mathbf{s} \in \mathbf{C}^n$ and $\Delta^* \subset \Delta_+$ such that for any fundamental weight λ_i there exists a root $\alpha \in \Delta^*$ satisfying $\langle \alpha^\vee, \lambda_i\rangle > 0$, we define

$$\zeta_r(\mathbf{s}, \mathbf{y}; \Delta^*) = \sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta^*} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}},$$

which is called the zeta-function of the roots Δ^* with exponential factors, introduced in [7,8]. When $\mathbf{y} = \mathbf{0}$ and $\Delta^* = \Delta_+$ is of type X_r, where $X = A, B, \ldots, G$, we denote it simply by $\zeta_r(\mathbf{s}; \Delta)$ or $\zeta_r(\mathbf{s}; X_r)$ which is called the zeta-function of the root system X_r. In particular when $\mathbf{s} = (s)$, namely $s_\alpha = s$ for each α, this coincides with (1.1) up to some exponential function part.

In the case of rank one, $\zeta_1(s; A_1)$ is just the Riemann zeta-function $\zeta(s)$. In the case of rank two, analytic properties of $\zeta_2(\mathbf{s}; A_2)$ and $\zeta_2(\mathbf{s}; B_2)$ have been studied in, for example, [12,14,17–19,21]. In the case of rank three, $\zeta_3(\mathbf{s}; A_3)$ has been studied in [2,5,15]. Now we consider the cases of B_3 and C_3 types, namely

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9; B_3) \\
&\quad = \sum_{m_1,m_2,m_3=1}^{\infty} m_1^{-s_1} m_2^{-s_2} m_3^{-s_3} (m_1+m_2)^{-s_4} \\
&\quad \times (m_2+m_3)^{-s_5} (2m_2+m_3)^{-s_6} (m_1+m_2+m_3)^{-s_7} \\
&\quad \times (m_1+2m_2+m_3)^{-s_8} (2m_1+2m_2+m_3)^{-s_9},
\end{aligned}$$

and

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9; C_3) \\
&\quad = \sum_{m_1,m_2,m_3=1}^{\infty} m_1^{-s_1} m_2^{-s_2} m_3^{-s_3} (m_1+m_2)^{-s_4} \\
&\quad \times (m_2+m_3)^{-s_5} (m_2+2m_3)^{-s_6} (m_1+m_2+m_3)^{-s_7} \\
&\quad \times (m_1+m_2+2m_3)^{-s_8} (m_1+2m_2+2m_3)^{-s_9}.
\end{aligned}$$

By the same method as introduced in the papers [11–14] of the second named author, we see that there is a certain recursive structure in the family of those zeta-functions corresponding to inclusion relations among certain sets of roots. This consideration gives the analytic continuation of these functions to the whole complex space, and furthermore, determines the location of possible singularities (cf. [7,15,16]). For example, we obtain

Theorem 1.1 [7]**.** *The possible singularities of $\zeta_3(\mathbf{s}; B_3)$ and of $\zeta_3(\mathbf{s}; C_3)$ are located only on the subsets of $\mathbf{C}^9$ defined by one of the following*:

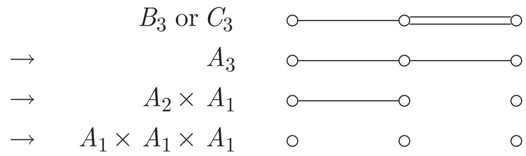
$$\begin{aligned}
&s_1+s_4+s_7+s_8+s_9 = 1-\ell, \\
&s_3+s_5+s_6+s_7+s_8+s_9 = 1-\ell, \\
&s_2+s_4+s_5+s_6+s_7+s_8+s_9 = 1-\ell, \\
&s_1+s_2+s_4+s_5+s_6+s_7+s_8+s_9 = 2-\ell, \\
&s_1+s_3+s_4+s_5+s_6+s_7+s_8+s_9 = 2-\ell, \\
&s_2+s_3+s_4+s_5+s_6+s_7+s_8+s_9 = 2-\ell, \\
&s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8+s_9 = 3,
\end{aligned}$$

where $\ell \in \mathbf{N}_0$.

