Lindelöf theorems for monotone Sobolev functions with variable exponent

By Toshihide FUTAMURA*) and Tetsu SHIMOMURA**)

(Communicated by Shigefumi MORI, M.J.A., Jan. 15, 2008)

Abstract: Our aim in this note is to deal with Lindelöf theorems for monotone Sobolev functions with variable exponent.

Key words: Monotone Sobolev functions; Lindelöf theorem; variable exponent.

1. Introduction. Let $\mathbf{B}$ be the unit ball of the n-dimensional Euclidean space $\mathbf{R}^n$. We denote by $\delta_{\mathbf{B}}(x)$ the distance of x from the boundary $\partial\mathbf{B}$, that is, $\delta_{\mathbf{B}}(x) = 1 - |x|$. We denote by $B(x, r)$ the open ball centered at x with radius r and set $\lambda B(x, r) = B(x, \lambda r)$ for $\lambda > 0$.

A continuous function u on $\mathbf{B}$ is called monotone in the sense of Lebesgue (see [8]) if the equalities

$$\max_{\overline{D}} u = \max_{\partial D} u \quad \text{and} \quad \min_{\overline{D}} u = \min_{\partial D} u$$

hold whenever D is a domain with compact closure $\overline{D} \subset \mathbf{B}$. If u is a monotone function on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} |\nabla u(z)|^p \, dz < \infty \quad \text{for some} \quad p > n - 1,$$

then

$$(1.1) \qquad |u(x) - u(y)| \le C(n,p) r^{1-n/p} \left(\int_{2B} |\nabla u(z)|^p \, dz \right)^{1/p}$$

whenever $y \in B = B(x, r)$ with $2B \subset \mathbf{B}$, where $C(n, p)$ is a positive constant depending only on n and p (see [11, Chapter 8] and [15, Section 16]). Using this inequality (1.1), the first author and Mizuta proved Lindelöf theorems for monotone Sobolev functions on the half space of $\mathbf{R}^n$ in [2]. For related results, see Koskela-Manfredi-Villamor [6], Manfredi-Villamor [9,10], Mizuta [11,12], the first author and Mizuta [3,4] and the first author [1].

Our aim in this note is to establish Lindelöf theorems for monotone Sobolev functions u on $\mathbf{B}$ satisfying

$$(1.2) \qquad \int_{\mathbf{B}} |\nabla u(z)|^{p(z)} dz < \infty$$

with variable exponents $p(\cdot)$ satisfying so called a log-Hölder condition. For generalized Lebesgue spaces, we refer to Orlicz [13], Kováčik-Rákosník [7] and Růžička [14]. In this note, we are concerned with a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ satisfying the following conditions:

(p1) $p_-(\mathbf{B}) \equiv \inf_{\mathbf{B}} p(x) > n - 1$,

(p2) $|p(x) - p(y)| \le \dfrac{C}{\log(1/|x-y|)}$ whenever $|x - y| < 1/e$, $x \in \mathbf{B}$ and $y \in \mathbf{B}$, for some constant $C > 0$.

Theorem. *Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). Define a set E of all $\xi \in \partial\mathbf{B}$ which satisfies*

$$\limsup_{r \to 0} r^{p(\xi)-n} \int_{B(\xi,r)\cap\mathbf{B}} |\nabla u(z)|^{p(z)} \, dz > 0.$$

If $\xi \in \partial\mathbf{B} \setminus E$ and there exists a rectifiable curve γ in $\mathbf{B}$ tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

Remark 1. We know that E is of $C_{1,p(\cdot)}$-capacity zero. For the definition of $(1, p(\cdot))$-capacity $C_{1,p(\cdot)}$ and this fact, we refer to [5].

2. Proof of the Theorem. Throughout this paper, let C denote various constants independent of the variables in question.

For a proof of the Theorem, we prepare the following lemmas.

Lemma 1. *Let u be a monotone function on $\mathbf{B}$ satisfying*

2000 Mathematics Subject Classification. Primary 31B25, 46E35.

*) Department of Mathematics, Daido Institute of Technology, 10-3 Takiharu, Nagoya 457-8530, Japan.

**) Department of Mathematics, Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima 739-8524, Japan.

† Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday.

$$\int_{\mathbf{B}} |\nabla u(z)|^{p(z)} dz \le 1. \tag{2.1}$$

Then

$$|u(x) - u(y)| \le C\delta_{\mathbf{B}}(x) + C\delta_{\mathbf{B}}(x)^{1-n/p(x)} \times \left(\int_{2B(x)} |\nabla u(z)|^{p(z)} dz \right)^{1/p(x)} \tag{2.2}$$

for whenever $x \in \mathbf{B}$ *and* $y \in B(x)$*, where* $B(x) = B(x, \delta_{\mathbf{B}}(x)/4)$.