It is to be noted that the above recursive structure can be explained in terms of Dynkin diagrams; a recursive step corresponds to a cut of one edge of the diagram. For example, by cutting one of the rightmost edges in the Dynkin diagram of type B_3 or C_3, we obtain that of A_3 type, which corresponds to the equation

$$\zeta_3(\mathbf{s}; A_3) = \zeta_3(s_1, s_2, s_3, s_4, s_5, 0, s_6, 0, 0; B_3 \text{ or } C_3).$$

In fact, $\zeta_3(\mathbf{s}, B_3 \text{ or } C_3)$ can be expressed as an integral involving $\zeta(\cdot, A_3)$ in the integrand. Consequently, we have the following recursion diagram



by repeating the same type of procedure.

Define

$$S(\mathbf{s}, \mathbf{y}; \Delta) = \sum_{w\in W} \left(\prod_{\alpha\in\Delta_{w^{-1}}} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\mathbf{s}, w^{-1}\mathbf{y}; \Delta).$$

This $S(\mathbf{s}, \mathbf{y}; \Delta)$ is a "Weyl group symmetric" linear combination of zeta-functions of root systems, which plays a fundamental role in the study of value-relations and functional relations in [8].

Let χ_α be a Dirichlet character modulo $f_\alpha \in \mathbf{N}$ for $\alpha \in \overline{\Delta}$. Set $\boldsymbol{\chi} = (\chi_\alpha)_{\alpha\in\overline{\Delta}}$. We define an action of $\mathrm{Aut}(\Delta)$ on characters by

$$(w\boldsymbol{\chi})_\alpha = \chi_{w^{-1}\alpha}.$$

Now we define the L-function by

$$L_r(\mathbf{s},\boldsymbol{\chi};\Delta) = \sum_{\lambda\in P_{++}} \prod_{\alpha\in\Delta_+} \frac{\chi_\alpha(\langle\alpha^\vee,\lambda\rangle)}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},$$

and more generally, define the L-function of Δ^* by

$$L_r(\mathbf{s},\boldsymbol{\chi};\Delta^*) = \sum_{\lambda\in P_{++}} \prod_{\alpha\in\Delta^*} \frac{\chi_\alpha(\langle\alpha^\vee,\lambda\rangle)}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}$$

for any $\Delta^* \subset \Delta_+$ such that for any fundamental weight λ_i, there exists a root $\alpha\in\Delta^*$ satisfying $\langle\alpha^\vee,\lambda_i\rangle > 0$. Using the method introduced in [11–14], we have

Theorem 1.2 [9]**.** *The L-function $L_r(\mathbf{s},\boldsymbol{\chi};\Delta^*)$ can be continued meromorphically to the whole $\mathbf{C}^{n^*}$ space, where $n^* = |\Delta^*|$.*

2. Bernoulli polynomials. Let $\mathscr{V}$ be the set of all linearly independent subsets $\mathbf{V} = \{\beta_1,\ldots,\beta_r\} \subset \Delta_+$ and let $L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}} \mathbf{Z}\beta^\vee$. For $\mathbf{V}\in\mathscr{V}$, let $\{\mu_\beta^{\mathbf{V}}\}$ be the dual basis of $\mathbf{V}^\vee = \{\beta^\vee\}$. Let $\mathscr{R}$ be the set of all linearly independent subsets $\mathbf{R} = \{\beta_1,\ldots,\beta_{r-1}\} \subset \Delta$, $\mathfrak{H}_{\mathbf{R}^\vee} = \bigoplus_{i=1}^{r-1} \mathbf{R}\,\beta_i^\vee$ the hyperplane passing through $\mathbf{R}^\vee \cup \{0\}$ and

$$\mathfrak{H}_{\mathscr{R}} := \bigcup_{\substack{\mathbf{R}\in\mathscr{R}\\ q\in Q^\vee}} (\mathfrak{H}_{\mathbf{R}^\vee} + q).$$

Then it can be shown that $V \setminus \mathfrak{H}_{\mathscr{R}}$ is a disjoint union of open subsets. Hence we denote by $\mathfrak{D}^{(\nu)}$ each open connected component of $V\setminus\mathfrak{H}_{\mathscr{R}}$ so that

$$V\setminus\mathfrak{H}_{\mathscr{R}} = \coprod_{\nu\in\mathfrak{J}} \mathfrak{D}^{(\nu)},$$

where $\mathfrak{J}$ is a set of indices. Fix a vector $\phi\in V$ such that

$$\phi \notin \bigcup_{\mathbf{R}\in\mathscr{R}} \mathfrak{H}_{\mathbf{R}^\vee} \subset \mathfrak{H}_{\mathscr{R}}.$$

Then $\langle\phi,\mu_\beta^{\mathbf{V}}\rangle \neq 0$ for all $\mathbf{V}\in\mathscr{V}$ and $\beta\in\mathbf{V}$. For $x\in\mathbf{R}$, we denote its fractional part $x-[x]$ by $\{x\}$. For $\mathbf{y}\in V$, $\mathbf{V}\in\mathscr{V}$ and $\beta\in\mathbf{V}$, we define

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases} \{\langle\mathbf{y},\mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi,\mu_\beta^{\mathbf{V}}\rangle > 0), \\ 1-\{-\langle\mathbf{y},\mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi,\mu_\beta^{\mathbf{V}}\rangle < 0). \end{cases}$$

We note that $\{x\} = 1-\{-x\}$ holds for $x\in\mathbf{R}\setminus\mathbf{Z}$ and that $\{x\}$ is right continuous while $1-\{-x\}$ is left continuous. For $\mathbf{y}\in V$ and $\mathbf{t} = (t_\alpha)_{\alpha\in\overline{\Delta}} \in \mathbf{C}^n$, we define

$$\begin{aligned} F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{V}\in\mathscr{V}} &\left(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle}\right) \\ &\times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \\ &\times \left(\prod_{\beta\in\mathbf{V}} \frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\right), \end{aligned}$$

and in particular $F(\mathbf{t};\Delta) = F(\mathbf{t},\mathbf{0};\Delta)$. It should be noted that in the A_1 case, we have

$$\begin{aligned} F(\mathbf{t},\mathbf{y};A_1) &= \frac{te^{t\{y\}}}{e^t-1} \\ &= \sum_{k=0}^\infty B_k(\{y\})\frac{t^k}{k!}, \end{aligned}$$

with $y = \langle\mathbf{y},\lambda_1\rangle$, $t = t_{\alpha_1}$ and $\phi = \alpha_1^\vee$, where $\{B_k(x)\}$ are the classical Bernoulli polynomials. Let $\mathbf{T} = \{t\in\mathbf{C} \mid |t| < 2\pi\}^n$.

Theorem 2.1 [8,9]**.** *Fix $\mathbf{y}\in V$. Then $F(\mathbf{t},\mathbf{y};\Delta)$ is holomorphic on $\mathbf{T}$ with respect to $\mathbf{t}$.*

For $\mathbf{k} = (k_\alpha)_{\alpha\in\overline{\Delta}} \in \mathbf{N}_0^n$ and $\mathbf{y}\in V$, we define $P(\mathbf{k},\mathbf{y};\Delta)$ and $B_{\mathbf{k}}(\Delta)$ by

$$F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{k}\in\mathbf{N}_0^n} P(\mathbf{k},\mathbf{y};\Delta) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$F(\mathbf{t};\Delta) = \sum_{\mathbf{k}\in\mathbf{N}_0^n} B_{\mathbf{k}}(\Delta) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

Let $y_i = \langle\mathbf{y},\lambda_i\rangle$ for $1\le i\le r$ and we identify $\mathbf{y}$ with $(y_i)_{1\le i\le r} \in \mathbf{R}^r$. We set $\mathbf{Q}[\mathbf{y}] = \mathbf{Q}[(y_i)_{1\le i\le r}]$.