Proof. For $x \in \mathbf{B}$, consider the function $p_*(x) = \inf_{z \in 2B(x)} p(z)$. Since u is monotone in $4B(x)$ and $p_*(x) \ge p_-(\mathbf{B}) > n-1$, we see that

$$|u(x) - u(y)| \le C\delta_{\mathbf{B}}(x)^{1-n/p_*(x)} \left(\int_{2B(x)} |\nabla u(z)|^{p_*(x)} dz \right)^{1/p_*(x)}$$

for every $y \in B(x)$. First note that

$$\begin{aligned} \delta_{\mathbf{B}}(x)^{1-n/p_*(x)} &= \delta_{\mathbf{B}}(x)^{1-n/p(x)} \delta_{\mathbf{B}}(x)^{-n(p(x)-p_*(x))/(p(x)p_*(x))} \\ &\le \delta_{\mathbf{B}}(x)^{1-n/p(x)} \delta_{\mathbf{B}}(x)^{-C/\log(1/\delta_{\mathbf{B}}(x))} \\ &\le C\delta_{\mathbf{B}}(x)^{1-n/p(x)}. \end{aligned}$$

Set $G = \{z \in 2B(x) : |\nabla u(z)| \ge 1\}$. Then

$$\begin{aligned} \int_{2B(x)} |\nabla u(z)|^{p_*(x)} dz &= \int_G |\nabla u(z)|^{p(z)} |\nabla u(z)|^{p_*(x)-p(z)} dz + \int_{2B(x) \setminus G} |\nabla u(z)|^{p_*(x)} dz \\ &\le \int_{2B(x)} |\nabla u(z)|^{p(z)} dz + C\delta_{\mathbf{B}}(x)^n, \end{aligned}$$

so that we obtain by (2.1)

$$\begin{aligned} |u(x) - u(y)| &\le C\delta_{\mathbf{B}}(x)^{1-n/p(x)} \left(\int_{2B(x)} |\nabla u(z)|^{p(z)} dz \right)^{1/p_*(x)} + C\delta_{\mathbf{B}}(x) \\ &\le C\delta_{\mathbf{B}}(x)^{1-n/p(x)} \left(\int_{2B(x)} |\nabla u(z)|^{p(z)} dz \right)^{1/p(x)} + C\delta_{\mathbf{B}}(x), \end{aligned}$$

as required. □

The following lemma can be proved by (2.2).

Lemma 2 (cf. [2, Lemma 1]). *Let* u *be a monotone function on* $\mathbf{B}$ *satisfying* (1.2). *If* $\xi \in \partial\mathbf{B} \setminus E$ *and there exists a sequence* $\{r_j\}$ *such that* $2^{-j-1} \le r_j < 2^{-j}$ *and* $u((1-r_j)\xi)$ *has a finite limit* L*, then* u *has a nontangential limit* L *at* ξ.

Lemma 3. *Let* $\{p_j\}$ *be a sequence such that* $p_* = \inf p_j > 1$ *and* $p^* = \sup p_j < \infty$. *Then*

$$\sum |a_j b_j| \leqq 2 \Big(\sum |a_j|^{p_j} \Big)^{1/q} \Big(\sum |b_j|^{p_j'} \Big)^{1/q'},$$

where $1/p_j + 1/p_j' = 1$, $q = p_*$ *if* $\sum |a_j|^{p_j} \ge \sum |b_j|^{p_j'}$ *and* $q = p^*$ *if* $\sum |a_j|^{p_j} \le \sum |b_j|^{p_j'}$.

Proof. Let $A = \sum |a_j|^{p_j}$ and $B = \sum |b_j|^{p_j'}$. In case $A \ge B$, for $0 < k \le 1$, we have

$$\begin{aligned} \sum |a_j b_j| &\le k \Big(\sum |a_j|^{p_j} + \sum |b_j/k|^{p_j'} \Big) \\ &\le k \Big\{ A + k^{-(p_*)'} B \Big\}. \end{aligned}$$

Here considering k such that $k^{(p_*)'} = B/A$, we find

$$\sum |a_j b_j| \le 2A^{1/p_*} B^{1/(p_*)'},$$

as required.

The remaining case can be proved similarly. □

Now we can prove the Theorem.

Proof of the Theorem. Without loss of generality we may assume that (2.1) holds. For $r > 0$ sufficiently small, take $x(r) \in \gamma \cap \partial B(\xi, r)$ and set $y(r) = (1-r)\xi$. We can take a finite chain of balls B_0, B_1, …, B_N such that

(i) $B_j = B(x_j)$, $x_j \in \partial B(\xi, r) \cap \mathbf{B}$, $x_0 = x(r)$ and $y(r) \in B_N$;

(ii) $\{\delta_{\mathbf{B}}(x_j)\}$ increase and $\delta_{\mathbf{B}}(x_j) \ge C|x(r) - x_j|$ for some constant $C > 0$;

(iii) $B_j \cap B_k \ne \emptyset$ if and only if $|j-k| \le 1$.

See [3, Lemma 2.2]. Set $p_j = p(x_j)$ and pick $z_j \in B_{j-1} \cap B_j$ for $1 \le j \le N$; set $z_0 = x(r)$ and $z_{N+1} = y(r)$. Since $p(\xi) > n-1$, there exists $\alpha > 0$ such that $n - p(\xi) < \alpha < 1$. Further, by the continuty of $p(\cdot)$, we may assume that $p(x(r)) - (n-\alpha) > (p(\xi) - (n-\alpha))/2$. By Lemmas 1 and 3, we see that

$$\begin{aligned} |u(x(r)) - u(y(r))| &\le \sum_{j=0}^{N} |u(z_{j+1}) - u(z_j)| \\ &\le C \sum_{j=0}^{N} \delta_{\mathbf{B}}(x_j)^{1-n/p_j} \left(\int_{2B_j} |\nabla u(z)|^{p(z)} dz \right)^{1/p_j} \end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{j=0}^{N}\delta_{\mathbf{B}}(x_j)\\
&\le C\sum_{j=0}^{N}\delta_{\mathbf{B}}(x_j)^{1-(n-\alpha)/p_j}\\
&\quad\times\left(\int_{2B_j}|\nabla u(z)|^{p(z)}\delta_{\mathbf{B}}(z)^{-\alpha}dz\right)^{1/p_j} + Cr\\
&\le C\left(\sum_{j=0}^{N}\delta_{\mathbf{B}}(x_j)^{p_j'\{1-(n-\alpha)/p_j\}}\right)^{1/q'(r)}\\
&\quad\times\left(\sum_{j=0}^{N}\int_{2B_j}|\nabla u(z)|^{p(z)}\delta_{\mathbf{B}}(z)^{-\alpha}dz\right)^{1/q(r)} + Cr\\
&\le C\left(\int_{B(\xi,2r)\cap\mathbf{B}}|\nabla u(z)|^{p(z)}|r-|z-\xi||^{-\alpha}dz\right)^{1/q(r)}\\
&\quad\times I^{(q(r)-1)/q(r)} + Cr,
\end{aligned}
$$

where $I = \sum_{j=0}^{N}\delta_{\mathbf{B}}(x_j)^{p_j'\{1-(n-\alpha)/p_j\}}$, $\min p_j \le q(r) \le \max p_j$ and $1/q(r)+1/q'(r)=1$. Here note that

$$
\begin{aligned}
&\frac{p_j-(n-\alpha)}{p_j-1}\\
&\quad=\frac{p(x(r))-(n-\alpha)}{p(x(r))-1}-\frac{(n-\alpha-1)\{p(x(r))-p_j\}}{\{p(x(r))-1\}(p_j-1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{(n-\alpha-1)\{p(x(r))-p_j\}}{\{p(x(r))-1\}(p_j-1)}\right| &\le \frac{C}{\log(1/|x(r)-x_j|)}\\
&\le \frac{C}{\log(1/\delta_{\mathbf{B}}(x_j))}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
I &\le \sum_{j=0}^{N}\delta_{\mathbf{B}}(x_j)^{\frac{p(x(r))-(n-\alpha)}{(p(x(r))-1)}}\delta_{\mathbf{B}}(x_j)^{-\frac{C}{\log(1/\delta_{\mathbf{B}}(x_j))}}\\
&\le C\sum_{j=0}^{N}\delta_{\mathbf{B}}(x_j)^{\{p(x(r))-(n-\alpha)\}/(p(x(r))-1)}\\
&\le Cr^{\{p(x(r))-(n-\alpha)\}/(p(x(r))-1)},
\end{aligned}
$$

since $p(x(r))-(n-\alpha) > (p(\xi)-(n-\alpha))/2 > 0$. Further, since

$$
\begin{aligned}
&\left|\frac{\{p(x(r))-(n-\alpha)\}(q(r)-1)}{p(x(r))-1}-\{p(\xi)-(n-\alpha)\}\right|\\
&\quad\le \frac{C}{\log(1/r)},
\end{aligned}
$$

we have

$$
I^{q(r)-1} \le Cr^{p(\xi)-(n-\alpha)}r^{-C/\log(1/r)} \le Cr^{p(\xi)-(n-\alpha)}.
$$

Then we obtain

$$
\begin{aligned}
&|u(x(r))-u(y(r))|^{q(r)} \le Cr^{p(\xi)-(n-\alpha)}\\
&\quad\times\int_{B(\xi,2r)\cap\mathbf{B}}|\nabla u(z)|^{p(z)}|r-|z-\xi||^{-\alpha}dz + Cr.
\end{aligned}
$$