Theorem 2.2 [8,9]**.** *The function $P(\mathbf{k},\mathbf{y};\Delta)$ is analytically continued to a polynomial function $B_{\mathbf{k}}^{(\nu)}(\mathbf{y};\Delta) \in \mathbf{Q}[\mathbf{y}]$ from each $\mathfrak{D}^{(\nu)}$ to the whole space $\mathbf{C}\otimes V$ with its total degree at most $|\mathbf{k}| = \sum_{\alpha\in\Delta_+} k_\alpha$.*

Let $\mathcal{S} = \{\mathbf{s} = (s_\alpha)\in\mathbf{C}^n \mid \Re s_\alpha > 1 \text{ for } \alpha\in\Delta_+\}$ and $\mathcal{K} = \mathcal{S}\cap\mathbf{N}^n$. Note that both $\mathcal{S}$ and $\mathcal{K}$ are $\mathrm{Aut}(\Delta)$-invariant sets.

Theorem 2.3 [8]**.**

$$S(\mathbf{k},\mathbf{y};\Delta) = (-1)^n\left(\prod_{\alpha\in\Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\right) P(\mathbf{k},\mathbf{y};\Delta)$$

for $\mathbf{k}\in\mathcal{K}$.

In the A_1 case, this theorem reduces to the formula

$$\sum_{j\in\mathbf{Z}\setminus\{0\}} \frac{e^{2\pi\sqrt{-1}jy}}{j^k} = -\frac{(2\pi\sqrt{-1})^k}{k!} B_k(\{y\}) \tag{2.1}$$

for $k \geq 2$. Hence the function $P(\mathbf{k}, \mathbf{y}; \Delta)$ may be regarded as a generalization of the Bernoulli periodic functions, $B_{\mathbf{k}}(\Delta) = P(\mathbf{k}, 0; \Delta)$ the Bernoulli numbers and $B_{\mathbf{k}}^{(\nu)}(\mathbf{y}; \Delta)$ the Bernoulli polynomials (see [1]). We have shown in [8] that $P(\mathbf{k}, \mathbf{y}; \Delta)$ is continuous in $\mathbf{y}$ on V and $F(\mathbf{t}, \mathbf{y}; \Delta)$ is continuous on $\mathbf{T} \times V$ if Δ is not of type A_1.

We define generalized Bernoulli numbers $B_{\mathbf{k},\boldsymbol{\chi}}(\Delta)$ by its generating function $G(\mathbf{t}, \boldsymbol{\chi}; \Delta)$ as

$$\begin{aligned} &G(\mathbf{t}, \boldsymbol{\chi}; \Delta) \\ &= \sum_{\substack{a_\alpha = 1 \\ \alpha \in \Delta_+}}^{f_\alpha} \left(\prod_{\alpha \in \Delta_+} \chi_\alpha(a_\alpha)/f_\alpha \right) F(\mathbf{f}\,\mathbf{t}, \mathbf{y}(\mathbf{a}; \mathbf{f}); \Delta) \\ &= \sum_{\mathbf{k} \in \mathbf{N}_0^n} B_{\mathbf{k},\boldsymbol{\chi}}(\Delta) \prod_{\alpha \in \Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}, \end{aligned}$$

where $\mathbf{f}\,\mathbf{t} = (f_\alpha t_\alpha)_{\alpha \in \Delta_+}$ and

$$\mathbf{y}(\mathbf{a}; \mathbf{f}) = \sum_{\alpha \in \Delta_+} \frac{a_\alpha}{f_\alpha} \alpha^\vee.$$

Theorem 2.4 [9]. *Let* $\mathbf{k} \in \mathcal{K}$. *Assume* $k_\alpha = k_\beta$, $\chi_\alpha = \chi_\beta$ *if* $\|\alpha\| = \|\beta\|$, *and assume* $(-1)^{-k_\alpha}\chi_\alpha(-1) = 1$ *for all* $\alpha \in \Delta_+$. *Then*

$$\begin{aligned} &L_r(\mathbf{k}, \boldsymbol{\chi}; \Delta) \\ &= \frac{(-1)^{|\mathbf{k}|+n}}{|W|} \left(\prod_{\alpha \in \Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha! f_\alpha^{k_\alpha}} g(\chi_\alpha) \right) B_{\mathbf{k},\overline{\boldsymbol{\chi}}}(\Delta), \end{aligned}$$

where $g(\chi)$ *is the Gauss sum.*

Theorem 2.5 [9]. *Assume that* Δ *is an irreducible root system. Moreover assume that* $f_\alpha > 1$ *if* Δ *is of type* A_1. *Then for* $w \in \mathrm{Aut}(\Delta)$,

$$B_{w^{-1}\mathbf{k}, w^{-1}\boldsymbol{\chi}}(\Delta) = \left(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{-k_\alpha} \chi_\alpha(-1) \right) B_{\mathbf{k},\boldsymbol{\chi}}(\Delta).$$

Theorem 2.6 [9]. *We have* $B_{\mathbf{k},\boldsymbol{\chi}}(\Delta) = 0$ *if there exists an element* $w \in \mathrm{Aut}(\Delta)_{\mathbf{k}} \cap \mathrm{Aut}(\Delta)_{\boldsymbol{\chi}}$ *such that*

$$\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{-k_\alpha} \chi_\alpha(-1) \neq 1,$$

where $\mathrm{Aut}(\Delta)_{\mathbf{k}}$ *and* $\mathrm{Aut}(\Delta)_{\boldsymbol{\chi}}$ *are the stabilizers of* $\mathbf{k}$ *and* $\boldsymbol{\chi}$ *respectively.*

A more explicit form of the generating function $F(\mathbf{t}, \mathbf{y}; \Delta)$ can be calculated. For example, $F(\mathbf{t}, \mathbf{y}; B_2)$ is given as follows:

Example 2.7. The set of positive roots of type B_2 consists of α_1, α_2, $2\alpha_1 + \alpha_2$ and $\alpha_1 + \alpha_2$. Let $t_1 = t_{\alpha_1}$, $t_2 = t_{\alpha_2}$, $t_3 = t_{2\alpha_1+\alpha_2}$ and $t_4 = t_{\alpha_1+\alpha_2}$. Let $\phi = \alpha_1^\vee + \varepsilon\alpha_2^\vee$ where $\varepsilon > 0$ is sufficiently small. Then we have

$$\begin{aligned} &F(\mathbf{t}, \mathbf{y}; B_2) = t_1 t_2 t_3 t_4 \\ &\times \Bigg(\frac{e^{\{y_1\}t_1 + \{y_2\}t_2}}{(e^{t_1}-1)(e^{t_2}-1)(t_1+t_2-t_3)(t_1+2t_2-t_4)} \\ &+ \frac{e^{\{y_1-y_2\}t_1 + \{y_2\}t_3}}{(e^{t_1}-1)(e^{t_3}-1)(t_1+t_2-t_3)(t_1-2t_3+t_4)} \\ &- \frac{2(e^{\{y_1-\frac{y_2}{2}+\frac{1}{2}\}t_1 + \{\frac{y_2}{2}+\frac{1}{2}\}t_4} + e^{\{y_1-\frac{y_2}{2}\}t_1 + \{\frac{y_2}{2}\}t_4})}{(e^{t_1}-1)(e^{t_4}-1)(t_1+2t_2-t_4)(t_1-2t_3+t_4)} \\ &- \frac{e^{(1-\{y_1-y_2\})t_2 + \{y_1\}t_3}}{(e^{t_2}-1)(e^{t_3}-1)(t_1+t_2-t_3)(t_2+t_3-t_4)} \\ &+ \frac{e^{(1-\{2y_1-y_2\})t_2 + \{y_1\}t_4}}{(e^{t_2}-1)(e^{t_4}-1)(t_1+2t_2-t_4)(t_2+t_3-t_4)} \\ &+ \frac{e^{\{2y_1-y_2\}t_3 + (1-\{y_1-y_2\})t_4}}{(e^{t_3}-1)(e^{t_4}-1)(t_2+t_3-t_4)(t_1-2t_3+t_4)} \Bigg). \end{aligned}$$

By using the generating functions, we can explicitly calculate $B_{\mathbf{k},\boldsymbol{\chi}}(\Delta)$. Hence, from Theorem 2.4, we obtain the following examples.