Moreover, since $0<\alpha<1$, we see that

$$
\int_{2^{-j-1}}^{2^{-j}}|r-|z-\xi||^{-\alpha}dr \le C2^{-j(1-\alpha)}.
$$

Hence it follows that

$$
\begin{aligned}
&\inf_{2^{-j-1}\le r<2^{-j}}|u(x(r))-u(y(r))|^{q(r)}\\
&\le C\int_{2^{-j-1}}^{2^{-j}}\Bigg(r^{p(\xi)-(n-\alpha)}\int_{B(\xi,2r)\cap\mathbf{B}}|\nabla u(z)|^{p(z)}\\
&\qquad\times|r-|z-\xi||^{-\alpha}dz\Bigg)\frac{dr}{r}+C2^{-j}\\
&\le C2^{-j\{p(\xi)-(n-\alpha)-1\}}\int_{B(\xi,2^{-j+1})\cap\mathbf{B}}|\nabla u(z)|^{p(z)}\\
&\qquad\times\left(\int_{2^{-j-1}}^{2^{-j}}|r-|z-\xi||^{-\alpha}dr\right)dz + C2^{-j}\\
&\le C2^{-j\{p(\xi)-n\}}\int_{B(\xi,2^{-j+1})\cap\mathbf{B}}|\nabla u(z)|^{p(z)}dz + C2^{-j}.
\end{aligned}
$$

Since $\xi\notin E$ and u has a finite limit L at ξ along γ, we find a sequence $\{r_j\}$ such that $2^{-j-1}\le r_j<2^{-j}$ and

$$
\lim_{j\to\infty}u(y(r_j)) = \lim_{j\to\infty}u(x(r_j)) = L.
$$

Thus u has a nontangential limit L at ξ by Lemma 2. □

Remark 2. Let u be a monotone function on $\mathbf{B}$ satisfying (1.2). Then u has a nontangential limit at $\xi\in\partial\mathbf{B}$ except in a set of $C_{1,p(\cdot)}$-capacity zero.

In fact, to show this, we define

$$
E_1 = \left\{\xi\in\partial\mathbf{B} : \int_{\mathbf{B}}|\xi-y|^{1-n}|\nabla u(y)|dy = \infty\right\}
$$

and set $F = E\cup E_1$, where E is as in the Theorem. Note here from [5, Lemmas 4.1 and 4.4] that F is of $C_{1,p(\cdot)}$-capacity zero. If $\xi\notin E_1$, then u has a finite limit L along a line γ. In view of the Theorem, we see that if $\xi\in\partial\mathbf{B}\setminus F$, then u has a nontangential limit L at ξ.

References

[1] T. Futamura, Lindelöf theorems for monotone Sobolev functions on uniform domains, Hiroshima Math. J. **34** (2004), no. 3, 413–422.
[2] T. Futamura and Y. Mizuta, Lindelöf theorems

for monotone Sobolev functions, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 2, 271–277.

[3] T. Futamura and Y. Mizuta, Boundary behavior of monotone Sobolev functions on John domains in a metric space, Complex Var. Theory Appl. **50** (2005), no. 6, 441–451.

[4] T. Futamura and Y. Mizuta, Continuity of weakly monotone Sobolev functions of variable exponent, in *Potential theory in Matsue*, 127–143, Adv. Stud. Pure Math. **44**, Math. Soc. Japan, Tokyo, 2006.

[5] T. Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for Riesz potential space of variable exponent, Math. Nachr. **279** (2006), no. 13, 1463–1473.

[6] P. Koskela, J. J. Manfredi and E. Villamor, Regularity theory and traces of $\mathcal{A}$-harmonic functions, Trans. Amer. Math. Soc. **348** (1996), no. 2, 755–766.

[7] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (**116**) (1991), no. 4, 592–618.

[8] H. Lebesgue, Sur le probléme de Dirichlet, Rend. Cir. Mat. Palermo **24** (1907), 371–402.

[9] J. J. Manfredi and E. Villamor, Traces of monotone Sobolev functions, J. Geom. Anal. **6** (1996), no. 3, 433–444.

[10] J. J. Manfredi and E. Villamor, Traces of monotone Sobolev functions in weighted Sobolev spaces, Illinois J. Math. **45** (2001), no. 2, 403–422.

[11] Y. Mizuta, *Potential theory in Euclidean spaces*, Gakkōtosyo, Tokyo, 1996.

[12] Y. Mizuta, Tangential limits of monotone Sobolev functions, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **20** (1995), no. 2, 315–326.

[13] W. Orlicz, Über konjugierte Exponentenfolgen, Studia Math. **3** (1931), 200–211.

[14] M. Růžička, *Electrorheological fluids : modeling and Mathematical theory*, Lecture Notes in Math., 1748, Springer, Berlin, 2000.

[15] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lectures Notes in Math., 1319, Springer, Berlin, 1988.