Example 2.8. Let $\mathbb{1}$ be the trivial character. In the case when $\boldsymbol{\chi} = \{\mathbb{1}\} = (\mathbb{1}, \ldots, \mathbb{1})$, $\mathbf{k} = \{2\} = (2, \ldots, 2)$ and $\mathbf{y} = \mathbf{0}$, we have

$$\begin{aligned} \zeta_2(\{2\}; B_2) &= \frac{\pi^8}{302400}; \\ \zeta_3(\{2\}; B_3) &= \frac{19}{8403115488768000}\pi^{18}; \\ \zeta_3(\{2\}; C_3) &= \frac{19}{8403115488768000}\pi^{18}, \end{aligned}$$

which are examples of Witten's volume formulas with explicit values of the constants. Let χ_5 the quadratic character of conductor 5. Then we have

$$\begin{aligned} &L_2(2,2,2,2; \chi_5, \chi_5, \chi_5, \chi_5; B_2) = \frac{92}{29296875}\pi^8; \\ &L_2(2,4,4,2; \chi_5, \chi_5, \chi_5, \chi_5; B_2) = \frac{133676}{17303466796875}\pi^{12}; \\ &L_2(2,2,2,2; \mathbb{1}, \chi_5, \chi_5, \mathbb{1}; B_2) = -\frac{3679}{1230468750}\pi^8; \\ &L_3(2,2,2,2,2,2; \chi_5, \chi_5, \chi_5, \chi_5, \chi_5, \chi_5; A_3) \\ &\quad = -\frac{1856}{213623046875}\pi^{12}. \end{aligned}$$

Also, let ρ_7 be the even cubic character of conductor 7 defined by

$$\rho_7(1) = 1,\ \rho_7(2) = e^{2\pi\sqrt{-1}/3},\ \rho_7(3) = e^{4\pi\sqrt{-1}/3}.$$

Then we obtain

$$L_2(2,2,2,2;\rho_7,\rho_7,\rho_7,\rho_7;B_2) = \frac{\pi^8}{g(\overline{\rho_7})^4}\left(-\frac{3406}{86472015}-\frac{1294\sqrt{-3}}{17294403}\right) = g(\rho_7)^4\pi^8\left(-\frac{3406}{207619308015}-\frac{1294\sqrt{-3}}{41523861603}\right);$$

$$L_2(2,4,4,2;\mathbb{1},\rho_7,\rho_7,\mathbb{1};B_2) = g(\rho_7)^2\pi^{12}\left(\frac{69967019}{181289027372537700} + \frac{102810289\sqrt{-3}}{181289027372537700}\right).$$

3. Functional relations. By using the method introduced in the papers [18,19] of the third named author, we can prove some functional relations among zeta-functions and also among L-functions of root systems which include Witten's volume formulas as follows:

Example 3.1. In the case of A_3 type, we have

$$2\zeta_3(2,2,s,2,2,2;A_3)+\zeta_3(2,s,2,2,2,2;A_3) + \zeta_3(2,2,2,2,s,2;A_3)+2\zeta_3(2,2,2,2,2,s;A_3) = 339\zeta(s+10)-256\zeta(2)\zeta(s+8) + 74\zeta(4)\zeta(s+6)+2\zeta(6)\zeta(s+4).$$

This equation, as well as the functional equations stated below, holds for all $s\in\mathbf{C}$ except for singular points of functions on the both sides.

In particular, putting $s=2$ in the above equation, we obtain

$$\zeta_3(\{2\};A_3)=\frac{23}{2554051500}\pi^{12}$$

which was obtained by Gunnells and Sczech [2]. Note that Nakamura [17] considers functional relations of A_3 type in a different way.

By using our method, we can further obtain

$$\zeta_3(\{1\};A_3)=-\frac{62}{105}\zeta(2)^3+2\zeta(3)^2,$$

which is not included in Witten's volume formulas.

Example 3.2. In the case of C_3 type, we have

$$\zeta_3(2,2,s,2,2,2,2,2,2;C_3) + \zeta_3(2,2,2,2,s,2,2,2,2;C_3) + \zeta_3(2,2,2,2,2,2,s,2,2;C_3) = \frac{184775}{4096}\zeta(s+16)-\frac{16875}{512}\zeta(2)\zeta(s+14) + \frac{513}{64}\zeta(4)\zeta(s+12)+\frac{25}{64}\zeta(6)\zeta(s+10) + \frac{1}{32}\zeta(8)\zeta(s+8).$$

Putting $s=2$, we obtain

$$\zeta_3(\{2\};C_3)=\frac{19}{8403115488768000}\pi^{18},$$

which coincides with a result stated in Example 2.8.

Example 3.3. We further consider the case of G_2 type in [10], for example,

$$\zeta_2(2,s,2,2,2,2;G_2)+\zeta_2(2,2,s,2,2,2;G_2) + \zeta_2(2,2,2,s,2,2;G_2) = -\frac{5}{1458}\left(2^{-s}+\frac{5519}{4}\right)\zeta(s+10) - \frac{1}{162}(2^{-s}-466)\zeta(2)\zeta(s+8).$$

Putting $s=2$, we obtain

$$\zeta_2(2,2,2,2,2,2;G_2)=\frac{23}{297904566960}\pi^{12}.$$

Example 3.4. Concerning the L-function of B_2 type, we obtain

$$\begin{aligned}
&L_2(2,2,s,2;\chi_5,\chi_5,\chi_5,\chi_5;B_2) + L_2(2,s,2,2;\chi_5,\chi_5,\chi_5,\chi_5;B_2)\\
&=\frac{1}{50}\Bigl[3\pi\sqrt{-1}\Bigl\{\mathrm{Li}\Bigl(s+5;e^{2\pi\sqrt{-1}/5}\Bigr) - \mathrm{Li}\Bigl(s+5;e^{-2\pi\sqrt{-1}/5}\Bigr)\Bigr\}\\
&\quad+6\pi\sqrt{-1}\Bigl\{\mathrm{Li}\Bigl(s+5;e^{4\pi\sqrt{-1}/5}\Bigr) - \mathrm{Li}\Bigl(s+5;e^{-4\pi\sqrt{-1}/5}\Bigr)\Bigr\}\\
&\quad-2\pi^2\Bigl\{\mathrm{Li}\Bigl(s+4;e^{2\pi\sqrt{-1}/5}\Bigr)+\mathrm{Li}\Bigl(s+4;e^{-2\pi\sqrt{-1}/5}\Bigr)\Bigr\}\\
&\quad-\frac{2}{5}\pi^2\Bigl\{\mathrm{Li}\Bigl(s+4;e^{4\pi\sqrt{-1}/5}\Bigr)-\mathrm{Li}\Bigl(s+4;e^{-4\pi\sqrt{-1}/5}\Bigr)\Bigr\}\\
&\quad+\frac{24}{5}\pi^2\zeta(s+4)\Bigr],
\end{aligned}$$

where $\mathrm{Li}(s;z)=\sum_{n\geq 1}z^n n^{-s}$. Putting $s=2$ and using $\zeta(6)=\pi^6/945$,

$$\sum_{m=1}^{\infty}\frac{\sin(2\pi m/5)}{m^7}=\frac{1112}{3515625}\pi^7,$$

and so on, we obtain

$$L_2(2,2,2,2;\chi_5,\chi_5,\chi_5,\chi_5;B_2)=\frac{92}{29296875}\pi^8$$

which is also a result in Example 2.8.

Remark 3.5. As mentioned above, the functional relations stated in this section can be obtained by the method in [18,19]. However we can also obtain them by using a certain generalization of the method stated in Section 2. This result will be given in a forthcoming paper.

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